Interaction Patterns and Web-Structures of Resonant Solitons of the Kadomtsev-Petviashvili Equation

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Interaction Patterns and Web-structures
of
Resonant Solitons of the Kadomtsev-Petviashvili Equation

by

Anupama Tippabhotla

A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
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DEDICATION

To my brother Siva Kumar
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Interaction Patterns and Web-structures of Resonant Solitons of the Kadomtsev-Petviashvili Equation

Anupama Tippabhotla

ABSTRACT

In this thesis, the interaction pattern for a class of soliton solutions of the Kadomtsev-Petviashvili (KP) equation \((-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0\) is analyzed. The complete asymptotic properties of the soliton solutions for \(y \to \pm \infty\) are determined. The resonance characteristic of two sub-classes of the soliton solutions, in which \(N_+\) incoming line solitons for \(y \to -\infty\) interact to form \(N_+\) outgoing line solitons for \(y \to \infty\), is described. These two specific sub-classes of \((N_-, N_+)\)-soliton solutions are the following:

1) \{(2, 3), (2, 4), (2, 5)\},
2) \{(3, 2), (3, 3), (3, 4)\}.

The intermediate solitons and the interaction regions of the above soliton solutions are determined, and their various interaction patterns are explored. Maple and Mathematica are used to get the 3 dimensional plots and contour plots of the soliton solutions to show their interaction patterns. Finally, the spider-web-structures of the discussed solitons of the KP equation are displayed.
1. Introduction

Waves and vibrations are among the most basic forms of motion, and their study goes a very long way back in time. Small amplitude waves are described mathematically by a linear differential equation, and their behaviour can be studied in detail. In contrast, when the amplitude is not restricted to being small, the differential equation becomes nonlinear, and its analysis becomes in general an extremely difficult problem. Localized large amplitude waves called solitons represent one of the most striking aspects of nonlinear phenomena, though there exist other types of phenomena like positons [14] and complexitons [10]. Solitons propagate without spreading and have particle-like properties, which retain their identities in a collision. During the collision of solitons the solution cannot be represented as a linear combination of two soliton solutions, but after collision solitons recover their shapes and the only result of collision is phase shift.

1.1 Discovery of solitary waves  The solitary wave [16,20] was first observed by J. Scott Russell on the Edinburgh-Glasgow canal in 1834; he called it the great wave of translation. Russell reported his observations to the British association in his 1844 Report on Waves in the following words:

“I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still on at a rate of some 8 or 9 miles an hour, preserving its original figure some 30 feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of 1 or 2 miles, I lost it in the windings of the channel.” Russell also performed some laboratory experiments, generating solitary waves by dropping a weight at one end of a water channel.

Mikio Sato was the first to discover that the Kadomtsev-Petviashvili (KP) equation is the most fundamental among many soliton equations. Sato discovered that polynomial solutions of the bilinear KP equation are equivalent to the characteristic polynomials of the general linear group. Later, he found a Lax pair for a hierarchy of KP-like equations by means of a pseudo-differential operator and came to the conclusion that the KP equation is equivalent to the motion of a point in a Grassmanian manifold and its bilinear equation is nothing but a plucker relation. Also, Junikichi Satsuma had discovered before Sato that the soliton solutions of the KdV equation could be expressed in terms of Wronskian determinants [19,13]. Later, in 1983, Freeman and Nimmo found that the KP bilinear
equation could be rewritten as a determinantal identity if one expresses its soliton solutions in terms of Wronskian.

The KdV equation is a 1+1-dimensional equation describing shallow water waves. The KP equation was introduced in order to discuss the stability of shallow water waves to perpendicular horizontal perturbations. Some physicists may object to the idea that the KP equation is the most fundamental of the soliton equations. First of all, the KdV equation is derived through a certain approximation, and then the KP equation is obtained from the KdV equation under the assumption that horizontal perturbations are small. The KP equation is fundamental because of the simple mathematical structure of its solutions and its relation to the other soliton equations arising from this simplicity.

The KP equation is the 2+1-dimensional (two-dimensional space, \((x, y)\) plus one-dimensional time, \(t\)) nonlinear partial differential equation:

\[
(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0
\]

Since we obtain the KdV equation (which is the most celebrated equation in the integral equations) by neglecting the \(y\)-derivative term, this is also called the two-dimensional KdV equation (shallow water wave equation proposed at the end of nineteenth century). At present, a huge number of concrete examples of integrable nonlinear differential (and difference) equations are known. Many techniques for finding exact solutions of these equations have also been discovered: inverse scattering theory which solves the initial value problem, the bilinear method initiated by Ryogo Hirota, the theory of quasi periodic solutions based on Riemann surface and theta functions, etc.

The investigation of soliton-like structures in realistic and intricate multidimensional world obviously is of great interest. The solitons in more dimensions have been found experimentally and theoretically in many branches of physics and applied mathematics. They exist in the ocean as waves bombarding oil wells, also exist in much smaller natural and laboratory systems as plasmas, molecular systems, laser pulses propagating in solids, super fluid helium magnetic system, structural phase transitions, liquid crystals, polymers and fluid flows as well in elementary particles.

1.2 Objectives of this thesis

The aim of this thesis is to study the interactions, structures, resonance and asymptotic properties for a class of soliton solutions of the KP equation, where \(N_-\) and \(N_+\) denote the numbers of incoming and outgoing line solitons, respectively. We will observe the changes in the interaction patterns and web-structures by varying the values of parameters in the above soliton solutions of the KP equation. We will find the intermediate solitons based on the work of [9], and describe the interaction between various intermediate solitons. Here we will be considering the following two specific sub-classes of \((N_-, N_+)\)-soliton solutions:

1) \{(2, 3), (2, 4), (2, 5)\},
2) \{(3, 2), (3, 3), (3, 4)\}.

By considering the idea of fundamental resonance [9], we describe the intermediate and
interaction patterns of the resonant solitons in the \((x, y)\) plane. Then we will analyze the asymptotic properties of the two above mentioned sub-classes of \((N_-, N_+)\)-soliton solutions in this thesis.

1.3 Summary In the chapter 1, we have seen the introduction about solitons, discovery of solitary waves and some applications of solitons. In the chapter 2, we will study some basic definitions and theorems required to understand this thesis. Most of the results of this chapter are from Ref [9]. Finally in the chapter 3, we will describe the actual interaction patterns, analysis of the asymptotic properties and web-structures of two specific sub-classes of \((N_-, N_+)\)-soliton solutions of the KP equation.
2. Basic Facts

2.1 The KP equation and its Wronskian solutions

The aim of this chapter is to study a family of soliton solutions of the Kadomtsev-Petviashvili (KP) equation

\[
(-4u_{t} + u_{xxx} + 6uu_{x})_{x} + 3u_{yy} = 0
\]  

which can be transformed into the bilinear form

\[
(-4D_{x}D_{t} + D_{x}^{4} + 3D_{y}^{2})\tau \cdot \tau = 0,
\]

Here \(D_{x}, D_{y}, \) and \(D_{t}\) are the Hirota derivatives, for example, the Hirota derivative \(D_{m}\) is defined as

\[
D_{m}f = (\partial_{x} - \partial_{t})^{m} f(x,y,t)g(x',y',t)|_{x=x'},
\]

and \(u\) is obtained from the tau-function \(\tau(x,y,t)\) through

\[
u(x,y,t) = 2\frac{\partial^{2}}{\partial x^{2}} \log \tau(x,y,t).
\]

It is well known that some solutions of the KP equation can be obtained by the Wronskian form \(\tau = \tau_{M}\) with \((3,6,7,13)\)

\[
\tau_{M} = W_{\tau}(f_{1}, \ldots, f_{M}) := \left| \begin{array}{ccc}
f_{1}^{(0)} & \cdots & f_{M}^{(0)} \\
\vdots & \ddots & \vdots \\
f_{1}^{(M-1)} & \cdots & f_{M}^{(M-1)} \end{array} \right|,
\]

where \(f_{i}^{(n)} = \partial^{n} f_{i} / \partial x^{n}\), and \(\{f_{i}(x,y,t) | i = 1, \ldots, M\}\) is a linearly independent set of \(M\) solutions of the equations

\[
\frac{\partial f_{i}}{\partial y} = \frac{\partial^{2} f_{i}}{\partial x^{2}}, \quad \frac{\partial f_{i}}{\partial t} = \frac{\partial^{3} f_{i}}{\partial x^{3}}, \quad \text{for} \ 1 \leq i \leq M.
\]

For example, the 2-soliton solution of the KP equation is obtained by the set \((f_{1}, f_{2})\) with

\[
f_{i} = e^{\theta_{2i-1}} + e^{\theta_{2j}}, \quad i, j = 1, 2,
\]

where the phases \(\theta_{j}\) are given by linear functions of \((x,y,t)\)

\[
\theta_{j}(x,y,t) = -k_{j}x + k_{j}^{2}y - k_{j}^{3}t + \theta_{0}, \quad j = 1, \ldots, 4
\]

with \(k_{1} < k_{2} < k_{3} < k_{4}\) and \(k_{j}\)'s being constant.

