2008

Lagrange interpolation on Leja points

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Lagrange Interpolation on Leja Points

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
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Date of Approval:
April 1, 2008

Keywords: Equilibrium distribution, Fekete points, Lebesgue constants, Newton interpolation, potential theory

Acknowledgments

I wish to thank the Department of Mathematics at The University of South Florida for giving me the opportunity and the support to finish my doctorate studies. I would also like to sincerely thank the members of my graduate committee, Dr. Totik, Dr. Shehktman, Dr. Rakhmanov, and Dr. Danyielan for their support. However I think special thanks is due to Dr. Totik, for his valuable help, support, encouragement, and patience in letting me find my own way. He truly wanted this to be a dissertation to be proud of.

Table of Contents

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ABSTRACT

In this dissertation we investigate Lagrange interpolation. Our first result will deal with a hierarchy of interpolation schemes. Specifically, we will show that given a triangular array of points in a regular compact set K, such that the corresponding Lebesgue constants are subexponential, one always has the uniform convergence of $L_n(f)$ to f for all functions analytic on K. We will then show that uniform convergence of $L_n(f)$ to f for all analytic functions f is equivalent to the fact that the probability measures $\gamma_n = \frac{1}{n}\sum_{j=1}^{n}\delta_{z_{n,j}}$, which are associated with our triangular array, converge weak star to the equilibrium distribution for K.

Motivated by our hierarchy, we will then come to our main result, namely that the Lebesgue constants associated with Leja sequences on fairly general compact sets are subexponential. More generally, considering Newton interpolation on a sequence of points, we will show that the weak star convergence of their corresponding probability measures to the equilibrium distribution, together with a certain distancing rule, implies that their corresponding Lebesgue constants are sub-exponential.

1 Introduction

The focus of this dissertation is Lagrange interpolation. We here give a brief overview of our results, with technical details omitted.

From the abundance of literature on polynomial approximation, one of the most classical results, Weierstrass's theorem (see [11], page 159), tells us that every continuous function on a compact set K can be uniformly approximated by polynomials. When K is a finite closed interval on the real line, we know that given an $f \in C(K)$, if p_n^* is the best approximant to f in the space $\mathbf{P}_n$, the space of polynomials of degree at most n, then the equation

$$p_n^*(x) = f(x) \tag{1.0.1}$$

has $n+1$ solutions on K. From this it follows that if for each n we interpolate at the solutions to (1.0.1), we will have

$$\|L_n(f) - f\|_K \to 0. \tag{1.0.2}$$

It is not true however, that (1.0.2) must hold for interpolation on an arbitrary triangular array of points. In fact when $K = [-1, 1]$, the celebrated theorem of Faber (see [7], page 27) claims that given any triangular array of points (a set of points $\{z_{n,k}\}_{1 \le k \le n; n=1,2,\ldots}$, where $z_{n,j} \ne z_{n,k}$ whenever $j \ne k$) there is always a function $f \in C(K)$ such that

$$\|L_n(f) - f\|_K \nrightarrow 0,$$

and even

$$\|L_n(f)\|_K \to \infty. \tag{1.0.3}$$

In light of (1.0.3), the question becomes what are *good* points at which to interpolate.

Our first result, a hierarchy of interpolation schemes, finds conditions that make a triangular array *good* points at which to interpolate. Specifically, we find conditions such that (1.0.2) holds for all functions f which are analytic on K. We will show that (1.0.2) is true if and only if certain measures associated with our interpolation points converge weak star to the equilibrium distribution for K (we say a sequence of measures $\{\gamma_n\}$ on a compact set K converges weak star to μ if $\int_K f(t)d\gamma_n(t) \to \int_K f(t)d\mu(t)$ for all $f \in C(K)$).

Aside from having (1.0.2) hold for a large class of functions, there is another property which makes an array of points desirable from an interpolation standpoint. If p_n^* is the best approximant in the space $\mathbf{P}_n$, the set of polynomials of degree less than or equal to n, to a function f, we have

$$\|L_n(f) - f\|_K \le \|p_{n-1}^* - f\|_K \left(\|L_n\| + 1\right).$$

It is thus desirable to keep the norm $\|L_n\|$ of L_n, which we call the n-th Lebesgue constant, as small as possible (when we speak of the norm of L_n, we are speaking of the standard operator norm). Generally speaking, the Lebesgue constants associated with a triangular array of points will be *small* when the points are spread out. The measure of *smallness* which we shall use in this dissertation is subexponentiality, that is, we will find interpolation points such that

$$\|L_n\|^{1/n} \to 1.$$

Despite the fact that points which are somewhat spread out will tend to have *small* Lebesgue constants, it is not true that for the array of equidistant nodes the Lebesgue constants are subexponential (see [15]). It is trivial to show however, that the Lebesgue constants associated with Fekete sets (see [12], page 142) are subexponential. An nth Fekete set for a compact set K is any set of points $z_1, z_2, \ldots, z_n$ which maximizes

$$V(z_1, \ldots, z_n) = \prod_{1 \le i < j \le n} |z_j - z_i|.$$

Since Fekete sets are difficult to determine, in this dissertation we will examine an analogous, yet more easily determined set on which to interpolate.

Our main result will concern Newton interpolation. Newton interpolation comes when the rows within a triangular array are nested sets, i.e., when we are given a sequence of points $z_1, z_2, \ldots,$ and L_n interpolates on the first n terms of the sequence. The Newton sequences which we shall consider are Leja sequences. Leja sequences are defined inductively. Once $z_1, z_2, \ldots, z_{n-1}$ have been determined, z_n is chosen so that

$$V(z) = \prod_{i=1}^{n-1} |z - z_i|$$

is maximized for $z = z_n$. The model for our main result will come when we show that the Lebesgue constants associated with Leja sequences on $[-1, 1]$ are subexponential. We will then extend this result to more general sets in the complex plane.

Interpolation on Leja points has been previously investigated. Lothar Reichel (see [10]) had performed numerical calculations which suggested that Leja points were *good* interpolation points. This dissertation then, gives a basis to the claim that Leja points are *good* points.

Incidentally, in spite of the fact that for all $f \in C[-1, 1]$ (1.0.2) holds for some triangular array, it is not known if for all f there exists a Newton sequence so that (1.0.2) holds. However, it is known that for all analytic f there exists a Newton sequence such that (1.0.2) holds. Even further, given a compact set K, there exists a Newton sequence such that (1.0.2) holds for all functions analytic on K (see [6]). This dissertation exhibits Leja sequences as such *good* sequences.

2 Preliminaries

2.1 Lagrange Interpolation

As stated in the introduction, the Weierstrass theorem is non-constructive, and thus the question becomes how to generate good polynomial approximants. One way to generate polynomial approximants to a continuous function is by Lagrange interpolation. Again to repeat from the introduction, there exists an immensely vast amount of literature on Lagrange interpolation, see e.g. [13] and the references there. We here present only the basic knowledge which we shall need in this dissertation.

Lagrange interpolation comes when we interpolate to function values, i.e., when we are given n distinct points in the complex plane, $z_1, z_2, \ldots, z_n$, and we find the unique polynomial, $p_{n-1}(z)$, of degree $(n-1)$, such that

$$p_{n-1}(z_i) = f(z_i), \quad i = 1, 2, \ldots, n. \tag{2.1.1}$$

That such a polynomial exists is easily seen through the functions

$$l_k(z) = \prod_{j=1, j\neq k}^{n} \frac{(z - z_j)}{(z_k - z_j)}, \quad k = 1, 2, \ldots, n. \tag{2.1.2}$$

Indeed, $l_k(z_j) = \delta_{j,k}$, and it is thus easily seen that

$$p_{n-1}(z) = \sum_{k=1}^{n} f(z_k) l_k(z) \tag{2.1.3}$$

is a solution to (2.1.1). That this solution is unique is also easily seen. Indeed, if p_1 and p_2 are both solutions to (2.1.1), then $p = p_1 - p_2$ is an $(n-1)$th degree polynomial having n zeros, and thus it is the zero polynomial.

In general, when one speaks of Lagrange interpolation, one speaks of a compact set K and a triangular array of points which lie in K. That is, one speaks of a set of points of the form $z_j = z_{n,j}$, where there are exactly n distinct points for each $n = 1, 2, \ldots$:

$$\begin{array}{ccc} z_{1,1} & & \\ z_{2,1} & z_{2,2} & \\ z_{3,1} & z_{3,2} & z_{3,3} \\ \vdots & \vdots & \ddots \end{array}$$

Associated with this triangular array of points is then a sequence, $\{L_n\}$, of Lagrange operators, where L_n is defined as in (2.1.3) using the nth row of the triangular array, i.e.,

$$L_n f(z) = L_n(f; z) = \sum_{k=1}^{n} f(z_{n,k}) l_{n,k}(z). \tag{2.1.4}$$

If we wish to emphasize the points which we are interpolating at, we shall use the notation

$$L_n(f; z) = L_n(f; z_1, \ldots, z_n; z).$$

As an operator from $C(K)$ into $\mathbf{P}_{n-1}$, the set of polynomials of degree less than or equal to $(n-1)$, L_n is given the usual norm:

$$\|L_n\| = \sup_{\|f\|=1} \|L_n(f)\|,$$

where $\|\cdot\| = \|\cdot\|_K$ on the right is the supremum norm on K. We call this norm the nth Lebesgue constant associated with the array. It is clear from (2.1.4) that L_n is a linear map, and that $L_n(p) = p$ for all $p \in \mathbf{P}_{n-1}$. These two properties allow us to use the Lebesgue constant as a first estimate in determining how well $L_n(f)$ approximates f. If p_{n-1}^* is the best approximant to f in $\mathbf{P}_{n-1}$, we have the following

$$\|L_n(f) - f\| = \|L_n(f) - p_{n-1}^* + p_{n-1}^* - f\| = \tag{2.1.5}$$

$$\|L_n(f - p_{n-1}^*) + p_{n-1}^* - f\| \leq$$

$$\|L_n(f - p_{n-1}^*)\| + \|p_{n-1}^* - f\| \leq \|L_n\| \|p_{n-1}^* - f\| + \|p_{n-1}^* - f\| =$$

$$\|p_{n-1}^* - f\| \left(\|L_n\| + 1\right).$$

For given points, $z_1, z_2, \ldots, z_n$, it is easily seen that

$$\|L_n\| = \sup_{z \in K} \sum_{k=1}^{n} |l_k(z)|. \tag{2.1.6}$$

Indeed,

$$\|L_n(f)\| = \sup_{z \in K} \left| \sum_{k=1}^{n} f(z_k) l_k(z) \right| \leq \|f\| \sup_{z \in K} \sum_{k=1}^{n} |l_k(z)|, \tag{2.1.7}$$

from where it follows that

$$\|L_n\| \leq \sup_{z \in K} \sum_{k=1}^{n} |l_k(z)|. \tag{2.1.8}$$

But if the right hand side of (2.1.8) attains its maximum at z^*, then for a function f with

$$\|f\| = 1 \quad \text{and} \quad f(z_k) = \frac{\overline{l_k(z^*)}}{|l_k(z^*)|},$$

we actually have equality in (2.1.7), thereby giving us (2.1.6).