This ordering is sufficient for the solution \(u\) to be nonsingular. The ordering \(k_{1} \neq k_{2} < k_{3} \neq k_{4}\) is needed for the positivity of \(\tau_{2}\). Note, for example, that if
\( \hat{k}_1 < k_2 < k_3 < k_4 \), \( \tau_2 \) takes zero and the solution blows up at some points in formula (2.6) can be extended to the \( M \)-soliton solution with \( \{ f_1, \ldots, f_M \} \)[17].

On the other hand, it is also known that the solutions of the finite Toda lattice hierarchy are obtained by the set of tau-functions \( \{ \tau_{Mi} \mid i = 1 \ldots M \} \) with the choice of \( f \)-functions,

\[
\begin{cases}
  f_1 = \sum_{i=1}^{N} e^{\theta_i} = : f , \\
  f_i = f^{(i-1)} , \quad 1 < i \leq M \leq N ,
\end{cases}
\]

where the phases \( \theta_i , \ 1 \leq i \leq N , \) are given in the form (2.7)[8]. This implies that each \( \tau_{Mi} \) gives a solution of the KP equation. If the \( f \)-functions are chosen according to (2.8), the tau-functions are then given by the Hankel determinants

\[
\tau_M = \begin{vmatrix}
  f^{(0)} & f^{(1)} & \cdots & f^{(M-1)} \\
  f^{(1)} & f^{(2)} & \cdots & f^{(M)} \\
  \vdots & \ddots & \vdots & \vdots \\
  f^{(M-1)} & f^{(M)} & \cdots & f^{(2M-2)}
\end{vmatrix},
\]

for \( 1 \leq M \leq N \). Note here that \( \tau_M = c \exp(\theta_1 + \ldots + \theta_N) \), with \( c=\text{constant} \), yielding the trivial solution. Note also that \( \tau_M \) and \( \tau_{N-M} \) produce the same solution with the symmetry \( (x, y, t) \rightarrow (-x, -y, -t) \), due to duality of the determinants.

In this thesis, we are concerned with the behavior of the KP solutions (2.3) whose tau functions are given by (2.9). We describe the patterns of the solutions in the \( x - y \) plane where each soliton solution of the KP equation is asymptotically expressed as a line, namely,

\[
x = c_\pm y + \xi_\pm , \quad \text{for} \quad y \rightarrow \pm \infty ,
\]

with appropriate constants \( c_\pm \) and \( \xi_\pm \) for a fixed \( t \). In particular, we find that all the solutions are ‘resonant’ solitons in the sense that these solutions are different from ordinary multi-soliton solutions. The difference appears in the interaction patterns.

Suppose the numbers of line solitons in asymptotic stages as \( y \rightarrow \pm \infty \) are denoted by \( N_\pm \), respectively. Therefore, the number of incoming solitons is represented by \( N_- \) and the number of outgoing solitons is represented by \( N_+ \), which is given by the size of the hankel determinant (2.9), i.e., \( N_+ = M \).

Thus, the total number \( N \) of exponential terms in the function \( f \) in (2.8) gives the total number of solitons presented in both asymptotic limits, i.e., \( N = (N_- + N_+) \). We call these solutions ‘(\( N_- , N_+ \))-solitons’.
In particular, if \( N = 2 N_+ = 2 N_- \), the solution describes an \( N_- \) -soliton having the same set of line solitons in each asymptotics for \( y \to \pm \infty \). However these multi-soliton solutions also differ from the ordinary multi-soliton solutions of the KP equation. The ordinary \( n \) -soliton solution of the KP equation is described by ‘\( n \)’ intersecting line solitons with a phase shift at each interaction point. If we ignore the phase shifts, these \( n \) lines form \((n - 1)(n - 2)/2\) bounded regions in the generic situation. However, the number of bounded regions for the (resonant) \( N_- \)-soliton solution with (2.9) is found to be \((N_+ - 1)^2\); We will see later that for the case of a \((N_-, N_+)\)-soliton solution, the number of bounded regions (holes) in the graph of the solution is given by \((N_- - 1)(N_+ - 1)\), except at finite values of \( t \) in the temporal evolution.

These resonant [6] \( N_- \)-soliton solutions are similar to some of the solitons of the coupled KP (cKP) hierarchy recently studied. These solutions are called ‘spider-web-like’ [8] solutions. The analysis of finding web structure that we describe in this thesis may also be applied to the case of cKP hierarchy.

2.2 Asymptotic analysis of the solutions  
In the \((x, y)\) plane, the solution \( u \) defined by (2.3), (2.4) and (2.8) with \( N = 2 \) describes a plane wave as \([u = \phi(k_x x + k_y y - \omega t)])\) having the wavenumber vector \( k = (k_x, k_y) \) and the frequency \( \omega \),

\[
k = (k_1 + k_2, k_1^2 - k_2^2) =: k_{1,2}, \quad \omega = k_1^3 - k_2^3 =: \omega_{1,2}.
\]

Here \((k, \omega)\) satisfies the dispersion relation,

\[
4\omega k_x + k_x^4 + 3k_y^2 = 0. \tag{2.10}
\]

We refer to the one-soliton solution as a line soliton, which can be expressed by a (contour) line, \( \theta_1 = \theta_2 \), in the \((x, y)\) plane. In this paper, since we discuss the pattern of soliton solutions in the \((x, y)\) plane, we refer to \( c = dx/dy \) as the velocity of the line soliton in the \( x \) direction; that is, \( c = 0 \) indicates the direction of the positive \( y \)-axis. The results of this section are from Ref [9]. We list and discuss them for later reference in chapter 3.

**Theorem 2.2.1**  
Let \( f \) be given by \( f = \sum_{i=1}^{N} e^{\theta_i} \) with \( \theta_i = -k_x x + k_i y - k_i^3 t + \theta_i^0 \). Then for \( N = N_+ + N_- \) and \( 1 \leq N_+ \leq N - 1 \), the tau-function defined by the Hankel determinant has the form

\[
\tau_{N_+} = \sum_{1 \leq i_1 < \ldots < i_{N_+} \leq N} \Delta(i_1, \ldots, i_{N_+}) \exp\left( \sum_{j=1}^{N_+} e^{\theta_j} \right),
\]

where \( \Delta(i_1, \ldots, i_{N_+}) \) is the square of the VanderMonde determinant
\[
\Delta(i_1, \ldots, i_{N_+}) = \prod_{1 \leq j < l \leq N_+} (k_{ij} - k_{lj})^2.
\]

**Proof.** Apply the Binet-Cauchy theorem for

\[
\tau_{N_+} = \det \begin{pmatrix}
e^{\theta_1} & e^{\theta_2} & \cdots & e^{\theta_{N_+}} \\
k_1 e^{\theta_1} & k_2 e^{\theta_2} & \cdots & k_N e^{\theta_{N_+}} \\
\vdots & \vdots & \ddots & \vdots \\
k_1^{-1} e^{\theta_1} & k_2^{-1} e^{\theta_2} & \cdots & k_N^{-1} e^{\theta_{N_+}}
\end{pmatrix}
\begin{pmatrix}
1 & k_1 & \cdots & k_{N_+}^{-1} \\
1 & k_2 & \cdots & k_{N_+}^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & k_N & \cdots & k_{N_+}^{-1}
\end{pmatrix}
\]

One should note from that the \( \tau_{N_+} \)-function contains all possible combinations of \( N_+ \) phases from the set \( \{\theta_j \mid j = 1, \ldots, N\} \), unlike the case of ordinary multi-soliton solutions of the KP equation. For example, the \( \tau_2 \)-function for the 2-soliton solution with include only four terms, and is missing the combinations \( \theta_1 + \theta_2 \) and \( \theta_3 + \theta_4 \). This makes a crucial difference on the interaction patterns of soliton solutions, as explained in this thesis. In particular, we will see that the \( (N_-, N_+) \)-solitons are all of resonant type\[15\] in the sense that the local structure of each interaction point in those solitons consists of either \( (2, 1) \)- or \( (1, 2) \)-solitons.

**Theorem 2.2.2** The \( \tau_{N_+} \)-function given by (2.9) is positive definite, and therefore, the solution \( u \) has no singularity. In general, the Wronskian takes zeros at some points in the flow parameters.

Let us now define a local coordinate frame \((\xi, y)\) in order to study the asymptotic behaviour for large \(|y|\) with

\[
x = cy + \xi.
\]

Then the phase functions \( \theta_i \) in \( f \) of (2.10) becomes

\[
\theta_i = -k_i \xi + \eta_i(c)y + \theta_i^0, \text{ for } i = 1, \ldots, N,
\]

with

\[
\eta_i(c) = k_i(k_i - c).
\]

Without loss of generality, we assume the ordering for the parameters \( \{k_i \mid i = 1, \ldots, N\} \) as follows:

\[
k_1 < k_2 < \ldots < k_N.
\]

Then one can easily show that the lines \( \eta = \eta_i(c) \) are in general position; that is, each line \( \eta = \eta_i(c) \) intersects with all other lines at \( N - 1 \) distinct points in the \((c, \eta)\) plane; in other words, only two lines meet at each intersection point. Now the purpose is to find the dominant exponential terms in the \( \tau_{N_+} \)-function for \( y \to \pm \infty \) as a function of the velocity \( c \). First note that if only one exponential is dominant, and then \( \omega_1 = -\partial_x \log \tau_{N_+} \) is just a
constant, and therefore the solution \( u = -2\partial_x \omega_i \) is zero. Then, nontrivial contributions to \( u \) arise when one can find two exponential terms, which dominate over the others. Note that because the intersections of the \( \eta_i \)'s are always pair wise, three or more terms cannot make a dominant balance for large \( |y| \).

In the general case, \( N_+ \neq 1 \), the \( \tau_{N_+} \)-function involves exponential terms having combinations of phases, and two exponential terms that make a dominant balance can be found as follows. Let us first define the level of intersection of \( \eta_i(c) \).

**Definition 2.2.3** Let \( \eta_i(c) \) and \( \eta_j(c) \) intersect at the value \( c = c_{i,j} = k_i + k_j \), i.e., \( \eta_i(c_{i,j}) = \eta_j(c_{i,j}) \). The level of intersection, denoted by \( \sigma_{i,j} \), is defined, as the number of other \( \eta_j \)'s that at \( c = c_{i,j} \) are larger than \( \eta_i(c_{i,j}) = \eta_j(c_{i,j}) \). That is, 
\[
\sigma_{i,j} = |\{ \eta_l \mid \eta_l(c_{i,j}) > \eta_i(c_{i,j}) = \eta_j(c_{i,j}) \}|.
\]
We also define \( I(n) \) as the set of pairs \( (n_i,n_j) \) having the level \( \sigma_{i,j} = n \), namely 
\[
I(n) := \{(\eta_i,\eta_j) \mid \sigma_{i,j} = n \text{ for } i \leq j \}
\]
The level of intersection can take the range \( 0 \leq \sigma_{i,j} \leq N - 2 \). Then one can show:

**Theorem 2.2.4** The set \( I(n) \) is given by 
\[
I(n) = \{(\eta_i,\eta_{N-n_{i+1}}) \mid i = 1,\ldots,n+1 \}.
\]

**Proof.** From the assumption \( k_1 < k_2 < \ldots < k_N \), we have the following inequality at \( c = c_{i,j} \) (i.e., \( \eta_i = \eta_j \)) for \( i < j \), 
\[
\eta_{i+1},\ldots,\eta_{j-1} < \eta_i = \eta_j < \eta_1,\ldots,\eta_{i-1},\eta_{j+1},\ldots,\eta_N.
\]
Then taking \( j = N-n-1 \) leads to the assertion of the theorem 2.2.4. \( \Box \)

Note here that the total number of pairs \( (n_i,n_j) \) is 
\[
\binom{N}{2} = \frac{1}{2} N(N-1) = \sum_{n=0}^{N-2} |I(n)|.
\]
For the case of \( (N_-,N_+)-\)solitons, the following formulae are useful:
\[
\begin{aligned}
I(N_- - 1) &= \{\eta_i,\eta_{N_-(N_++1)} \mid i = 1,\ldots,N_- \}, \\
I(N_+ - 1) &= \{\eta_i,\eta_{N_+(N_-+1)} \mid i = 1,\ldots,N_+ \}.
\end{aligned}
\]
Here recall that \( N_+ + N_- = N \). These formulae indicate that, for each intersecting pair \( (n_i,n_j) \) with the level \( N_- - 1 \) \( (N_+ - 1) \), there are \( N_+ - 1 \) terms \( \eta_i \)'s that are smaller (larger) than \( \eta_i = \eta_j \). Then the sum of those \( N_+ - 1 \) terms with either \( \eta_j \) or \( \eta_i \) provides two dominant exponents in the \( \tau_{N_+} \)-function for \( y \to -\infty \) \( (y \to \infty) \) (see more detail in the proof of Theorem 2.2.5). Note also that \( |I(N_- - 1)| = N_- \). Now we can state our main theorem:

**Theorem 2.2.5** Let \( \omega_i \) be a function defined by 
\[
\omega_i = -\frac{\partial}{\partial \xi} \log \tau_{N_-},
\]
with $\tau_{N_i}$ being given by (2.10). Then $\omega_1$ has the following asymptotic for $y \to \pm \infty$:

For $y \to -\infty$ and $x = c_{i,N_i-1} y + \xi$ for $i = 1, \ldots, N_-$,

$$\omega_1 \to \begin{cases} K_i(-,-) := \sum_{j=i}^{N_i+1} k_j & \text{as } \xi \to -\infty, \\
K_i(+,-) := \sum_{j=i}^{N_i-1} k_j & \text{as } \xi \to \infty; \end{cases}$$

and for $y \to \infty$ and $x = c_{i,N_i+1} y + \xi$ for $i = 1, \ldots, N_+$,

$$\omega_1 \to \begin{cases} K_i(-,+) := \sum_{j=1}^{i-1} k_j + \sum_{j=i+1}^{N_i-1} k_{N_i-j+1} & \text{as } \xi \to -\infty, \\
K_i(+,+) := \sum_{j=1}^{i} k_j + \sum_{j=i+1}^{N_i-1} k_{N_i-j+1} & \text{as } \xi \to \infty; \end{cases}$$

where $c_{i,j} = k_i + k_j$.

**Proof.** $N_+ + i$, i.e., $(n_i, n_{N_i+1}) \in \mathcal{I}(N_+ - 1)$, from the theorem 2.2.4 we have the inequality,

$$\eta_{i+1}, \eta_{i+2}, \ldots, \eta_{i+N_i-1} < \eta_i = \eta_{N_i+i}.$$

This implies that, for $c_{i,j} = k_i + k_{N_i+j}$, the following two exponential terms in the $\tau_{N_+}$-function in theorem 2.2.1,

$$\exp(\sum_{j=1}^{N_i+1} \theta_j), \quad \exp(\sum_{j=i+1}^{N_i+1} \theta_j),$$

provide the dominant terms for $y \to -\infty$. Note that the condition $\eta_i = \eta_{N_i+i}$ leads to $c = c_{i,N_i+j} = k_i + k_{N_i+j}$. Thus the function $\omega_1$ can be approximated by the following form along $x = c_{i,N_i+j} y + \xi$ for $y \to -\infty$:

$$\omega_1 \sim -\frac{\partial}{\partial \xi} \log(\Delta_i (+,-) e^{-K_i(+,-)\xi} + \Delta_i (-,-) e^{-K_i(-,-)\xi} )$$

$$= \frac{K_i (+,+) \Delta_i (+,-) e^{-K_i(+,-)\xi} + K_i (-,-) \Delta_i (-,-) e^{-K_i(-,-)\xi}}{\Delta_i (+,-) e^{-K_i(+,-)\xi} + \Delta_i (-,-) e^{-K_i(-,-)\xi}}$$

$$= \frac{K_i (+,-) \Delta_i (+,-) e^{(k_{N_i+j} - k_j)\xi} + K_i (-,-) \Delta_i (-,-)}{\Delta_i (+,-) e^{(k_{N_i+j} - k_j)\xi} + \Delta_i (-,-)},$$

where

$$\Delta_i (+,-) = \Delta(i, \ldots, N_+ + i - 1) \exp(\sum_{j=i}^{N_i+1} \theta_j^0)$$

$$\Delta_i (-,-) = \Delta(i + 1, \ldots, N_+ + i) \exp(\sum_{j=i+1}^{N_i+1} \theta_j^0).$$
Now, from \( k_i < k_{N_i} \), it is obvious that \( \omega_i \) has the desired asymptotics as \( \xi \to \pm \infty \) for \( y \to -\infty \). Similarly, for the case of \((\eta_i,\eta_{N_i+j}) \in I(N_--1)\) we have the inequality

\[
\eta_i = \eta_{N_i+j} < \eta_1, \eta_2, \ldots, \eta_{N_i+j+1}, \ldots, \eta_N.
\]

Then the dominant terms in the \( \tau_{N_i} \)-function on \( x = c_{i,N_i+j}, y + \xi \) for \( y \to \infty \) are given by the exponential terms

\[
\exp\left( \sum_{j=1}^i \theta_j + \sum_{j=i}^{N_i-j-1} \theta_{N_j-j+1} \right), \quad \exp\left( \sum_{j=1}^{i-1} \theta_j + \sum_{j=i}^{N_i-j-1} \theta_{N_j-j+1} \right).
\]

Then, following the previous argument, we obtain the desired asymptotics as \( \xi \to \pm \infty \) for \( y \to \infty \).