It is clear from (2.1.5) that it is desirable to choose interpolation nodes so that the Lebesgue constants are small. Inspection of (2.1.6) and (2.1.2) tells us that the Lebesgue constants will be *small* when the nodes are somewhat spread out. The notion of *smallness* which we shall use in this dissertation will be subexponentiality, i.e. $\|L_n\|^{1/n} \to 1$. Equally spaced nodes would be the easiest selection, however the Lebesgue constants associated with these nodes are not subexponential (see [15]). If however, the nth row of a triangular array is an nth Fekete set, it is trivial to show that the Lebesgue constants are subexponential. Recall that an nth Fekete set for a compact set K is any system of points maximizing

$$V(z_1, \ldots, z_n) = \prod_{1 \le i < j \le n} |z_j - z_i|. \tag{2.1.9}$$

By comparison of (2.1.2) and (2.1.9) it is easily seen that

$$|l_k(z)| \le 1$$

when we interpolate at Fekete sets, from where it follows that

$$\|L_n\| \le n.$$

In this sense Fekete sets are *almost ideal* for Lagrange interpolation.

The problem with Fekete sets is that they are difficult to determine. Analogous yet more easily determined sets are Leja sequences. Recall from the introduction that Leja sequences are defined inductively. Let $z_1 \in K$ be arbitrary. Once $z_1, z_2, \ldots, z_{n-1}$ have been determined, z_n is chosen so that

$$V(z) = \prod_{i=1}^{n-1} |z - z_i|$$

is maximized. While finding Fekete points is an extremal problem in n variables, finding Leja points is an extremal problem in a single variable. The main result of this dissertation will be in showing that the Lebesgue constants associated with Leja sequences on $[-1, 1]$ (and on more general sets) are subexponential.

As stated in the introduction, before coming to our main result, we will prove a theorem regarding a hierarchy of interpolation schemes, showing that our notion of *smallness*, subexponentiality, is actually quite useful. In proving our main result we will need nothing more than the definition of Lagrange interpolation, (2.1.2), and (2.1.6), and the potential theory which is to follow. But in proving our statement regarding hierarchy of interpolation schemes, we will need slightly more knowledge of Lagrange interpolation.

We now give a Lemma regarding interpolation error. This Lemma will be used in our hierarchy section. Of course, this is a classical result, see e.g. [2].

Lemma 2.1.1 *Let K be a compact set, let $t_1, t_2, \ldots, t_n$ be points in K, let Γ be a closed contour containing K, and let L_n be the Lagrange operator associated with these t_i. Then for any function f analytic on and inside Γ we have*

$$f(t) - L_n f(t) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z) p_n(t)}{(z-t) p_n(z)} dz,$$

where $p_n(t) = \prod_{i=1}^{n}(t - t_i)$.

Proof. We first note that

$$L_n f(t) = \sum_{i=1}^{n} \frac{p_n(t) f(t_i)}{(t - t_i) p_n'(t_i)}. \tag{2.1.10}$$

Next we note that the function

$$\frac{f(z)}{(z-t)p_n(z)}$$

has residue

$$\frac{f(t)}{p_n(t)}$$

at $z = t$, and residue

$$\frac{f(t_i)}{(t_i - t)p_n'(t_i)}$$

at $z = t_i$. By the residue theorem it then follows that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-t)p_n(z)}dz = \frac{f(t)}{p_n(t)} + \sum_{i=1}^{n} \frac{f(t_i)}{(t_i - t)p_n'(t_i)}.$$

Multiplying through by $p_n(t)$ we obtain

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)p_n(t)}{(z-t)p_n(z)}dz = f(t) + \sum_{i=1}^{n} \frac{f(t_i)p_n(t)}{(t_i - t)p_n'(t_i)}. \tag{2.1.11}$$

Inspection of (2.1.10) and (2.1.11) then gives us our desired result.

■

There is another estimate for interpolation error which we will need in our hierarchy section. We need to discuss Newton Interpolation before presenting this estimate (interpolation on Leja sequences is of course an example of Newton interpolation, but we do not need this estimate for our result on Leja sequences). Newton interpolation, we recall, is when we are given a sequence $\{x_n\}$ and for each n, $L_n(f)$ agrees with f on the first n terms of this sequence. In this scenario, if p_n is the polynomial obtained by interpolating on the first $n+1$ terms, and p_{n-1} is the polynomial obtained by interpolating on the first n terms, we must have

$$p_n(x) = p_{n-1}(x) + a_{n+1}(x - x_1)(x - x_2)\cdots(x - x_n). \tag{2.1.12}$$

Indeed, since p_{n-1} agrees with f at $x_1, x_2, \ldots, x_n$, the right hand side of (2.1.12) also must agree with f at $x_1, x_2, \ldots, x_n$. Therefore by a suitable choice of a_{n+1}, the right hand side of (2.1.12) will also agree with f at x_{n+1}. It follows that for Newton Interpolation, we have the following representation of the interpolation polynomial

$$\begin{aligned} L_{n+1}(f) &= a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2) + \cdots \\ &\quad + a_{n+1}(x - x_1)(x - x_2)\cdots(x - x_n). \end{aligned} \tag{2.1.13}$$

The constants a_i in (2.1.13) are called divided differences, and the following notation is often used

$$a_n = [x_1, x_2, \ldots, x_n]f, \quad n = 1, 2, \ldots,$$

It is obvious that $a_1 = [x_1]f = f(x_1)$, and an easy calculation gives that

$$a_2 = [x_1, x_2]f = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

In general we have the following useful and easily proven formula (see [5] page 98)

$$[x_1, x_2, \ldots, x_n]f = \frac{[x_2, x_3, \ldots, x_n]f - [x_1, x_2, \ldots, x_{n-1}]f}{x_n - x_1}. \tag{2.1.14}$$

In fact, (2.1.14) is where the term divided difference comes from.

Thus, in terms of divided differences, Newton interpolation takes the form

$$\begin{aligned} L_{n+1}(f) &= [x_1]f + [x_1, x_2]f(x - x_1) + [x_1, x_2, x_3]f(x - x_1)(x - x_2) + \cdots \\ &\quad + [x_1, x_2, \ldots, x_{n+1}]f(x - x_1)(x - x_2) \cdots (x - x_n). \end{aligned} \tag{2.1.15}$$

We now illustrate how to calculate divided differences with an example which will be of particular interest to us later on. We calculate the divided differences of

$$f(z) = \frac{1}{z - \alpha}.$$

We have

$$[x_1]f = \frac{1}{x_1 - \alpha} \quad \text{and,}$$

$$[x_1, x_2]f = \frac{[x_2]f - [x_1]f}{x_2 - x_1} = \frac{\frac{1}{x_2 - \alpha} - \frac{1}{x_1 - \alpha}}{x_2 - x_1} = \frac{-1}{(x_1 - \alpha)(x_2 - \alpha)}.$$

We now use $[x_1, x_2]f$ and $[x_1, x_3]f$ to calculate $[x_1, x_2, x_3]f$. We have the following

$$[x_1, x_2, x_3]f = \frac{[x_1, x_2]f - [x_1, x_3]f}{x_2 - x_3} =$$

$$\frac{\frac{-1}{(x_1 - \alpha)(x_2 - \alpha)} - \frac{-1}{(x_1 - \alpha)(x_3 - \alpha)}}{x_2 - x_3} = \frac{1}{(x_1 - \alpha)(x_2 - \alpha)(x_3 - \alpha)}.$$

It is clear that in general we will have

$$[x_1, x_2, \ldots, x_n]f = \frac{(-1)^{n+1}}{(x_1 - \alpha)(x_2 - \alpha) \cdots (x_n - \alpha)}.$$

Since this calculation will be important later in the dissertation, we record it as a Lemma.

Lemma 2.1.2 *Let $\{x_n\}$ be a sequence of distinct complex numbers and let*

$$f(\alpha) = \frac{1}{z-\alpha},$$

where α is different from every x_n. Then the n-th divided difference of f is given by

$$[x_1, x_2, \ldots, x_n]f = \frac{(-1)^{n+1}}{(x_1-\alpha)(x_2-\alpha)\cdots(x_n-\alpha)}.$$

Having had this discussion of Newton interpolation and divided differences, we are now ready to give the second interpolation error formula which we will need. This is a classical result (see e.g. [5], page 100), but we present a proof.

Lemma 2.1.3 *Let $\{x_n\}$ be a sequence of distinct points on the plane, and let x be a point different from every x_i. Then we have*

$$f(x) - L_{n+1}(f;x) = [x_1, x_2, \ldots, x_{n+1}, x]f \prod_{i=1}^{n+1}(x - x_i).$$

Proof. Let t be an arbitrary node not equal to any of the $x_1, x_2, \ldots, x_{n+1}$. Then we have

$$\begin{aligned} p_{n+1}(f; x_1, x_2, \ldots, x_{n+1}, t; x) &= p_n(f; x_1, \ldots, x_{n+1}; x) + \\ &\quad + [x_1, x_2, \ldots, x_{n+1}, t]f \prod_{i=1}^{n+1}(x - x_i). \end{aligned}$$

Now put $x = t$. Since the polynomial on the left interpolates to f at t, we get

$$f(t) = p_n(f;t) + [x_1, x_2, \ldots, x_{n+1}, t]f \prod_{i=1}^{n+1}(t - x_i).$$

Writing again x for t (which was arbitrary, after all), we find

$$f(x) - p_n(f;x) = [x_1, x_2, \ldots, x_{n+1}, x]f \prod_{i=1}^{n+1}(x - x_i).$$

■

2.2 Potential Theory

The focus of this dissertation is on Lagrange interpolation rather than potential theory. However, there is a connection between Lagrange interpolation and potential theory which we will need to exploit to obtain our results. What follows are the basic definitions and classical results from potential theory which are of interest to us in this dissertation. As the focus of this dissertation is not potential theory, the results are given without proof. A standard reference for potential theory in the plane are the books [9] and [12].