For other values of \( c \), i.e., \( c = c_{i,N_i+j} \) and \( c \neq c_{i,N_i+j} \), just one exponential term becomes dominant, and thus \( \omega_i \) approaches a constant as \( |y| \to \infty \). This completes the proof. \( \square \)

**Theorem 2.2.6** The above theorem can be summarized:

i) As \( y \to -\infty \), the function \( \omega_i \) has \( N_+ \) jumps, moving with velocities \( c_{j,N_j} \) for \( j = 1, \ldots, N_+ \); as \( y \to \infty \), \( \omega_i \) has \( N_- \) jumps, moving with velocities \( c_{i,N_i} \) for \( i = 1, \ldots, N_- \).

ii) Each jump represents a line soliton of the \( u \)-solution, and therefore the whole solution represents a \((N_-,N_+)-\)soliton.

iii) Each velocity of the asymptotic line solitons in the \((N_-,N_+)-\)soliton is determined from the \( c-\eta \) graph of the levels of intersections.

iv) This theorem determines the complete structure of asymptotic patterns of the solutions \( u(x,y,t) \) given by (2.3) for the Toda lattice equation. In the case of the ordinary multi-soliton solution of the KP equation, the \( \tau \)-function (2.4) does not contain all the possible combinations of phases, and therefore the theorem should be modified.

The key idea for the asymptotic analysis of using the levels of intersection is still applicable of ordinary multi-solitons. In fact, one can find from the same argument that the asymptotic velocities for the ordinary \( M \)-solitons are given by \( c_{2i-1,2i} = k_{2i-1} + k_{2i} \) where the \( \tau_M \)-function is the Wronskian with \( f_i = e^{\theta_{2i-1}} + e^{\theta_{2i}} \) for \( i = 1, \ldots, M \) and \( k_1 < k_2 < \ldots < k_{2M} \). Note that the velocities are different from those of the resonant \( M \)-soliton solution.

**Theorem 2.2.7** In the generic situation, the number of holes (bounded regions) in the graph of the \((N_-,N_+)-\)soliton solution is \((N_- - 1)(N_+ - 1)\).

**Proof.** We use mathematical induction. The case \( N_+ = 1 \) corresponds to the Burgers equation, and it is immediate to show that the graph of the \((N_-,1)-\)soliton solution has a tree shape; that is, no holes. Now suppose that the \((N_-,N_+)-\)soliton has \((N_- - 1)(N_+ - 1)\) holes...
h) holes. Add a new phase \( \theta_{N+1} \), with \( k_{N+1} < \ldots < k_N < k_{N+1} \), which produces a new, fastest, incoming \([N_+ + 1, N + 1]\) soliton, and assume that this solution intersects with the \([1, N_+ + 1]\) soliton, which is the slowest outgoing soliton. Then the resonant process of those solitons generates a \([1, N + 1]\) soliton as a \((2, 1)\) process, which then intersects with the new slowest \([1, N + 2]\) soliton to generate an intermediate \([N_+ + 2, N + 1]\) soliton. This intermediate soliton interacts with the second slowest outgoing soliton, the \([2, N + 2]\) soliton, to generate \([2, N + 3]\) and \([N + 3, N + 1]\) solitons, and so on. It is obvious that there are \(N_+ - 1\) newly created holes; that is, if \((N_-, N_+) \to (N_+ + 1, N_+)\), the number of holes increases as 
\[(N_- - 1)(N_+ - 1) \to (N_- - 1)(N_+ - 1) + (N_+ - 1) = N_+(N_+ - 1).\]

The case of the \((N_-, N_+ + 1)\) solution can be analyzed in the same way using the duality of the determinants. This completes the proof.

\(\Box\)

**Theorem 2.2.8** In the generic situation for \(N_+ + N_+ = N \geq 3\), the total numbers of intersection points and intermediate solitons in a \((N_-, N_+)-\)soliton solution are respectively given by \(2N_+N_+ - N\) and \(3N_+N_+ - 2N\).

**Proof.** By applying mathematical induction one can easily find that the number of new vertices (intersection points) is \(2N_+ - 1\) and that of new intermediate solitons is \(3N_+ - 2\). This yields the desired results.

\(\Box\)

One should compare these numbers with the case of ordinary \(M\)-soliton solution, where the total numbers of holes and intersection points are \(1/2(M - 1)(M - 2)\) and \(1/2M(M - 1)\), respectively. The resonant process blows up each vertex in an ordinary \(M\)-soliton solution to create a hole, so that the total number of holes in a resonant \(M\)-soliton solution is given by 
\[
\frac{1}{2}(M - 1)(M - 2) + \frac{1}{2}M(M - 1) = (M - 1)^2.
\]

Note also that the total number of vertices in a resonant \(M\)-soliton is four times of the vertices of an ordinary \(M\)-soliton, i.e., each vertex is blown up to make 4 vertices with one hole.

Finally, we would like to point out that the KP equation has a large variety of multi-soliton-type solutions. Among those solutions, we found that, since the \(\tau_M\)-function of the resonant \((N_-, N_+)-\)soliton for the Toda lattice hierarchy contains all possible combinations of phase terms \(\{\theta_i | i = 1, \ldots, N\}\), the interaction process for these solutions results in a **fully resonant** situation. On the other hand, the ordinary \(M\)-soliton solutions display a **nonresonant** case; that is, resonant triangles representing either \((2, 1)\)- or \((1, 2)\)-
solitons cannot be formed because of the missing exponential terms in the tau-function [17,19]. One can then find a partially resonant case consisting of ordinary multi-soliton interaction with the addition of some resonant interactions; one such example is the case having \( f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} \) and \( f_2 = e^{\theta_4} + e^{\theta_5} \) for the \( \tau_2 \)-function where the ordinary 2-soliton interaction coexists with resonant interactions. We will discuss the details of the general patterns for 2 specific classes of multi-soliton solutions for the KP equation in the next chapter.
3. Interaction Patterns and Web-structures

First we would like to start with the structure of (1, 2)-soliton solution case. We describe the intermediate patterns of the resonant solitons in the \((x, y)\) plane. The key idea is to consider the pattern as a collection of fundamental resonances. The fundamental resonance consists of three parameters: \(\{k_1, k_2, k_3\}\), that is, the case of \(N = 3\) with \(|N_+ - N_-| = 1\). Without loss of generality, let us take \(N_+ = 1\) and \(N_- = 2\), i.e., a (1,2)-soliton. Then, with \(k_1 < k_2 < k_3\), the pattern of the fundamental resonance is a \(Y\)-shape graph.

Here and in the following we denote with \([i, j]\) the asymptotic line soliton with \(c = c_{i,j} = k_i + k_j\). Notice that \(I(N_+ - 1) = I(0) = \{ (\eta_i, \eta_j) \}\) and \(I(N_- - 1) = I(1) = \{ (\eta_i, \eta_j), (\eta_i, \eta_k) \}\). One should note that at the vertex of the \(Y\)-shape graph each index appears exactly twice as the result of resonance, and in figure those vertices form a triangle, which we refer to as a “resonant triangle”. The resonant triangle is equivalent to the resonance condition for the wavenumber vectors. Since the vertex of the \(Y\)-shape graph consists of three line solitons, \(\theta_i = \theta_j, 1 \leq i \leq j \leq 3\), the location of the vertex is obtained from the solution of the equations \(\theta_1 = \theta_2 = \theta_3\).

Note here that the coefficient matrix is nonsingular for \(k_1 < k_2 < k_3\), and the location \((x, y)\) is uniquely determined by a function of \(t\). This implies that there always exists a \(Y\)-shape graph if there are three line solitons satisfying the resonance conditions given in chapter 2.

Since the \(\tau_{N_+}\)-function (2.9) contains all possible combinations of \(N_+\) phases, all the vertices in the graph form \(Y\)-shape intersections as a result of dominant balance of three exponential terms in the \(\tau_{N_+}\) at each vertex. One should also note that a vertex with 4 or more line solitons is not generic: A vertex with \(m\) distinct line solitons is obtained from the system of \(m\) equations, \(\{ \theta_{ik} = \theta_{jk} \mid i_k \neq j_k, k = 1,\ldots,m \}\), in which at least \(m - 1\) equations are linearly independent. Then for \(m \geq 4\), this system in \(x\) and \(y\) is overdetermined so that the solution exists only for specific choices of \(\theta_0^i\) for fixed values of \(t\). In the cases of both ordinary and resonant 2-soliton solutions, the two pairs of solitons as \(y \to \pm \infty\) are the same, and therefore there are only two independent equations.
The ordinary 2-soliton solution needs a balance of four exponential terms to realize an \(X\)-shape vertex. It was shown that the \(X\)-shape vertex of an ordinary 2-soliton solution is blown up into a hole with four \(Y\)-shape vertices for the resonant 2-soliton solution.