Definition 2.2.1 *Let μ be a finite Borel measure on C with compact support. Its* potential *is the function*

$$U^{\mu}(z) : C \to [-\infty, \infty),$$

defined by

$$U^{\mu}(z) = \int \log|z-w| d\mu(w) \qquad (z \in C).$$

In this dissertation we will often have sequences of measures which are convergent in the weak star sense, and we will need to make use of the following Theorem.

Theorem 2.2.2 ([9], page 59) *Let K be a compact set and let γ_n be a sequence of measures which converge weak star to some measure γ. Then the following is true*

$$\limsup_{n\to\infty} U^{\gamma_n}(z) \leq U^{\gamma}(z), \quad z \in C.$$

We will also need upper semi-continuous properties of potentials, given by the following definition and Theorem.

Definition 2.2.3 *A function $f : C \to [-\infty, \infty)$ is called upper semi-continuous if for all α the set*

$$U = \{z : f(z) < \alpha\},$$

is an open set.

Theorem 2.2.4 ([9], page 53) *Let K be a compact set. The potential of a measure μ with support in K is an upper semi-continuous function, and is harmonic on $C \backslash K$.*

Of particular interest to us in this dissertation will be the potentials of equilibrium distributions. More than just upper semi-continuous, the potential of an equilibrium distribution on a regular compact set is actually continuous. There are however, several definitions which need to be introduced before we discuss this, including of course the definition of regularity and equilibrium distributions.

Definition 2.2.5 *Let μ be a finite Borel measure on C with compact support. Its* energy *$I(\mu)$ is given by*

$$I(\mu) = \int\int \log|z-w| d\mu(z) d\mu(w) = \int U^{\mu}(z) d\mu(z).$$

Definition 2.2.6 *A subset E of C is called* polar *if $I(\mu) = -\infty$ for every finite Borel measure $\mu \neq 0$ for which the support of μ is a compact subset of E.*

Although we will not examine polar sets in this dissertation, this definition is given to aid in the understanding of the next definition and theorem.

Definition 2.2.7 *Let K be a compact subset of C, and denote by $\mathcal{M}(K)$ the collection of all Borel probability measures on K. If there exists $\mu_K \in \mathcal{M}(K)$ such that*

$$I(\mu_K) = \sup_{\mu \in \mathcal{M}(K)} I(\mu),$$

then μ_K is called an equilibrium measure for K.

Theorem 2.2.8 ([9], page 58) *Every non-polar compact set K has a unique equilibrium measure.*

Below we shall introduce the notion of logarithmic capacity, and with it a set is non-polar if and only if it is of positive capacity.

As stated above, equilibrium distributions will be of particular interest to us. We will need the following important theorem regarding equilibrium distributions.

Theorem 2.2.9 ([9], page 59) *Let K be a compact set in C, and let μ_K be the equilibrium measure for K. Then*

$$U^{\mu_K}(z) \geq I(\mu_K), \quad z \in C.$$

Since U^{μ_K} is harmonic on $C \backslash K$, and thus by the minimum principle for harmonic functions does not attain a minimum on $C \backslash K$, as a direct consequence of Theorem 2.2.9, we have

Theorem 2.2.10 *Let U^{μ_K} be the equilibrium distribution for K. Then*

$$U^{\mu_K}(z) > I(\mu_K), \quad z \in C \backslash K.$$

We now introduce our definition of regularity. Our definition is not one of the standard definitions given in literature. We are however using this definition because it is equivalent to the standard one used in the literature, and because it simplifies our discussion.

Definition 2.2.11 *A compact set K is called regular if*

$$U^{\mu_K}(z) = I(\mu_K), \quad z \in K.$$

Typically, regularity is defined in terms of the Dirichlet problem: namely it is required that the Dirichlet solution to a continuous boundary data be continuous on the closure of the domain. Our definition is equivalent to that one, but it is more convenient to use. As examples of regular sets consider compact sets of at least two points for which the complement is simply connected.

We also have the following Theorem regarding regular domains.

Theorem 2.2.12 ([12], page 54) *Let K be a regular domain with equilibrium distribution μ_K. Then $U^{\mu_K}(z)$ is a continuous function on K, and hence on the whole complex plane.*

We give one last definition in this section. The concept of capacity will be used throughout this dissertation.

Definition 2.2.13 *The logarithmic capacity of non-polar compact set K is given by*

$$\operatorname{Cap}(K) = \exp(I(\mu_K)).$$

Throughout this dissertation we will be converting polynomials to potentials by taking the natural log of these polynomials, and then converting these logs into integrals. We will discuss this more in our subsection on equilibrium measures, but the reader should make note of Definition 2.2.11 and Definition 2.2.13. Indeed, many of our integrals will be set equal to $\log(\operatorname{Cap}(K))$.

2.3 Leja Points

We recall from the introduction that Leja sequences associated with a compact subset K of the plane are defined inductively. Let $z_1 \in K$ be arbitrary. Once $z_1, z_2, \ldots, z_{n-1}$ have been determined, z_n is chosen so that

$$V(z) = \prod_{i=1}^{n-1} |z - z_i|$$

is maximized for $z = z_n$. We speak of *a* Leja sequence, rather than *the* Leja sequence, since there may be more than one choice for z_n. In fact, the choice of z_1 is arbitrary.

Since Leja points on a compact set K are determined by finding the maximum of a polynomial on K, by the maximum modulus theorem, it follows that a Leja sequence on a compact set K will necessarily lie on K's outer boundary, where the outer boundary of K is defined as the boundary of the unbounded component of the complement $C \setminus K$. In our hierarchy of interpolation schemes, one of the assumptions that we make on the system of nodes is that they lie on the outer boundary of K, which is then automatically satisfied for Leja points. In proving that the Lebesgue constants associated with Leja sequences on a closed domain are subexponential, our problem will be reduced to proving that the Lebesgue constants associated with Leja sequences on arcs are subexponential.

Another easily seen property of Leja sequences, and one which we shall use, is that their selection is invariant with respect to rotations and translations in the plane. If $z_1, \ldots, z_n$ are the first n points of a Leja sequence on a compact set K, and if the complex plane is rotated about a point w through an angle of θ, then a *new* Leja sequence will be obtained from the *new* K simply by rotating the original Leja points about the point w through an angle of θ. Similarly, if K is translated by the constant c, then a *new* Leja sequence will be $z_1 + c, z_2 + c, \ldots$.

The most important property of Leja points which we shall need is not easily seen. The following Theorem will be needed in our main result. It regards the connection between Leja sequences and equilibrium distributions. Since the focus of this dissertation is Lagrange interpolation, and we are simply using potential theory, this Theorem is given without proof. It's proof can be found in ([12], page 258]).

Theorem 2.3.1 *Let K be a compact set with equilibrium distribution μ_K. Let $\{z_j\}$ be a Leja sequence on K, and let γ_n be the normalized counting measure with respect to the first n terms of this Leja sequence, i.e.,*

$$\gamma_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}.$$

Then

$$\gamma_n \xrightarrow{*} \mu_K.$$

2.4 Equilibrium Measures

As stated in subsection 1.2, there is a connection between potential theory and polynomial interpolation. Having introduced Leja points as well as one of the major theorems which we will exploit (Theorem 2.3.1), we are now ready to explore the connection between potential theory and polynomials. In Theorem 2.3.1 we linked Leja sequences to equilibrium measures through the concept of weak star convergence. Here again, we will discuss weak star convergence, as it will facilitate much of our interplay between polynomials and potential theory. To see this how this will be done consider that the absolute value of a polynomial $p(z)$ with zeros at $z_1, z_2, \ldots, z_n \in K$ can be written as

$$\begin{aligned} |p(z)| = \prod_{j=1}^{n} |z - z_j| &= \exp(n \left(1/n \sum_{j=1}^{n} \log|z - z_j| \right) \\ &= \exp\left(n \int_K \log|z - t| d\gamma_n(t) \right), \end{aligned} \tag{2.4.16}$$

where γ_n is the probability measure taking the value $1/n$ at $t = z_1, \ldots, z_n$. On the left side of (2.4.16) we have a polynomial, while on the right hand side we have the exponential function raised to a potential, $U^{\gamma_n}(z)$. Thus finding where a polynomial is either *large* or *small* is equivalent to finding where a potential is either *large* or *small*. Further, when the probability measures γ_n associated with our sequence of points $\{z_j\}$ converge weak star to a measure γ, finding where a polynomial is either *large* or *small* will be equivalent to finding where the potential of γ is *large* or *small*. In many cases our measures, γ_n, will converge weak star to the equilibrium measure μ_K on K (as of course is the case when we are considering Leja sequences).

In this subsection we give several results regarding equilibrium measures which will help us in our discussion of polynomial interpolation. While we omitted proofs in section 1.2, here we will give proofs. There are two reasons for this. The first reason is that many of these results are too simple to be found in the literature on potential theory. The second reason is that we want to illustrate the importance of (2.4.16), the relationship between polynomials and potentials.

Before beginning we give a word about notation. In what follows, if we are given a sequence of points $\{z_j\}$, then γ_n will denote the probability measure which takes the value $1/n$ at the first n points, i.e.

$$\gamma_n = 1/n \sum_{j=1}^{n} \delta_{z_j}.$$

As before, we will use μ_K to denote the equilibrium measure on a compact set K. We also note that the reader should remember the definition of a potential associated with a given measure γ on a compact set K,

$$U^\gamma(z) = \int_K \log|z-t| d\gamma(t). \tag{2.4.17}$$

This is because we will often not use the explicit notation $U^\gamma(z)$ in the following lemmas, but only the right hand side of (2.4.17).

Lemma 2.4.1 *Let K be a compact set and let γ be a probability measure on K such that $\gamma \neq \mu_K$. Let*

$$M = \left\{ z \in K : \int_K \log|z-t| d\gamma(t) < \log(\mathrm{Cap}(K)) \right\}$$

Then $\gamma(M) > 0$, and furthermore, there exists a disc D with center in M such that $\gamma(D) > 0$ and such that for some $\epsilon > 0$ the following holds

$$\int_K \log|z-t| d\gamma(t) < \log(\mathrm{Cap}(K)) - \epsilon, \quad z \in D.$$

Proof. If M had measure 0, then we would have

$$\int_K \int_K \log|z-t| d\gamma(z) d\gamma(t) \geq \log(\mathrm{Cap}(K)),$$

and this contradicts the unicity of the equilibrium distribution since $\gamma \neq \mu_K$. Further, by upper semi-continuity, each point $z \in M$ lies in the center of a disc D such that for some $\epsilon_D > 0$ the following holds

$$\int_K \log|z-t| d\gamma(t) < \log(\mathrm{Cap}(K)) - \epsilon_D, \quad z \in D.$$

The set of all such D forms an open cover of M. From this cover we can find a countable subcover. Since $\gamma(M) > 0$, we must have $\gamma(D) > 0$ for one of the D's contained in the countable sub cover. This completes the proof.

■

Lemma 2.4.2 *Let K be a regular compact set, and let γ be a probability measure on K such that $\gamma \neq \mu_K$. Then there exists $z \in K$ such that*

$$\int_K \log|z-t| d\gamma(t) \geq \log(\mathrm{Cap}(K)).$$

Proof. Since K is regular, we have the following

$$\int_K \log|z-t|d\mu_K(t) = \log(\mathrm{Cap}(K)), \quad z \in K,$$

and thus that

$$\int_K \int_K \log|z-t|d\mu_K(t)d\gamma(z) = \log(\mathrm{Cap}(K)).$$

By an application of Fubini's Theorem, we must also have

$$\int_K \int_K \log|z-t|d\gamma(t)d\mu_K(z) = \log(\mathrm{Cap}(K)). \tag{2.4.18}$$

Now if

$$\int_K \log|z-t|d\gamma(t) < \log(\mathrm{Cap}(K))$$

held for all z, then we would have

$$\int_K \int_K \log|z-t|d\gamma(t)d\mu(z) < \log(\mathrm{Cap}(K)). \tag{2.4.19}$$

Since (2.4.19) contradicts (2.4.18), it follows that we must have

$$\int_K \log|z-t|d\gamma(t) \geq \log(\mathrm{Cap}(K))$$

for some z. This completes the proof.

■

Lemma 2.4.3 *Let K be a regular compact set with equilibrium distribution μ_K. Let Γ be a finite union of Jordan curves containing K. Then there exists $\epsilon > 0$ such that the following holds*

$$\int_K \log|z-t|d\mu_K(t) > \log(\mathrm{Cap}(K)) + \epsilon, \quad z \in \Gamma.$$

Proof. Since K is regular, by Theorem 2.2.12

$$U^{\mu_K}(z) = \int_K \log|z-t|d\mu_K(t)$$

is continuous on Γ. Further,

$$U^{\mu_K}(z) > \log(\mathrm{Cap}(K)), \quad z \in \Gamma. \quad \text{(see Theorem 2.2.10)}$$

Thus for each $z \in \Gamma$ there exists a neighborhood N_z and an ϵ_{N_z} such that

$$U^{\mu_K}(w) > \log(\mathrm{Cap}(K)) + \epsilon_{N_z}, \quad w \in N_z.$$

Since Γ is compact we need only finitely many N_z's to cover Γ. The smallest of the ϵ_{N_z}'s is our desired epsilon.

■

Lemma 2.4.4 *Let K be a regular compact set with equilibrium distribution μ_K, let $\{\gamma_n\}$ be a sequence of probability measures which converge weak star to μ_K, and let Γ be a finite union of Jordan curves containing K. Then there exists $\epsilon > 0$, and an N such that for $n > N$ we have the following*

$$\int_K \log|z-t|d\gamma_n(t) > \log(\text{Cap}(K)) + \epsilon, \quad z \in \Gamma.$$

Proof. Let ϵ_1 be as in Lemma 2.4.3. Then since $\log|z-t|$ is continuous on K for fixed $z \in \Gamma$, we have the following

$$\int_K \log|z-t|d\gamma_n(t) \to \int_K \log|z-t|d\mu_K(t) > \log(\text{Cap}(K)) + \epsilon_1. \tag{2.4.20}$$

Further, since $\log|z-t|$ is continuous on $\Gamma \times K$, if we fix $z \in \Gamma$ and $\epsilon_2 > 0$, then there exists a small neighborhood D_z of z such that for all probability measures m we have

$$\left|\int_K \log|z-t|dm(t) - \int_K \log|w-t|dm(t)\right| < \epsilon_2, \quad w \in D_z. \tag{2.4.21}$$

By (2.4.20), for each $z \in \Gamma$ we can find an N_z such that for $n > N_z$ we have

$$\int_K \log|z-t|d\gamma_n(t) > \log(\text{Cap}(K)) + \epsilon_1/2. \tag{2.4.22}$$

And now by (2.4.21) and (2.4.22) (taking $\epsilon_2 = \epsilon_1/4$), there exists D_z such that

$$\int_K \log|w-t|d\gamma_n(t) > \log(\text{Cap}(K)) + \epsilon_1/4, \quad n > N_z, \quad w \in D_z.$$

Since Γ is compact, we need only finitely many D_z's to cover Γ. Taking N to be the largest of the N_z's and ϵ to be $\epsilon_1/4$, we have completed our proof.

■

Lemma 2.4.5 *Let K be a compact set, μ_K the equilibrium distribution for K, and let $\{\gamma_n\}$ be a sequence of measures which converge weak star to μ_K. Then for all $\epsilon > 0$ there exists δ and an N, such that*

$$\gamma_n(V) < \epsilon, \quad \text{for} \quad n > N, \quad \text{whenever} \quad \textit{diameter}(V) < \delta.$$

Proof. Let $\epsilon > 0$ be given. Then there exists δ_1 such that diameter$(V) < \delta_1$ implies

$$\mu_K(V) < \frac{\epsilon}{2}.$$

To see this, consider that an immediate consequence of the definition of μ_K tells us that

$$\mu_K(\{z\}) = 0, \quad \text{for all } z. \tag{2.4.23}$$

From (2.4.23) it follows easily (because μ_K has finite energy and hence no point masses) that each point z has a neighborhood N_z such that

$$\mu_K(N_z) < \epsilon/2.$$

By compactness of K there exist $N_{z_1}, \ldots, N_{z_m}$ which cover K. Associated with this cover is then a δ_1 such that any disc of radius δ_1 and with center in K is contained in one of the N_{z_i}. This is our required δ_1.

Cover K with discs $D_1, \ldots, D_m$ of radii $\delta_1/2$ and centers at some x_i. Then there exist continuous functions f_i such that

$$f_i(x) = \begin{cases} 1 & \text{if } x \in D_i \\ 0 & \text{if } |x - x_i| > \delta_1 \end{cases}$$

Since $\gamma_n \xrightarrow{*} \mu_k$, there exists N such that $n > N$ implies that

$$\int_K f_i d\gamma_n < \int_K f_i d\mu_K + \frac{\epsilon}{2} < \mu_K(D_i) + \frac{\epsilon}{2} < \epsilon, \quad i = 1, 2, \ldots, m.$$

In particular, $n > N$ implies that $\gamma_n(D_i) < \epsilon$. Since $\bigcup D_i$ covers K, there exists a δ such that all sets of diameter δ are contained in some D_i. This is our required delta.

■

In a similar line of thought to Lemma 2.4.5, we also have the following Lemma.

Lemma 2.4.6 *Let K be a regular compact set with equilibrium distribution μ_K. Then for all $\epsilon > 0$, there exists δ such that*

$$\left| \int_{|t-z|<\delta} \log|t - z| d\mu_K(t) \right| < \epsilon, \quad z \in K.$$

Proof. For each $z \in K$, the integral

$$\int_K \log|t - z| d\mu_K$$

exists and is finite (in fact it is equal to $\log(\mathrm{Cap}(K))$ for all $z \in K$). If $B_r(z)$ is the ball of radius r and center at z, and if $b_r(z)$ is the characteristic function for the complement of $B_r(z)$, then, by the dominated convergence theorem, we have

$$\int_K \log|t - z| b_r(t) d\mu_K(t) \to \int_K \log|t - z| d\mu_K(t),$$

as $r \to 0$. This of course implies that for each $z \in K$ there exists $B_r(z)$ such that

$$\left| \int_{B_r(z)} \log|t - z| d\mu_K(t) \right| < \epsilon.$$

Since K is compact it can be covered by finitely many such balls $B_{r_1}(z_1), \ldots, B_{r_m}(z_m)$. Associated with this open cover is then a δ such that any disc of radius δ and center in K is contained in one of the $B_{r_i}(z_i)$. This is our required δ.

■

Lemma 2.4.7 *Let K be a compact set such that K is the union of finitely many Jordan arcs, let μ_K be the equilibrium distribution on K, and let $\{\gamma_n\}$ be a sequence of measures on K which converge weak star to μ_K. Then*

$$\gamma_n(\Gamma) \rightarrow \mu_K(\Gamma)$$

for all subarcs $\Gamma \subset K$.

Proof. Let $\epsilon > 0$ be given. We can cover the endpoints of Γ in open discs D_1, D_2 such that

$$\mu_K(D_1) + \mu_K(D_2) < \frac{\epsilon}{2}, \tag{2.4.24}$$

and such that for some N_1, we have

$$\gamma_n(D_1) + \gamma_n(D_2) < \frac{\epsilon}{2}, \qquad n > N_1. \tag{2.4.25}$$

We note that (2.4.25) is due to Lemma 2.4.5. By Urysohn's Lemma, there exists a continuous function f of norm 1 such that $f = 1$ on Γ and such that $f = 0$ on $K \backslash (\Gamma \bigcup D_1 \bigcup D_2)$. Since γ_n converges weak star to μ_K, there exists $N_2 > N_1$ such that $n > N_2$ implies

$$\left| \int_K f(t) d\gamma_n(t) - \int_K f(t) d\mu(t) \right| < \frac{\epsilon}{2}. \tag{2.4.26}$$

Comparison of (2.4.24), (2.4.25), and (2.4.26), and the definition of f tells us that for $n > N_2$ we have

$$|\gamma_n(\Gamma) - \mu(\Gamma)| < \epsilon.$$

This completes the proof.

■

As a direct consequence of Lemma 2.4.7, we have the following Lemma.

Lemma 2.4.8 *Let K be as in Lemma 2.4.7. Let s be a simple step-function defined on K, taking its finitely many values $a_1, \ldots, a_j$ on the subarcs $A_1, \ldots, A_j$. Also, let $\{\gamma_n\}$ be a sequence of measures on K which converge weak star to μ_K. Then we have*

$$\int_K s(t) d\gamma_n(t) \rightarrow \int_K s(t) d\mu(t).$$

Another direct consequence of Lemma 2.4.7 is the following lemma. This lemma will be important in our main result.