Now we discuss the asymptotic analysis of the (1, 2)-soliton solution. First, we consider the point \(\eta = \eta_{2+i}\), i.e., \((\eta, \eta_{2+i}) \in I(0)\). Then we get the inequality from the theorem (2.2.4) as \(\eta_{2+i} < \eta_{1} = \eta_{2+i}\) and \(c = k_{i} + k_{i+2}\). Therefore the 2 exponential terms

\[
\exp\left(\sum_{j=i}^{2i-1} \theta_{j}\right), \quad \exp\left(\sum_{j=i+1}^{2i} \theta_{j}\right),
\]

which are dominant in the \(\tau_{2}\) function for \(y \to \infty\). Also the condition \(\eta_{i} = \eta_{2+i}\) leads to \(c = k_{i} + k_{i+2}\). Thus the function \(\omega_{i}\) will be approximated by the following form along \(x = c_{i,2+i}, y + \xi\) for \(y \to -\infty\)

\[
\omega_{i} \sim -\frac{\partial}{\partial \xi} \log(\Delta_{i}(+,+,-)e^{-K_{i}(+,-)\xi} + \Delta_{i}(-,+,-)e^{-K_{i}(-,+,-)\xi})
= \frac{K_{i}(+,+)\Delta_{i}(+,+,-)e^{-K_{i}(+,-)\xi} + K_{i}(-,+,-)\Delta_{i}(-,+,+)e^{-K_{i}(-,+,-)\xi}}{\Delta_{i}(+,+)e^{-K_{i}(+,-)\xi} + \Delta_{i}(-,+)e^{-K_{i}(-,+,-)\xi}}
= \frac{K_{i}(+,+)\Delta_{i}(+,+,-)e^{(K_{2+i} - K_{i})\xi} + K_{i}(-,+,-)\Delta_{i}(-,+)e^{(K_{2+i} - K_{i})\xi}}{\Delta_{i}(+,+)e^{(K_{2+i} - K_{i})\xi} + \Delta_{i}(-,+)}
\]

where

\[
\Delta_{i}(+,+) = \Delta(i,\ldots,2+i-1)\exp\left(\sum_{j=i}^{2i-1} \theta_{j}\right),
\]

\[
\Delta_{i}(-,+) = \Delta(i+1,\ldots,2+i)\exp\left(\sum_{j=i+1}^{2i} \theta_{j}\right).
\]

Since \(k_{i} < k_{2+i}\), \(\omega_{i}\) has the asymptotics as \(\xi \to \pm \infty\), mentioned in the theorem 2.2.5.

Now we proceed similarly for the case \((\eta_{i}, \eta_{2+i}) \in I(1)\) and we have the inequality \(\eta_{i} = \eta_{2+i} < \eta_{1}\) on \(x = c_{i,2+i}, y + \xi\) for \(y \to -\infty\) and \(\xi \to \pm \infty\). So the dominant terms are

\[
\exp\left(\sum_{j=i}^{i} \theta_{j} + \sum_{j=i+1}^{2i} \theta_{3-j}, \right), \quad \exp\left(\sum_{j=i}^{i-1} \theta_{j} + \sum_{j=i+1}^{2i-1} \theta_{3-j}\right).
\]
For all the other values of \( c \) just one exponential term becomes dominant, so \( \omega_1 \) approaches a constant as \( y \to \infty \). Therefore the function \( \omega_1 \) has one jump moving with velocities \( c_{j,2+j} \) for \( j = 1 \) as \( y \to -\infty \). When \( y \to \infty \), \( \omega_1 \) has two jumps moving with velocities \( c_{i,1+i} \) for \( i = 1,2 \). Every jump represents a line soliton solution \( u \) which represents the whole solution of a (1, 2)-soliton. The velocity of each line soliton in [1, 2] soliton is found by the graph of levels of interactions. So here we have 1 incoming solitons having the velocities \( c_{1,3} \) corresponding to the set \( I(0) \) and 2 outgoing solitons with the velocities \( c_{1,2}, c_{2,3} \) corresponding to the set \( I(1) \).

Now we construct the graph of the levels of interaction for the (1, 2)-soliton case. The level of interaction will take the range \( 0 \leq \sigma_{i,j} \leq 3 \).

a) The circle at the level \( I(0) \) corresponds to the incoming soliton, and the diamonds at the level \( I(1) \) correspond to the outgoing solitons.

![Figure 1. Levels of interaction of (1, 2)-soliton case](image)

b) 3-dimensional pattern
Figure 2. 3-dimensional pattern of $(1, 2)$-soliton

The Y-shape graph illustrates the fundamental resonance with the parameters $(k_1, k_2, k_3)$ where $k_1 < k_2 < k_3$.

Now we shall see the effect of parameters on the shape and region of the interaction pictures.

b) If $(k_1, k_2, k_3) = (-1, 0, 1)$ at $t = (-20, 0, 20)$ then the interaction pictures are:
c) Suppose we decrease the value of one of the parameters, for example, we set $k_1 = -3$ keeping $t=20$ then the interaction picture is:

![Contour plot of the interaction pattern of (1, 2)-soliton with $k_1 = -3$ at $t=20$]

**Figure 4.** Contour plot of the interaction pattern of (1, 2)-soliton with $k_1 = -3$ at $t=20$

d) Suppose we increase the value of one of the parameters, for example, we set $k_3 = 3$ keeping $t=20$ then the interaction picture is:
e) Suppose we set all the values of the parameters negative, for example, we set 
\((k_1, k_2, k_3) = (-1, -2, -3)\) keeping \(t=20\) then the interaction picture is:

f) Suppose we set all the values of the parameters positive, for example, we set 
\((k_1, k_2, k_3) = (1, 2, 3)\) keeping \(t=20\) then the interaction picture is:
3.1 Asymptotic analysis of the $(2, 3)$-soliton solution  By using the Theorem 2.2.5 we analyze the solution as follows:

For $y \to -\infty$ and $x = c_{i,3+j} y + \xi$ for $i = 1, 2$,

$$\omega_i \to \begin{cases} 
K_i(-,-) := \sum_{j=1}^{3+i} k_j & \text{as } \xi \to -\infty, \\
K_i(+,-) := \sum_{j=1}^{3+i-1} k_j & \text{as } \xi \to \infty.
\end{cases}$$

For $y \to \infty$ and $x = c_{i,2+j} y + \xi$ for $i = 1, 2, 3$,

$$\omega_i \to \begin{cases} 
K_i(-,+):= \sum_{j=1}^{i-1} k_j + \sum_{j=1}^{3-i-1} k_{i-j+1} & \text{as } \xi \to -\infty, \\
K_i(+,+):= \sum_{j=1}^{i} k_j + \sum_{j=1}^{3-i} k_{i-j+1} & \text{as } \xi \to \infty.
\end{cases}$$

Therefore the function $\omega_i$ has two jumps moving with velocities $c_{j,3+j}$ for $j = 1, 2$ as $y \to -\infty$. When $y \to \infty$, $\omega_i$ has three jumps moving with velocities $c_{j,2+j}$ for
\( i = 1, 2, 3 \). Every jump represents a line soliton of the solution \( u \) which represents the whole solution of a (2, 3)-soliton.

The velocity of each line soliton in (2, 3)-soliton is found by the graph of levels of interactions. So here we have two incoming solitons having the velocities \( c_{1,4}, c_{2,5} \) corresponding to the set \( I(1) \) and three outgoing solitons with the velocities \( c_{1,3}, c_{2,4}, c_{3,5} \) corresponding to the set \( I(2) \).

The level of interaction can take the range \( 0 \leq \sigma_{i,j} \leq 3 \). Then the set \( I(n) \) is given by

\[
I(n) = \{(\eta_i, \eta_{n+1-i}) \text{ for } i = 1, \ldots, n+1\} \quad \text{and} \quad n = 0, 1, 2, 3.
\]

Therefore the total number of pairs of \((\eta_i, \eta_j)\) is \((1/2.5.4)=10\), i.e.,

\[
I(0) = \{(\eta_1, \eta_5)\},
\]

\[
I(1) = \{(\eta_1, \eta_4), (\eta_2, \eta_5)\},
\]

\[
I(2) = \{(\eta_1, \eta_3), (\eta_2, \eta_4), (\eta_3, \eta_5)\},
\]

\[
I(3) = \{(\eta_1, \eta_2), (\eta_2, \eta_3), (\eta_3, \eta_4), (\eta_4, \eta_5)\}.
\]

The number of holes in the graph of the (2, 3)-soliton solution is 2. The total number of intersection points and intermediate solitons are 7 and 8 respectively.

Now we construct the graph of the levels of interaction for the (2, 3)-soliton case:

![Figure 8. Levels of interaction of (2, 3)-soliton](image)

In the above levels of interaction picture the level \( I(1) \)- stars represent the incoming line solitons and the level \( I(2) \)-boxes represent the outgoing line solitons.

Now we shall see the effect of parameters on the shape and region of the interaction picture of the (2,3)-soliton solution:
b) When \((k_1, k_2, k_3, k_4, k_5) = (-3, -2, -1, 1, 2)\) at \(t = (-20, 0)\) then the interaction pictures are:

![Contour plot of the interaction pattern of (2, 3)-soliton with \((k_1, k_2, k_3, k_4, k_5) = (-3, -2, -1, 1, 2)\) at \(t = (-20, 0)\)](image)

Figure 9. Contour plot of the interaction pattern of (2, 3)-soliton with \((k_1, k_2, k_3, k_4, k_5) = (-3, -2, -1, 1, 2)\) at \(t = (-20, 0)\)

![The interaction pattern of the intermediate solitons of the (2, 3)-soliton solution](image)

Figure 10. The interaction pattern of the intermediate solitons of the (2, 3)-soliton solution

We can observe the following facts:

1) The region of interaction widens on increasing the variable \(t\) from -20 to 0 keeping all the parameters \((k_i)\) constant.
2) The interaction becomes weak and the interaction pattern vanishes as if the value of one of the parameter is decreased.
3) If all the values are of parameters \( k_i \) are negative or positive then interaction pattern vanishes.

### 3.2 Asymptotic analysis of the (2, 4)-soliton solution

By using Theorem 2.2.5 we can analyze the solution as follows:

For \( y \to -\infty \) and \( x = c_i,4+i,y + \xi \) for \( i = 1, 2 \),

\[
\omega_1 \to \begin{cases} \sum_{j=1}^{4+i-1} k_j & \text{as } \xi \to -\infty, \\ \sum_{j=1}^{4+i} k_j & \text{as } \xi \to \infty. \end{cases}
\]

For \( y \to \infty \) and \( x = c_i,2+i,y + \xi \) for \( i = 1, 2, 3, 4 \),

\[
\omega_1 \to \begin{cases} \sum_{j=1}^{4+i-1} k_j + \sum_{j=1}^{4+i} k_{6-j+1} & \text{as } \xi \to -\infty, \\ \sum_{j=1}^{4+i} k_j + \sum_{j=1}^{4+i} k_{6-j+1} & \text{as } \xi \to \infty. \end{cases}
\]

Therefore the function \( \omega_1 \) has two jumps moving with velocities \( c_j,4+j \) for \( j = 1, 2 \) as \( y \to -\infty \). When \( y \to \infty \), \( \omega_1 \) has four jumps moving with velocities \( c_{i,2+i} \) for \( i = 1, 2, 3, 4 \). Every jump represents a line soliton solution \( u \) which represents the whole solution of a (2, 4)-soliton. The velocity of each line soliton in (2, 4)-soliton is found by the graph of levels of interactions. So here we have two incoming solitons having the velocities \( c_{1,5}, c_{2,6} \) corresponding to the set \( I(1) \) and four outgoing solitons with the velocities \( c_{1,3}, c_{2,4}, c_{3,5}, c_{4,6} \) corresponding to the set \( I(3) \).

The level of interaction can take the range \( 0 \leq \sigma_{i,j} \leq 4 \). Then the set \( I(n) \) is given by

\[
I(n) = \{(\eta_i, \eta_{6-n+i-1}) \text{ for } i = 1,\ldots,n+1 \} \text{ and } n = 0, 1, 2, 3, 4.
\]

Therefore the total number of pairs of \( (\eta_i, \eta_j) \) is 15:

\[
I(0) = \{(\eta_1, \eta_6)\}, \\
I(1) = \{(\eta_1, \eta_5), (\eta_2, \eta_6)\}, \\
I(2) = \{(\eta_1, \eta_4), (\eta_2, \eta_5), (\eta_3, \eta_6)\}, \\
I(3) = \{(\eta_1, \eta_3), (\eta_2, \eta_4), (\eta_3, \eta_5), (\eta_4, \eta_6)\}, \\
I(4) = \{(\eta_1, \eta_2), (\eta_2, \eta_3), (\eta_3, \eta_4), (\eta_4, \eta_5), (\eta_5, \eta_6)\}.
\]

The number of holes in the graph of the (2, 4)-soliton solution is 3. The total number of intersection points and intermediate solitons are 10 and 12 respectively.

Now we construct the graph of the levels of interaction for the [2, 4] case:
In the above levels of interaction picture the level $I(1)$-stars represent the incoming line solitons and the level $I(3)$-diamonds represent the outgoing line solitons.

Now we consider the effect of parameters in the interaction patterns of the (2, 4)-soliton solution:

a) When $(k_1, k_2, k_3, k_4, k_5, k_6) = (-3, -2, -1, 1, 2, 3)$ at
\( t = (-20, 0, 20) \)

Figure 13. Contour plot of the interaction pattern of (2, 4)-soliton with \( (k_1, k_2, k_3, k_4, k_5) = (-3, -2, -1, 1, 2, 3) \) at 
\( t = (-20, 0, 20) \)

b) If we decrease the value of one of the parameters, i.e., if we set \( k_1 = -5 \), then we will see the following interaction picture:
c) If we increase the value of one of the parameters, i.e., if we set $k_\phi = 5$, then we will see the following interaction picture:

![Contour plot](image)

**Figure 15.** Contour plot of the interaction pattern of (2, 4)-soliton with $k_\phi = 5$ at $t=20$

We can observe the following facts:
1) The region of interaction widens on increasing the variable $t$ from -20 to 20 keeping all the parameters ($k_i$’s) constant.
2) The region of interaction widens on increasing or decreasing one of the parameter values.
3) The interaction becomes weak and the interaction pattern vanishes if we increase or decrease more than one parameter value.
4) If all the values are of parameters \((k_i)\) are negative or positive then the interaction pattern vanishes.

3.3 Asymptotic analysis of the \((2, 5)\)-soliton solution  

By using the theorem 2.2.5 we analyze the solution as follows:

For \(y \to -\infty\) and \(x = c_{i,5+i} y + \xi\) for \(i = 1, 2,\)

\[
\omega_1 \rightarrow \begin{cases} 
K_i(-,-) := \sum_{j=i+1}^{5+i} k_j & \text{as } \xi \to -\infty, \\
K_i(+,-) := \sum_{j=i}^{5+i-1} k_j & \text{as } \xi \to \infty.
\end{cases}
\]

For \(y \to \infty\) and \(x = c_{i,2+i} y + \xi\) for \(i = 1,\ldots, 5,\)

\[
\omega_1 \rightarrow \begin{cases} 
K_i(-,+):= \sum_{j=1}^{i-1} k_j + \sum_{j=1}^{5-i} k_{7-j+1} & \text{as } \xi \to -\infty, \\
K_i(+,+):= \sum_{j=1}^{5-i} k_j + \sum_{j=1}^{5-i} k_{7-j+1} & \text{as } \xi \to \infty.
\end{cases}
\]

Therefore the function \(\omega_1\) has two jumps moving with velocities \(c_{j,5+j}\) for \(j = 1, 2\) as \(y \to -\infty\). When \(y \to \infty\), \(\omega_1\) has five jumps moving with velocities \(c_{i,5+i}\) for \(i = 1, 2, 3, 4, 5\).

Every jump represents a line soliton solution \(u\) which represents the whole solution of a \((2, 5)\)-soliton. The velocity of each line soliton in \((2, 5)\)-soliton is found by the graph of levels of interactions. So here we have two incoming solitons having the velocities \(c_{1,6}, c_{2,7}\) corresponding to the set \(I(1)\) and five outgoing solitons with the velocities \(c_{1,3}, c_{2,4}, c_{3,5}, c_{4,6}, c_{5,7}\) corresponding to the set \(I(4)\).

The level of interaction can take the range \(0 \leq \sigma_{i,j} \leq 5\). Then the set \(I(n)\) is given by \(I(n) = \{(\eta_i, \eta_{7-n+i-1})\} \text{ for } i = 1,\ldots, n+1\) and \(n = 0, 1, 2, 3, 4, 5\).

Therefore the total number of pairs of \((\eta_i, \eta_j)\) is 10, i.e.,

\[
I(0) = \{(\eta_1, \eta_7)\}, \quad I(1) = \{(\eta_1, \eta_6), (\eta_2, \eta_7)\}, \\
I(2) = \{(\eta_1, \eta_5), (\eta_2, \eta_6), (\eta_3, \eta_7)\}, \quad I(3) = \{(\eta_1, \eta_4), (\eta_2, \eta_5), (\eta_3, \eta_6), (\eta_4, \eta_7)\}, \\
I(4) = \{(\eta_1, \eta_3), (\eta_2, \eta_4), (\eta_3, \eta_5), (\eta_4, \eta_6), (\eta_5, \eta_7)\}, \quad I(5) = \{(\eta_1, \eta_2), (\eta_2, \eta_3), (\eta_3, \eta_4), (\eta_4, \eta_5), (\eta_5, \eta_6), (\eta_6, \eta_7)\}.
\]

The number of holes in the graph of the \((2, 5)\)-soliton solution is 4. The total number of intersection points and intermediate solitons are 13 and 16, respectively.
Now we construct the graph of the levels of interaction for the (2, 5)-soliton case:

![Graph of levels of interaction for (2, 5)-soliton case](image)

**Figure 16. Levels of interaction of (2, 5)-soliton**

In the above levels of interaction picture the level $I(1)$ - diamonds represent the incoming line solitons and the level $I(4)$ - stars represent the outgoing line solitons. Now we construct the graph of the levels of interaction for the (2, 5)-soliton case.