Lemma 2.4.9 *Let $K = [-1, 1]$, let $\{x_j\}$ be a sequence of Leja points on $[-1, 1]$, let μ_K be the equilibrium distribution on $[-1, 1]$, and let γ_n be the probability measure taking the value $1/n$ at the first n Leja points. Now let $R \geq 1$ be given and let $x \in (0, 1)$ be such that $1 - 2^R(1 - x) > 0$. Finally, let k be the number of the first n Leja points contained in the interval $I = [1 - 2^R(1 - x), 1 - 2(1 - x)]$. Then there exists N such that for $n > N$ we have*

$$k/n > \frac{1}{4\pi} \int_{1-2^R(1-x)}^{1-2(1-x)} \frac{1}{\sqrt{1 - t^2}} dt.$$

Proof. The equilibrium distribution for $[-1,1]$ is

$$\mu_{[-1,1]} = \frac{1}{2\pi}\frac{1}{\sqrt{1-t^2}}.$$

Since $\gamma_n \overset{*}{\rightharpoonup} \mu_{[-1,1]}$ (see Theorem 2.3.1), by Lemma 2.4.7 we have

$$k/n = \gamma_n(I) \to \mu_{[-1,1]}(I) = \frac{1}{2\pi}\int_{1-2^R(1-x)}^{1-2(1-x)} \frac{1}{\sqrt{1-t^2}}dt.$$

Thus there exists N such that $n > N$ implies

$$k/n > \frac{1}{4\pi}\int_{1-2^R(1-x)}^{1-2(1-x)} \frac{1}{\sqrt{1-t^2}}dt.$$

■

When we have a sequence of measures $\{\gamma_n\}$ which converge weak star to a measure $\gamma \neq \mu_K$, then Lemma 2.4.7 does not apply, but we do have the following Lemma, which we will need in our Hierarchy section.

Lemma 2.4.10 *Let K be a compact set, let γ be a measure on K, and let $\{\gamma_n\}$ be a sequence of measures which converge weak star to γ. Now let D be an open disc such that $\gamma(D) > 0$. Then there exists N such that $n > N$ implies*

$$\gamma_n(D) > 0.$$

Proof. Let R be the radius of D and assume w.l.o.g. that D is centered at the origin. Define $f(z) = 1 - |z|/R$ if $|z| \le R$ and $f(z) = 0$ otherwise. Then f is continuous, and by assumption

$$\int f d\gamma > 0.$$

Hence, by weak star convergence, for large n

$$\int f d\gamma_n > 0,$$

which proves the claim.

■

We will now present two theorems which we will use in our main result. One of these will also be important to us in our hierarchy section. The proofs of each of these theorems illustrate the importance of potential theory to this dissertation. In each proof we either convert a polynomial or a product into a potential, and then *use* potential theory to achieve our results.

Theorem 2.4.11 *Let K be a regular compact set, let $\{z_j\}$ be a sequence of Leja points on K, and let $P_{n,k}(z) = \prod_{j=1,j\neq k}^{n}(z - z_j)$. Then $\|P_{n,k}\|_K^{1/n} \to \mathrm{Cap}(K)$ uniformly in k.*

Proof. We first show that

$$\limsup_n \|P_{n,k}\|_K^{1/n} \geq \text{Cap}(K). \tag{2.4.27}$$

We note however, that showing (2.4.27) is equivalent to showing

$$\limsup_n \|P_{n,k}\|_K^{1/(n-1)} \geq \text{Cap}(K), \tag{2.4.28}$$

and that to show (2.4.28), it is enough to verify that for all n, k there exists $z \in K$ such that

$$\log |P_{n,k}(z)|^{1/(n-1)} \geq \log\left(\text{Cap}(K)\right). \tag{2.4.29}$$

To show (2.4.29) we write

$$\log |P_{n,k}(z)|^{1/(n-1)} = \int_K \log|z-t| d\gamma_{n,k}(t), \tag{2.4.30}$$

where $\gamma_{n,k}$ is the measure which takes the value $\frac{1}{n-1}$ at the points $t = z_j, j \neq k$, $1 \leq j \leq n$. Since $\gamma_{n,k}$ is not the equilibrium distribution for K, by Lemma 2.4.2 the integral in (2.4.30) is indeed greater than or equal to $\log\left(\text{Cap}(K)\right)$ for some $z \in K$. This proves (2.4.27).

Having shown (2.4.27), to complete the proof it suffices to show that for all ϵ there exists N, chosen independently of k, such that $n > N$ implies that

$$\|P_{n,k}\|_K^{1/n} < \text{Cap}(K) + \epsilon. \tag{2.4.31}$$

By the maximum modulus theorem however, in order to show (2.4.31), it is enough to prove that for all $\epsilon > 0$ there exists a finite union of Jordan curves Γ containing K, and an N such that for $n > N$

$$\|P_{n,k}\|_\Gamma^{1/n} < \text{Cap}(K) + \epsilon. \tag{2.4.32}$$

We will indeed use the strategy of approximating our polynomial on such a Γ and applying the maximum modulus theorem. Rather than proving (2.4.32) directly, we will show that the log of the left side of (2.4.32) approximates the log of the right side of (2.4.32). To be precise, we will complete our proof by showing that for all $\epsilon > 0$, there exists a Γ containing K and an N such that

$$\frac{1}{n}\log|P_{n,k}(z)| < \log\left(\text{Cap}(K)\right) + \epsilon, \quad n > N, \quad z \in \Gamma. \tag{2.4.33}$$

(This does indeed show (2.4.32). To see this replace the ϵ in (2.4.33) with $\log(1+\delta)$, where δ is such that $\text{Cap}(K)\delta < \epsilon$.) To show (2.4.33) we first write

$$\frac{1}{n}\log|P_{n,k}(z)| = \frac{1}{n}\sum_{j\neq k}^{n} \log|z - z_j| = \frac{n-1}{n}\int_K \log|z-t| d\gamma_{n,k}(t). \tag{2.4.34}$$

Next, since

$$\gamma_n \overset{*}{\rightharpoonup} \mu_K,$$

(this is by Theorem 2.3.1) where μ_K is the equilibrium distribution for K, and γ_n is the probability measure taking the value $1/n$ at $t = z_j, j = 1, 2, \dots n$, it is clear that

$$\gamma_{n,k} \xrightarrow{*} \mu_K. \tag{2.4.35}$$

Further, since K is regular, by Theorem 2.2.12

$$U^{\mu_K}(z) = \int_K \log|z-t| d\mu_K(t)$$

is continuous in C, and by definition of regularity $U^{\mu_K}(z) = \log(\text{Cap}(K))$ for $z \in K$. It follows that we can choose Γ sufficiently close to K such that

$$\int_K \log|z-t| d\mu_K(t) < \log(\text{Cap}(K)) + \epsilon/2, \quad z \in \Gamma. \tag{2.4.36}$$

We claim that (2.4.34), (2.4.35), and (2.4.36) complete the proof. Indeed, since $\log|z-t|$ is continuous on K for $z \notin K$, by (2.4.34), (2.4.35), and (2.4.36) it follows that for fixed $z^* \in \Gamma$ there exists N^* such that $n > N^*$ implies

$$\frac{1}{n}\log|P_{n,k}(z^*)| < \log(\text{Cap}(K)) + \epsilon. \tag{2.4.37}$$

We note that this N^* can be chosen independently of k since $\gamma_{n,k_1} = \gamma_{n,k_2}$ except on a set of measure $2/(n-1)$, and this set is negligible for large values of N. We further note that small perturbations in z create small perturbations in $\log|z-t|$ and thus in $\int_K \log|z-t| d\gamma_{n,k}(t)$. It follows that we can actually choose N^* so that

$$\frac{1}{n}\log|P_{n,k}(z)| < \log(\text{Cap}(K)) + \epsilon$$

holds for all z in a disc D_{z^*} about z^*. Since Γ is compact, it can be covered by finitely many such discs: $D_{z_1^*}, \dots, D_{z_m^*}$. Now if N_i^* is such that $N_i^* < n$ implies

$$\frac{1}{n}\log|P_{n,k}(z)| < \log(\text{Cap}(K)) + \epsilon, \quad z \in D_{z_i^*},$$

it follows that for

$$n > \max\{N_1^*, \dots, N_m^*\},$$

we will have

$$\frac{1}{n}\log|P_{n,k}(z)| < \log(\text{Cap}(K)) + \epsilon, \quad n > N, \quad z \in \Gamma.$$

We have thus shown (2.4.33) and completed our proof.

■

Theorem 2.4.12 *Let K be the union of finitely many Jordan arcs, let $\{z_j\}$ be a sequence of Leja points on K, and for $1 \le k \le n$ set*

$$P(n,k,\delta) = \prod_{|x_j - x_k| \ge \delta,\ 1 \le j \le n} |x_j - x_k|.$$

Then for all $\epsilon > 0$ there exists δ, N, such that for $n > N$ we have the following

$$\left|(P(n,k,\delta))^{1/n} - \text{Cap}(K)\right| < \epsilon, \quad k = 1, 2, \dots, n.$$

Proof. Rather than directly proving our desired result, we will instead show that for all $\epsilon > 0$ there exists δ, N, such that for $n > N$ we have

$$\left|\log\left(P(n,k,\delta)\right)^{1/n} - \log\left(\mathrm{Cap}(K)\right)\right| < \epsilon, \quad k = 1, 2, \ldots, n.$$

The two statements clearly imply each other. We write

$$\frac{1}{n}\log\left(P(n,k,\delta)\right) = \frac{1}{n}\sum_{|z_j - z_k| \geq \delta} \log|z_j - z_k| = \int_{|z_k - t| \geq \delta} \log|t - z_k| d\gamma_n(t),$$

where γ_n is the probability measure concentrated on the first n Leja points, giving the weight $1/n$ to each of these points. Our assumptions about K imply that K is regular, and we thus have

$$\int_K \log|z_k - t| d\mu_K(t) = \log\left(\mathrm{Cap}(K)\right),$$

where μ_K is the equilibrium distribution for K (this is by our definition of regularity). By Lemma 2.4.6 we also know that

$$\int_{|z_k - t| \geq \delta} \log|z_k - t| d\mu_K(t) \to \int_K \log|z_k - t| d\mu_K(t) = \log\left(\mathrm{Cap}(K)\right),$$

uniformly in k as $\delta \to 0$. It follows that we will have completed our proof if we can show that

$$\int_{|z_k - t| \geq \delta} \log|t - z_k| d\gamma_n(t) \to \int_{|z_k - t| \geq \delta} \log|z_k - t| d\mu_K(t),$$

and that this convergence occurs uniformly in k.