![Graph showing interaction pattern of intermediate solitons](image)

**Figure 17. The interaction pattern of the intermediate solitons**
Now we consider the effect of parameters in the interaction patterns of the (2, 5)-soliton solution:

a) When \((k_1, k_2, k_3, k_4, k_5, k_6, k_7) = (-3, -2, -1, 0, 1, 2, 3)\) and \(t = (-10, 20, 0)\) we have the interaction picture as:

![Contour plot of the interaction pattern of (2, 5)-soliton with \((k_1, k_2, k_3, k_4, k_5, k_6, k_7) = (-3, -2, -1, 0, 1, 2, 3)\) and \(t = (-10, 20, 0)\)](image)

Figure 18. Contour plot of the interaction pattern of (2, 5)-soliton with \((k_1, k_2, k_3, k_4, k_5, k_6, k_7) = (-3, -2, -1, 0, 1, 2, 3)\) and \(t = (-10, 20, 0)\)
b) If we decrease the value of one of the parameters, i.e., if we set $k_i = -10$, then we will see the following interaction picture:

![Figure 19. Contour plot of the interaction pattern of (2, 5)-soliton with $k_i = -10$](image)

We can observe the following facts:

1) The region of interaction widens on increasing the variable $t$ from -10 to 20 keeping all the parameters ($k_i$) the parameter values constant.

c) If we increase the value of one of the parameters, i.e., if we set $k_7 = 5$, then we will see the following interaction picture:

![Figure 20. Contour plot of the interaction pattern of (2, 5)-soliton with $k_7 = 5$](image)
2) The region of interaction widens on increasing or decreasing one of the parameter values.
3) The interaction becomes weak and the interaction pattern vanishes if we increase or decrease more than one parameter value.
4) If all the values of parameters \((k_i)\) are negative or positive, then the interaction pattern vanishes.

3.4 Asymptotic analysis of the \((3, 2)\)-soliton solution

By using the theorem 2.2.5 we analyze the solution as follows:

For \(y \to -\infty\) and \(x = c_{i,2+i} y + \xi\) for \(i = 1, 2, 3\),

\[
\omega_1 \rightarrow \begin{cases} \sum_{j=i+1}^{2+i} k_j & \text{as } \xi \to -\infty, \\ \sum_{j=i+1}^{2+i-1} k_j & \text{as } \xi \to \infty, \end{cases}
\]

For \(y \to \infty\) and \(x = c_{i,3+i} y + \xi\) for \(i = 1, 2\),

\[
\omega_1 \rightarrow \begin{cases} \sum_{j=1}^{i-1} k_j + \sum_{j=1}^{2-i+1} k_{5-j+1} & \text{as } \xi \to -\infty, \\ \sum_{j=1}^{i} k_j + \sum_{j=1}^{2-i} k_{5-j+1} & \text{as } \xi \to \infty, \end{cases}
\]

Therefore the function \(\omega_1\) has three jumps moving with velocities \(c_{j,2+j}\) for \(j = 1, 2, 3\) as \(y \to -\infty\).

When \(y \to \infty\), \(\omega_1\) has two jumps moving with velocities \(c_{i,2+i}\) for \(i = 1, 2\). Every jump represents a line soliton solution \(u\) which represents the whole solution of a \((3, 2)\)-soliton. The velocity of each line soliton in \((3, 2)\)-soliton is found by the graph of levels of interactions. So here we have three incoming solitons having the velocities \(c_{1,3}, c_{2,4}, c_{3,5}\) corresponding to the set \(I(2)\) and two outgoing solitons with the velocities \(c_{1,4}, c_{2,5}\) corresponding to the set \(I(1)\).

The level of interaction can take the range \(0 \leq \sigma_{i,j} \leq 3\). Then the set \(I(n)\) is given by \(I(n) = \{(\eta_i, \eta_{5-n+i})\} \) for \(i = 1, \ldots, n+1\) and \(n = 0, 1, 2, 3\). Therefore the total number of pairs of \((\eta_i, \eta_j)\) is 10, i.e.,

\[
I(0) = \{(\eta_1, \eta_5)\}, \quad I(1) = \{(\eta_1, \eta_4), (\eta_2, \eta_5)\}, \\
I(2) = \{(\eta_1, \eta_3), (\eta_2, \eta_4), (\eta_3, \eta_5)\}, \\
I(3) = \{(\eta_1, \eta_2), (\eta_2, \eta_3), (\eta_3, \eta_4), (\eta_4, \eta_5)\}.
\]

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The number of holes in the graph of (3, 2)-soliton solution is 2. The total number of intersection points and intermediate solitons are 7 and 8 respectively.

Now we construct the graph of the levels of interaction for the (3, 2)-soliton case:

![Levels of interaction of (3, 2)-soliton](image1)

**Figure 21. Levels of interaction of (3, 2)-soliton**

In the above levels of interaction picture the level $I(2)$ - boxes represent the incoming line solitons and the level $I(1)$ - stars represent the outgoing line solitons.

![Interaction pattern of intermediate solitons](image2)

**Figure 22. The interaction pattern of the intermediate solitons of the (3, 2)-soliton solution**
Now we consider the effect of parameters in the interaction patterns of the (3, 2)-soliton solution:

a) When \((k_1, k_2, k_3, k_4, k_5) = (-3, -2, -1, 1, 2)\) at \(t = (-20, 0, 20)\) we get the following interaction picture:

![Contour plot](image)

Figure 23. Contour plot of the interaction pattern of (3, 2)-soliton \((k_1, k_2, k_3, k_4, k_5) = (-3, -2, -1, 1, 2)\) at \(t = (-20, 0, 20)\)

b) If we decrease the value of one of the parameters, i.e., if we set \(k_1 = -5\), at \(t = -20\) then we will see the following interaction picture:
c) If the difference between two of the values $k_i's$ is decreased; suppose we set $k_1 = -5$ and $k_2 = -9/2$ at $t = -20$ then we get the interaction picture as:

![Contour plot of the interaction pattern of (3, 2)-soliton](image)

**Figure 24.** Contour plot of the interaction pattern of (3, 2)-soliton $\ k_1 = -5 \text{ at } t = -20$

d) If we increase the value of one of the parameters, i.e., if we set $k_6 = 5$, then we will see the following interaction picture:

![Contour plot of the interaction pattern of (3, 2)-soliton](image)

**Figure 25.** Contour plot of the interaction pattern of (3, 2)-soliton $\ k_1 = -5 \text{ and } k_2 = -9/2 \text{ at } t = -20$
We can observe the following facts:

1) The region of interaction widens on increasing the variable $t$ from -20 to 20 keeping all the parameters ($k_i$) the parameter values constant.
2) The region of interaction widens on increasing or decreasing one or two of the parameter values.
3) The interaction becomes weak and the interaction pattern vanishes as if we increase or decrease more than two parameter values.
4) If all the value are of parameters ($k_i$) are negative or positive then the interaction pattern vanishes.

### 3.4 Asymptotic analysis of the (3, 3)-soliton solution

By using the theorem 2.2.5 we analyze the solution as follows:

For $y \to -\infty$ and $x = c_i(x_{i+1}y + \xi)$ for $i = 1, 2, 3$,

$$
\omega_i \to \begin{cases}
K_i(-,-) := \sum_{j=i+1}^{3i+1} k_j & \text{as } \xi \to -\infty, \\
K_i(+,-) := \sum_{j=i}^{3i-1} k_j & \text{as } \xi \to \infty.
\end{cases}
$$
For \( y \to \infty \) and \( x = c_{i,3+i} y + \xi \) for \( i = 1, 2, 3 \),

\[
\omega_1 \to \begin{cases} 
K_1(-,+):= \sum_{j=1}^{i-1} k_j + \sum_{j=1}^{3-i+1} k_{6-j+1} \quad \text{as} \quad \xi \to -\infty, \\
K_1(+,+):= \sum_{j=1}^{i} k_j + \sum_{j=1}^{3-i} k_{6-j+1} \quad \text{as} \quad \xi \to \infty.
\end{cases}
\]

Therefore the function \( \omega_1 \) has three jumps moving with velocities \( c_{j,3+j} \) for \( j = 1, 2, 3 \) as \( y \to -\infty \). When \( y \to \infty \), \( \omega_1 \) has three jumps moving with velocities \( c_{i,3+i} \) for \( i = 1, 2, 3 \). Every jump represents a line soliton solution \( u \) which represents the whole solution of a (3, 3)-soliton. The velocity of each line soliton in (3, 3)-soliton is found by the graph of levels of interactions. So here we have three incoming solitons having the velocities \( c_{1,4}, c_{2,5}, c_{3,6} \) corresponding to the set \( I(2) \) and three outgoing solitons with the velocities \( c_{1,4}, c_{2,5}, c_{3,6} \) corresponding to the set \( I(2) \).

The level of interaction can take the range \( 0 \leq \sigma_{i,j} \leq 3 \). Then the set \( I(n) \) is given by

\[
I(n) = \{ (\eta_i, \eta_{6-n+i-1}) \} \quad \text{for} \quad i = 1, \ldots, n+1 \} \quad \text{and} \quad n = 0, 1, 2, 3, 4.
\]

Therefore the total number of pairs of \( (\eta_i, \eta_j) \) is 10, i.e.,

\[
I(0) = \{ (\eta_1, \eta_6) \}, \\
I(1) = \{ (\eta_1, \eta_3), (\eta_2, \eta_6) \}, \\
I(2) = \{ (\eta_1, \eta_4), (\eta_2, \eta_5), (\eta_3, \eta_6) \}, \\
I(3) = \{ (\eta_1, \eta_5), (\eta_2, \eta_3), (\eta_3, \eta_5), (\eta_4, \eta_6) \}, \\
I(4) = \{ (\eta_1, \eta_6), (\eta_2, \eta_5), (\eta_3, \eta_4), (\eta_4, \eta_5), (\eta_5, \eta_6) \}.
\]

The number of holes in the graph of (3, 3)-soliton solution is 4. The total number of intersection points and intermediate solitons are 12 and 15.

Now we construct the graph of the levels of interaction for the (3, 3)-soliton case:
Figure 27 Levels of interaction of (3, 3)-soliton

In the above levels of interaction picture the level $I(2)$ - boxes represent the incoming line solitons and the level $I(2)'$ boxes represent the outgoing line solitons.

Figure 28. Interaction pattern of (3, 3)-soliton

Now we consider the effect of parameters in the interaction patterns of the (3, 3)-soliton soliton:
a) When \( (k_1, k_2, k_3, k_4, k_5, k_6) = (-3, -2, -1, 1, 2, 3) \) and \( t = (-10, 0, 10) \) we will see the following interaction pictures:

![Contour plot of the interaction pattern of (3, 3)-soliton with \( (k_1, k_2, k_3, k_4, k_5, k_6) = (-3, -2, -1, 1, 2, 3) \) and \( t = (-10, 0, 10) \)](image)

b) If we decrease the value of one of the parameters, i.e., if we set \( k_1 = -10 \), at \( t = 10 \) then we will see the following interaction picture:
c) If we increase the value of one of the parameters, i.e., if we set \( k_0 = 6 \), at \( t = 10 \) then we will see the following interaction picture:

We can observe the following facts:

1) The region of interaction widens on increasing the variable \( t \) from -10 to 10 keeping all the parameters \( (k_i) \) the parameter values constant.
2) The region of interaction widens on increasing or decreasing one or two of the parameter values.
3) The interaction becomes weak and the interaction pattern vanishes as if we increase or decrease more than two parameter values.
4) If all the values are of parameters \( k_i \) are negative or positive then the interaction pattern vanishes.

3.5 Asymptotic analysis of the \((3, 4)\)-soliton solution  

By using the theorem 2.2.5 we analyze the solution as follows:

For \( y \to -\infty \) and \( x = c_{i,4+i} y + \xi \) for \( i = 1, 2, 3 \),

\[
\omega_1 \to \begin{cases} 
K_i(-,+) := \sum_{j=i+1}^{i+4} k_j \quad \text{as} \quad \xi \to -\infty, \\
K_i(+,-) := \sum_{j=i}^{i+4-1} k_j \quad \text{as} \quad \xi \to \infty.
\end{cases}
\]

For \( y \to \infty \) and \( x = c_{i,3+i} y + \xi \) for \( i = 1, 2, 3 \),

\[
\omega_1 \to \begin{cases} 
K_i(-,+) := \sum_{j=1}^{i-1} k_j + \sum_{j=1}^{4-i+1} k_{7-j+1} \quad \text{as} \quad \xi \to -\infty, \\
K_i(+,-) := \sum_{j=1}^{i} k_j + \sum_{j=1}^{4-i} k_{7-j+1} \quad \text{as} \quad \xi \to \infty.
\end{cases}
\]

Therefore, the function \( \omega_i \) has three jumps moving with velocities \( c_{j,4+j} \) for \( j = 1, 2, 3 \) as \( y \to -\infty \). When \( y \to \infty \), \( \omega_i \) has four jumps moving with velocities \( c_{i,3+i} \) for \( i = 1, 2, 3, 4 \).

Every jump represents a line soliton solution \( u \) which represents the whole solution of a \((3, 4)\)-soliton. The velocity of each line soliton in \([3, 4]\) soliton is found by the graph of levels of interactions. So here we have three incoming solitons having the velocities \( c_{1,5} \), \( c_{2,6} \), \( c_{3,7} \) corresponding to the set \( I(2) \) and four outgoing solitons with the velocities \( c_{1,4} \), \( c_{2,5} \), \( c_{3,6} \), \( c_{4,7} \) corresponding to the set \( I(3) \).

The level of interaction can take the range \( 0 \leq \sigma_{i,j} \leq 5 \). Then the set \( I(n) \) is given by

\( I(n) = \{(\eta_1, \eta_{7-n+i}) \} \) for \( i = 1, \ldots, n+1 \) and \( n = 0, 1, 2, 3, 4, 5 \).

Therefore the total number of pairs of \( (\eta_i, \eta_j) \) are:

\[
\begin{align*}
I(0) &= \{(\eta_1, \eta_7)\}, \\
I(1) &= \{(\eta_1, \eta_6), (\eta_2, \eta_7)\}, \\
I(2) &= \{(\eta_1, \eta_5), (\eta_2, \eta_6), (\eta_3, \eta_7)\}, \\
I(3) &= \{(\eta_1, \eta_4), (\eta_2, \eta_5), (\eta_3, \eta_6), (\eta_4, \eta_7)\}, \\
I(4) &= \{(\eta_1, \eta_3), (\eta_2, \eta_4), (\eta_3, \eta_5), (\eta_4, \eta_6), (\eta_5, \eta_7)\}, \\
I(5) &= \{(\eta_1, \eta_2), (\eta_2, \eta_3), (\eta_3, \eta_4), (\eta_4, \eta_5), (\eta_5, \eta_6), (\eta_6, \eta_7)\}.
\end{align*}
\]

The number of holes in the graph of \((3, 4)\)-soliton solution is 6. The total number of intersection points and intermediate solitons are 17 and 22.
Now we construct the graph of the levels of interaction for the \((3, 3)\)-soliton case:

![Diagram of levels of interaction for \((3, 3)\)-soliton case](image)

**Figure 32. Levels of interaction of \((3, 4)\)-soliton**

In the above levels of interaction picture the level \(I(2)\) - boxes represent the incoming line solitons and the level \(I(3)\) - triangles represent the outgoing line solitons.

![Diagram of levels of interaction for \((3, 4)\)-soliton case](image)

**Figure 33. Levels of interaction of \((3, 4)\)-soliton**
Now we consider the effect of parameters in the interaction patterns of the (3, 4)-soliton solution:

a) When \((k_1, k_2, k_3, k_4, k_5, k_6, k_7) = (-3, -2, -1, 0, 1, 2, 3)\) at \(t = (-20, 0, 20)\) we have the following interaction pictures:

![Contour plot of the interaction pattern of (3, 4)-soliton](image1)

b) If we decrease the value of one of the parameters, i.e., if we set \(k_i = -6\), at \(t=20\) then we will see the following interaction picture:

![Contour plot of the interaction pattern of (3, 4)-soliton](image2)

Figure 34. Contour plot of the interaction pattern of (3, 4)-soliton
\((k_1, k_2, k_3, k_4, k_5, k_6, k_7) = (-3, -2, -1, 0, 1, 2, 3)\) at \(t = (-20, 0, 20)\)
c) If we increase the value of one of the parameters, i.e., if we set $k_1 = 6$, at $t = 20$ then we will see the following interaction picture:
We can observe the following facts:

1) The region of interaction widens on increasing the variable $t$ from -10 to 10 keeping all the parameters $(k_i)$ the parameter values constant.
2) The region of interaction widens on increasing or decreasing one or two of the parameter values.
3) The interaction becomes weak and the interaction pattern vanishes as if we increase or decrease more than two parameter values.
4) If all the value are of parameters $(k_i)$ are negative or positive then the interaction pattern vanishes.
References