With this in mind, we define (for $x, t \in K$)

$$g_x(t) = \begin{cases} \log|x - t| & \text{if} \quad |x - t| \geq \delta \\ 0 \text{ elsewhere} \end{cases}$$

Then by our assumptions on K, for all $x \in K$, there exist simple functions $\overline{s_x}, \underline{s_x}$, which take their non-zero values on arcs contained in K, and there exists a disc $D_x > 0$ centered at x such that the following is true:

1. $\underline{s_x}(t) \leq g_y(t) \leq \overline{s_x(t)}, \quad y \in D_x, \quad t \in K$
2. $\int_K \overline{s_x}(t) d\mu_K(t) - \epsilon \leq \int_K g_y(t) d\mu_K \leq \underline{s_x}(t) d\mu_K(t) + \epsilon, \quad y \in D_x$

Since Lemma 2.4.8 tells us that

$$\int_K \overline{s_x}(t) d\gamma_n(t) \to \int_K \overline{s_x}(t) d\mu_K(t),$$

and that

$$\int_K \underline{s_x}(t) d\gamma_n(t) \to \int_K \underline{s_x}(t) d\mu_K(t),$$

it follows that there exists N_x such that $n > N_x$ implies that

$$\left| \int_K g_y(t) d\gamma_n(t) - \int_K g_y(t) d\mu_K(t) \right| < 3\epsilon, \quad y \in D_x$$

Now since K is compact, it follows that there exists $x_1, x_2, \ldots, x_m \in K$ such that $K \subset \bigcup D_{x_i}$. Then for $N = \max\{N_{x_1}, N_{x_2}, \ldots, N_{x_m}\}$, and $n > N$ we have

$$\left| \int_K g_y(t) d\gamma_n(t) - \int_K g_y(t) d\mu_K(t) \right| < 3\epsilon, \quad y \in K.$$

Based on our definition of $g_y(t)$, it follows that we have shown that

$$\int_{|x_k - t| \geq \delta} \log |t - x_k| d\gamma_n(t) \to \int_{|x_k - t| \geq \delta} \log |x_k - t| d\mu_K(t) \text{ uniformly in } k.$$

This completes our proof.

∎

3 Hierarchy of Interpolation Schemes

With our preliminaries completed, we are ready to give our first result, a hierarchy of interpolation schemes.

Let K be a regular compact set and let $\{z_{n,k}\}_{1\leq k\leq n;\ n=1,2,\ldots}$ be a triangular array of points lying in K. Let $L_n f$ be the Lagrange interpolation to f based on the points in the n-th row:

$$L_n f(z) = L_n(f;z) = \sum_{k=1}^{n} f(z_{n,k}) l_{n,k}(z).$$

We shall also use the normalized counting measures on the points in the n-th row:

$$\gamma_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{z_{n,k}}.$$

We recall the notion of outer boundary. The complement $C\setminus K$ consists of connected components, one of them, denoted by Ω, is unbounded. Now the boundary $\partial\Omega$ of Ω is called the outer boundary of K. We shall assume of the points $z_{n,k}$ that they lie on the outer boundary of K.

Consider the following statements:

(i) The Lebesgue constants Λ_n associated with L_n are subexponential:

$$\Lambda_n^{1/n} \to 1,$$

(ii) $\gamma_n \to \mu_K$ in the weak star sense, where μ_K is the equilibrium distribution of K,

(iii) $L_n f \to f$ uniformly on K for all functions f which are analytic on and inside a finite union of closed contours $\Gamma_i, i = 1, 2\ldots, v$, such that $K \subset \Gamma = \cup\Gamma_i$.,

(iv) $L_n f \to f$ uniformly on K for all f of the form $f(z) = 1/(z-\alpha)$, $\alpha \in \Omega$.

Clearly, **(iii)** implies **(iv)**, but actually we now show that **(ii)**-**(iv)** are equivalent, and **(i)** implies any of them.

Theorem 3.0.13 *Let K be a compact set with positive capacity, and assume that all $z_{n,k}$ lie on the outer boundary of K. Then* **(ii)**, **(iii)**, *and* **(iv)** *are equivalent, and* **(i)** *implies any of them.*

3.1 Proof of (i) $\Rightarrow$(ii)

We show that if γ_n does not converge weak star to μ_K then the Lebesgue constants associated with $\{z_{n,k}\}$ are not subexponential. If $\{\gamma_n\}$ does not converge weak star to μ_K, then by Helly's selection theorem, we can find a subsequence $\{\gamma_{n_s}\}$ such that $\gamma_{n_s} \overset{*}{\rightharpoonup} \gamma$, where γ is some probability measure not equal to μ_K. Let us relabel this subsequence as $\{\gamma_n\}$.

Recalling that

$$\Lambda_n = \sup_{z\in K}\left(\sum_{j=1}^{n} \frac{\prod_{j=1}^{n} |(z - z_{n,j})|}{\prod_{j=1}^{n} |(z_{n,k} - z_{n,j})|}\right), \tag{3.1.1}$$

the idea of our proof will be to use results from potential theory to somewhat easily complete our proof. Specifically, we will find a sequence $\{z_n^*\} \subset K$, and a sequence $\{z_{n,k^*}\}$, so that

$$\begin{aligned}\left(\prod_{j=1,j\neq k^*}^{n} |z_n^* - z_{n,j}|\right)^{1/n} &= \exp\left(\frac{1}{n}\sum_{j\neq k^*}^{n} \log|z_n^* - z_{n,j}|\right)\\ &= \exp\left(\int_K \log|z_n^* - t|d\gamma_n(t)\right),\end{aligned}$$

is *large* and so that

$$\begin{aligned}\left(\prod_{j=1,j\neq k^*}^{n} |z_{n,k^*} - z_{n,j}|\right)^{1/n} &= \exp\left(\frac{1}{n}\sum_{j\neq k^*}^{n} \log|z_{n,k^*} - z_{n,j}|\right)\\ &= \exp\left(\int_K \log|z_{n,k^*} - t|d\gamma_n(t)\right),\end{aligned}$$

is *small*. The quotient of these two terms will then tell us that (3.1.1) is not subexponential.

We first find our sequence $\{z_{n,k^*}\}$ and we begin to do this by defining the set

$$M = \left\{z \in K : \int_K \log|z-t|d\gamma(t) < \log(\mathrm{Cap}(K))\right\}.$$

Since $\gamma \neq \mu_K$, by Lemma 2.4.1 there is a disc D with center in M such that $\gamma(D) > 0$ and such that

$$\int_K \log|z-t|d\gamma(t) < \log(\mathrm{Cap}(K)) - \epsilon, \quad z \in D$$

for some $\epsilon > 0$. Since

$$\gamma(D) > 0,$$

by Lemma 2.4.10 there exists N such that for $n > N$ we have

$$\gamma_n(D) > 0.$$

In particular, for $n > N$ we can always find a $z_{n,k^*} \in D$. For each n we choose such a z_{n,k^*} and we form the measures

$$\gamma_n^* = \frac{1}{n-1} \sum_{j=1, j\neq k^*} \delta_{z_{n,j}^{(n)}}.$$

Since γ_n converges weak star to γ, it is clear that γ_n^* must converge weak star to γ. Now by our choice of z_{n,k^*}, we have

$$\int_K \log |z_{n,k^*} - t| d\gamma(t) < \log (\mathrm{Cap}(K)) - \epsilon,$$

and since γ_n^* converges weakly star to γ, by Theorem 2.2.2, we must have

$$\limsup_n \left(\int_K \log |z_{n,k^*} - t| d\gamma_n^*(t) \right) \leq \log (\mathrm{Cap}(K)) - \epsilon. \tag{3.1.2}$$

And this of course implies that

$$\limsup_n \left(\prod_{j=1, j\neq k^*}^{n} |z_{n,k^*} - z_{n,j}| \right)^{1/n} \leq \exp \left(\log (\mathrm{Cap}(K)) - \epsilon \right). \tag{3.1.3}$$

We now find our $\{z_n^*\}$. We will again consider the probability measure $\gamma_n^* = \frac{1}{n-1} \sum_{i\neq k^*}^{n} \delta_{z_i}$. Since $\gamma_n^* \neq \mu_K$, by Lemma 2.4.2 we can find a $z_n^* \in K$ such that

$$\int_K \log |z_n^* - t| d\gamma_n^*(t) \geq \log(\mathrm{Cap}(K)).$$

For z_n^* we then have

$$\left(\prod_{j\neq k_n}^{n} |z_n^* - z_{n,j}| \right)^{1/n} = \exp \left(\frac{1}{n} \sum_{j\neq k_n}^{n} \log |z_n^* - z_{n,j}| \right) = \tag{3.1.4}$$

$$\exp \left(\frac{n-1}{n} \int_K \log |z_n^* - z| d\gamma_n^*(z)) \right) \geq \exp \left(\frac{n-1}{n} \log(\mathrm{Cap}(K)) \right).$$

With (3.1.3) and (3.1.4) we now have that

$$\limsup_n \frac{\left(\prod_{j\neq k_n}^{n} |z_n^* - z_{n,j}| \right)^{1/n}}{\left(\prod_{j=1, j\neq k_n}^{n} |z_{n,k^*} - z_{n,j}| \right)^{1/n}} \geq e^{\epsilon} > 1.$$

This completes the proof.

■

3.2 Proof of (ii)⇒(iii)

Let Γ be a union of finitely many smooth Jordan curves which contain K and which are contained themselves in f's domain of analyticity. Then by the remainder theorem (Theorem 2.1.1) we have for $z \in K$

$$f(z) - L_n f(z) = f(z) - \sum_{j=1}^{n} \frac{f(z_{n,j}) P_n(z)}{P_n'(z_j)(z - z_{n,j})} = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)P_n(z)}{(t-z)P_n(t)} dt,$$

where $P_n(z) = \prod_{j=1}^{n}(z - z_{n,j})$. Thus, in order to show that $L_n(f)$ converges uniformly to f, it suffices to show that

$$\left| \int_\Gamma \frac{f(t)P_n(z)}{(t-z)P_n(t)} dt \right| \to 0 \tag{3.2.5}$$

uniformly in $z \in K$.

Now to show (3.2.5), since we know that $f(t)$ is bounded on Γ, and that $(t-z)$ is bounded on $\Gamma \times K$, we will focus on the quotient $P_n(z)/P_n(t)$. Using potential theory, we will find an upper bound for

$$|P_n(z)|^{1/n}, \quad z \in K,$$

and a lower bound for

$$|P_n(t)|^{1/n}, \quad t \in \Gamma,$$

and we will then complete our proof by showing that

$$\left| \frac{P_n(z)}{P_n(t)} \right| = \left(\frac{|P_n(z)|^{1/n}}{|P_n(t)|^{1/n}} \right)^n < r^n$$

for some $r < 1$.

We first find a lower bound for

$$|P_n(t)|^{1/n}, \quad t \in \Gamma.$$

Since K is regular, by Lemma 2.4.4, there exists $\epsilon > 0$ and N_1 such that for $n > N_1$ we have

$$\int_K \log|t - z| d\gamma_n(z) > \log(\mathrm{Cap}(K)) + \epsilon, \quad t \in \Gamma.$$

We then have that

$$\begin{aligned} |P_n(t)|^{1/n} = \prod_{j=1}^{n} |t - z_j|^{1/n} &= \exp\left(\int_K \log|t-z| d\gamma_n(z) \right) \\ &> \exp\left(\log(\mathrm{Cap}(K)) + \epsilon\right), \qquad t \in \Gamma, \quad n > N_1. \end{aligned} \tag{3.2.6}$$

We now find an upper bound for $|P_n(z)|^{1/n}$, $z \in K$. Since K is regular, we can find a contour Γ_1 around K such that on Γ_1 we have

$$\int_K \log|z - t| d\mu_K(t) < \log(\mathrm{Cap}(K)) + \epsilon/2.$$

Since γ_n tends to μ_K in weak star sense, on Γ_1 we have uniformly

$$\lim_{n\to\infty} \frac{1}{n} \log |P_n(z)| = \int_K \log |z-t| d\mu_K(t),$$

From these two statements we get

$$\limsup_{n\to\infty} \|P_n\|_{\Gamma_1}^{1/n} \leq \mathrm{Cap}(K) e^{\epsilon/2},$$

and hence by the maximum modulus theorem,

$$\limsup_{n\to\infty} \|P_n\|_K^{1/n} \leq \mathrm{Cap}(K) e^{\epsilon/2}. \tag{3.2.7}$$

With (3.2.6) and (3.2.7) we are now ready to complete our proof. Let N_2 be such that for $n > N_2$ the inequality in (3.2.7) holds. If $n > \max\{N_1, N_2\}$, (3.2.6) and (3.2.7) imply that

$$\left| \frac{P_n(z)}{P_n(t)} \right|^{1/n} < \exp\left(-\epsilon/2\right).$$

Now if

$$M = \|f\|_\Gamma,$$

and

$$\delta = \mathrm{distance}\{\Gamma, K\},$$

we then have the following (for $n > \max\{N_1, N_2\}$)

$$\left| \int_\Gamma \frac{f(t)P_n(z)}{(t-z)P_n(t)} dt \right| \leq \int_\Gamma \frac{M}{\delta} \left(\left| \frac{P_n(z)}{P_n(t)} \right|^{1/n} \right)^n d|t| \leq \Lambda(\Gamma) \frac{M}{\delta} \exp(-n\epsilon/2),$$

where $\Lambda(\Gamma)$ is the length of Γ. Since this last term tends to 0 as $n \to \infty$, this completes our proof.

■

3.3 Proof of (iv)$\Longrightarrow$ (ii)

We assume that γ_n does not converge weak star to μ_K, and show that there exists an $\alpha \in \Omega$ such that for $f(z) = 1/(z-\alpha)$, $L_n(f)$ does not converge uniformly to f.

We first note that if $\{\gamma_n\}$ does not converge weak star to μ_K then by Helly's selection theorem, we can find a subsequence of $\{\gamma_{n_s}\}$ that converges weak star to some measure $\gamma \neq \mu_K$. For convenience, let us relabel this subsequence as $\{\gamma_n\}$. Since γ_n lies on the outer boundary of K, so does γ. We next note that by Lemma 2.1.3

$$f(t) - L_n f(t) = [z_1, z_2, \ldots, ..z_n, t] f \prod_{j=1}^{n} (t - z_j)$$

Thus our proof will be complete if we find an $\alpha \in \Omega$ and a sequence $\{t_n\} \subset K$, such that for $f(z) = 1/(z-\alpha)$ we have

$$[z_1, z_2, \ldots, ..z_n, t_n]f \prod_{j=1}^{n}(t_n - z_j) \nrightarrow 0. \tag{3.3.8}$$

We will use the γ mentioned above and our lemmas on potential theory to find the proper α as well as a sequence $\{t_n\}$ so that (3.3.8) holds.

We first find α. We begin by noting that by Lemma 2.1.2, for $f(z) = 1/(z-\alpha)$, we have

$$[z_1, z_2, \ldots, z_n]f = \frac{(-1)^{n+1}}{\prod_{j=1}^{n}(z_j - \alpha)},$$

so that

$$|[z_1, z_2, \ldots, z_n]f| = \frac{1}{\left(\prod_{j=1}^{n}|z_j-\alpha|^{1/n}\right)^n} = \frac{1}{\exp\left(n\int_K \log|z-\alpha| d\gamma_n(z)\right)}.$$

This also shows

$$|[z_1, z_2, \ldots, z_n, t_n]f| = \frac{1}{|t_n-\alpha| \exp\left(n\int_K \log|z-\alpha| d\gamma_n(z)\right)}. \tag{3.3.9}$$

Another easy calculation give us that

$$\prod_{j=1}^{n}|t_n - z_j| = \left(\left(\prod_{j=1}^{n}|t_n - z_j|\right)^{1/n}\right)^n = \exp\left(n\int_K \log|t_n - z| d\gamma_n(z)\right). \tag{3.3.10}$$

Inspection of (3.3.8), (3.3.9), and (3.3.10) now tells us we will be done if we can find an α and a sequence $\{t_n\} \subset K$ such that

$$\frac{\exp\left(n\int_K \log|t_n - z| d\gamma_n(z)\right)}{|t_n-\alpha| \exp\left(n\int_K \log|z-\alpha| d\gamma_n(z)\right)} \nrightarrow 0.$$

But we are going to choose our α so that $\alpha \in \Omega \subset K^c$, and our $\{t_n\}$ so that $\{t_n\} \subset K$. We will thus have

$$|\alpha - t_n| \geq \text{distance}\{\alpha, K\} > 0.$$

It follows that we will be done as soon as we find $\alpha \in \Omega$, $\{t_n\} \subset K$ such that

$$\frac{\exp\left(n\int_K \log|t_n - z| d\gamma_n(z)\right)}{\exp\left(n\int_K \log|z-\alpha| d\gamma_n(z)\right)} \nrightarrow 0. \tag{3.3.11}$$

Now since $\gamma_n \neq \mu_K$, by Lemma 2.4.2, we can always find a t_n such that

$$\int_K \log|t_n - z| d\gamma_n(z) \geq \log(\text{Cap}(K)).$$

We will thus have shown (3.3.11) if we can find α such that

$$\int_K \log|z-\alpha| d\gamma_n(z) < \log(\text{Cap}(K)) \tag{3.3.12}$$

for large values of n. To find such an α we refer back to our γ from the beginning of the proof. Since $\gamma \neq \mu_K$, by Lemma 2.4.1

$$\int_K \log|z'-z| d\gamma(z) < \log(\text{Cap}(K)) - \epsilon$$

for some ϵ and $z' \in \text{supp}(\gamma)$. Recall now that the support of γ lies on the outer boundary of K, hence so is z'. By upper semicontinuity of the function

$$f(t) = \int_K \log|t-z| d\gamma(z),$$

we can find a z'' lying in the unbounded component of the complement of K such that

$$\int_K \log|z''-z| d\gamma(z) < \log(\text{Cap}(K)).$$

Since $\gamma_n \stackrel{*}{\rightharpoonup} \gamma$, and since

$$f(z) = \log|z''-z|$$

is continuous on K,

$$\int_K \log|z''-z| d\gamma_n(z) \to \int_K \log|z''-z| d\gamma(z) < \log(\text{Cap}(K))$$

Thus we choose as our α this z'', and by (3.3.12), this completes the proof.

■

Comments: We needed regularity in this proof. Indeed, if we only required compactness in our assumptions, then we could have a set of the form $K \cup \{z_0\}$ where $z_0 \notin K$. In this scenario if our triangular array is such that $z_{n,k} \neq z_0$ for all n, k (for such an array we could still have weak star convergence to μ_K), then we could not possibly have $L_n f \to f$ for all analytic functions f. To see this just take two functions f_1, f_2 which coincide on K, but such that $f_1(z_0) \neq f_2(z_0)$.

We also wish to note that **(i)** is a stronger condition than conditions **(ii)-(iv)**. To see this consider that for Lebesgue constants, Λ_n, we have

$$\Lambda_n = \sup_{z\in K} \sum_{k=1}^{n} |l_k(z)|, \tag{3.3.13}$$

where

$$l_k(z) = \prod_{j=1, j\neq k}^{n} \frac{(z-z_j)}{(z_k-z_j)}. \tag{3.3.14}$$

It is obvious from (3.3.14) that (3.3.13) can be made arbitrarily large by choosing interpolation points which are arbitrarily close. The distribution of the points become irrelevant then, if the points are chosen too closely together.

4 Lebesgue Constants on Leja Points on [-1,1]

In this section we shall present the model for our main result. Namely, we will prove the following Theorem:

Theorem 4.0.1 *Let $\{x_j\}$ be a sequence of Leja points on $[-1,1]$. Then the Lebesgue constants associated with these Leja points are subexponential, i.e.*

$$\lim_{n\to\infty} \Lambda_n^{1/n} = 1.$$

However, since the proof is rather long, it will be convenient to divide it into two parts. The first part of our proof will be contained in our subsection entitled **Reduction of the main proof**, and it will begin with a simple observation that when considering the nth roots of Λ_n, one actually need only to consider the quotient of a certain polynomial and a certain product. Following this observation we will cite a result from potential theory telling us that the polynomial involved in our quotient can be entirely handled with potential theory. We will conclude part one by showing that the product involved in our quotient can *almost* be handled entirely with potential theory. Thus this subsection, in essence, will show that potential theory *almost* gives us our entire result.

Part two of our proof will consist of dealing with the *small* portion of our proof that potential theory does not help us with. However, this *small* portion will be quite difficult to handle. A product of the form

$$\prod_{|x_j - x_k| < \delta} |x_j - x_k|,$$

will be our main difficulty. Part two of our proof will be contained in our subsection entitled **Estimate for the main product**. In the absence of potential theory, we shall use a distancing rule between the Leja points to help us in this subsection. We give this distancing rule now.

4.1 Separation of Leja Points

As explained in our hierarchy section, when there is no distancing rule between a given sequence of points, the associated Lebesgue constants may not be subexponential. Thus in proving our main result, that the Lebesgue constants associated with the Leja points are subexponential, we shall need a distancing rule.

We will obtain our distancing rule with the help of two well known inequalities. Following the statements of these inequalities will be three simple Lemmas. The third Lemma is our distancing rule for Leja points.

Theorem 4.1.1 (Bernstein's inequality) *Let P_n be a polynomial of degree n. Then for $t \in [-1,1]$ we have the following inequality*

$$|P_n'(t)| \leq \frac{n}{\sqrt{1-t^2}} \|P_n\|_{[-1,1]}.$$

See e.g. [3].

Theorem 4.1.2 (Markov's inequality) *Let P_n be a polynomial of degree n. Then for $t \in [-1,1]$ we have the following inequality*

$$|P_n'(t)| \leq n^2 \|P_n\|_{[-1,1]}.$$

See e.g. [3].

Lemma 4.1.3 *Let $\{x_j\}$ be a sequence of Leja points on $[-1,1]$ and let $n > j$. Then $x_n \geq 0$ and $x_j \geq 0$ imply that*

$$|x_n - x_j| \geq \frac{\sqrt{1-x_n} + \sqrt{1-x_j}}{2n}.$$

Proof. Let

$$P_{n-1}(x) = \prod_{i=1}^{n-1} (x - x_i),$$

and

$$M_{n-1} = \|P_{n-1}\|_{[-1,1]}.$$

Then by Bernstein's inequality we have

$$|P_{n-1}'(t)| \leq \frac{n-1}{\sqrt{1-t^2}} M_{n-1} \leq \frac{n}{\sqrt{1-t}} M_{n-1}.$$

Since x_n is the nth Leja point, we have $M_{n-1} = |P_{n-1}(x_n)|$. Also, let $x_j \geq 0$, $1 \leq j \leq n$, be given (we are assuming $x_n \geq 0$). Then we have the following:

$$M_{n-1} = |P_{n-1}(x_n)| = \left| \int_{x_j}^{x_n} P_{n-1}'(t) dt \right| \leq n M_{n-1} \left| \int_{x_j}^{x_n} \frac{dt}{\sqrt{1-t}} \right| =$$

$$n M_{n-1} 2 \left| \sqrt{1-x_n} - \sqrt{1-x_j} \right| = 2n M_{n-1} \frac{|x_n - x_j|}{\sqrt{1-x_j} + \sqrt{1-x_n}}$$

By looking at the first and last terms above, it follows that

$$|x_n - x_j| \geq \frac{1}{2} \frac{\sqrt{1-x_n} + \sqrt{1-x_j}}{n},$$

which completes the proof.

■

Lemma 4.1.4 *Let $\{x_j\}$ be a sequence of Leja points on $[-1,1]$ and let $n > j$. Then $|x_n - x_j| \geq 1/n^2$.*

Proof. We again let

$$P_{n-1}(x) = \prod_{i=1}^{n-1}(x - x_i).$$

Then Markov's inequality tells us that

$$\left|P'_{n-1}(t)\right| \leq n^2 \|P_{n-1}\|_{[-1,1]}.$$

This inequality combined with the mean value theorem gives us:

$$\frac{|P_{n-1}(x_j) - P_{n-1}(x_n)|}{|x_j - x_n|} \leq n^2 \|P_{n-1}||_{[-1,1]}.$$

Since $P_{n-1}(x_j) = 0$, and since by the definition of the nth Leja point we also have $P_{n-1}(x_n) = \|P_{n-1}\|_{[-1,1]}$, it follows that

$$\frac{\|P_{n-1}\|_{[-1,1]}}{|x_j - x_n|} \leq n^2 \|P_{n-1}\|_{[-1,1]},$$

from where the claim follows.

■

Lemma 4.1.5 *Let $\{x_j\}$ be a sequence of Leja points on $[-1, 1]$, let $i, j \leq n$ and let*

$$\Delta_n(x) = \frac{\sqrt{1 - |x|}}{n} + \frac{1}{n^2}$$

Then $x_i \geq 0$ and $x_j \geq 0$ imply that

$$|x_i - x_j| \geq \frac{1}{4} \left(\Delta_n(x_i) + \Delta_n(x_j)\right)$$

Proof. Let $i, j \leq n$, be such that $x_i \geq 0$ and $x_j \geq 0$, and assume without loss of generality that $i < j$. Then by Lemma 1 and Lemma 2, it follows that we have each of the following:

$$|x_j - x_i| \geq \frac{\sqrt{1 - x_j} + \sqrt{1 - x_i}}{2j},$$

$$|x_j - x_i| \geq \frac{1}{j^2}.$$

Adding the above two inequalities together we obtain

$$2|x_j - x_i| \geq \frac{1}{2}\frac{\sqrt{1 - x_j}}{j} + \frac{1}{2j^2} + \frac{1}{2}\frac{\sqrt{1 - x_i}}{j} + \frac{1}{2j^2},$$

i.e.

$$|x_j - x_i| \geq \frac{1}{4}\left(\frac{\sqrt{1 - x_j}}{j} + \frac{1}{j^2}\right) + \frac{1}{4}\left(\frac{\sqrt{1 - x_i}}{j} + \frac{1}{j^2}\right).$$

Since $j \leq n$, this implies

$$|x_j - x_i| \geq \frac{1}{4}\left(\frac{\sqrt{1-x_j}}{n} + \frac{1}{n^2}\right) + \frac{1}{4}\left(\frac{\sqrt{1-x_i}}{n} + \frac{1}{n^2}\right),$$

as we claimed.

■

4.2 Reduction of the Main Proof

We are now ready to begin our proof. As stated in the introduction to this section, the proof will be divided into two parts. This is part one.

Let $\{x_j\}$ be a sequence of Leja points. We must show that

$$\lim_{n\to\infty} \Lambda_n^{1/n} = 1, \tag{4.2.1}$$

where

$$\Lambda_n = \sup_{x\in[-1,1]} \left(\sum_{j=1}^{n} \frac{\prod_{j=1,j\neq k}^{n} |x - x_j|}{\prod_{j=1,j\neq k}^{n} |x_k - x_j|} \right) \tag{4.2.2}$$

However, we wish to find a simpler expression than (4.2.2). With this in mind, we make the following observation:

$$1 \leq \sup_{x\in[-1,1]} \left(\sum_{j=1}^{n} \frac{\prod_{j=1,j\neq k}^{n} |x - x_j|}{\prod_{j=1,j\neq k}^{n} |x_k - x_j|} \right) \leq n \left(\max_{k=1,\ldots,n} \left(\frac{\|P_{n,k}\|_{[-1,1]}}{\prod(n,k)} \right) \right),$$

where

$$P_{n,k}(x) = \prod_{j=1,j\neq k}^{n} (x - x_j),$$

and where

$$\prod(n,k) = \prod_{j=1,j\neq k}^{n} |x_k - x_j|.$$

Since the left inequality gives $1 \leq \Lambda_n$, in order to show (4.2.1) it suffices to show

$$\left(n \left(\max_{k=1,\ldots,n} \left(\frac{\|P_{n,k}\|_{[-1,1]}}{\prod(n,k)} \right) \right) \right)^{1/n} \to 1.$$

But $n^{1/n} \to 1$, and by Theorem 2.4.11 we already know that as $n \to \infty$,

$$\|P_{n,k}\|_{[-1,1]}^{1/n} \to 1/2$$

uniformly in k. Thus in order to verify (4.2.1), it actually suffices to prove that

$$\left(\prod(n,k)\right)^{1/n} \to 1/2$$

uniformly in k as $n \to \infty$.

We pause here to note that showing that the nth root of $\prod(n,k)$ converges uniformly in k to $1/2$ is not so easy as showing the convergence for finitely many values of k. This is because k runs from 1 to n. However, just as uniformity came easily in the proof of

$$\left(\|P_{n,k}\|_{[-1,1]}\right)^{1/n} \to 1/2,$$

uniformity will also come in the proof of

$$\left(\prod(n,k)\right)^{1/n} \to 1/2.$$

To prove this convergence we begin by defining

$$P_1(n,k,\delta) = \prod_{|x_j - x_k| \geq \delta} |x_j - x_k|,$$

and

$$P_2(n,k,\delta) = \prod_{|x_j - x_k| < \delta} |x_j - x_k|.$$

Now since

$$\prod(n,k) = P_1(n,k,\delta) P_2(n,k,\delta)$$

for all δ, we observe that if

$$(P_1(n,k,\delta))^{1/n}$$

can be shown to approximate $1/2$ uniformly in k as $n \to \infty$ (for sufficiently small δ) and if

$$(P_2(n,k,\delta))^{1/n}$$

can be shown to approximate 1 uniformly in k as $n \to \infty$ (for sufficiently small δ) then we will have also shown that

$$\left(\prod(n,k)\right)^{1/n}$$

approximates $1/2$ uniformly in k as $n \to \infty$. To be more precise, by an elementary argument (we omit the details), one can see that we will have completed our proof if we can show each of the following:

A) For all $\epsilon > 0$ there exists a δ_ϵ and an N such that

$$1/2 - \epsilon < \left(\prod_{|x_j - x_k| \geq \delta_\epsilon} |x_j - x_k| \right)^{1/n} < 1/2 + \epsilon$$

for $k = 1, \ldots, n$ whenever $n > N$.

B) For all $\epsilon > 0$ there exists a δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{|x_j - x_k| < \delta_\epsilon} |x_j - x_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ **whenever** $n > N$.

But statement **A** is the content of Theorem 2.4.12 (note that the capacity of $[-1, 1]$ is $1/2$, see [9]). Therefore to complete the proof we only have to show statement **B**.

We have now completed the first part of our proof. We have reduced our problem to showing statement **B**. We will handle statement **B** in part two of our proof. (Note: there is no upper bound given in statement **B** because we are assuming we have $\delta < 1$ and hence the product in question is ≤ 1. Thus part two of our proof will consist of finding a lower bound for the nth root of $P_2(n, k, \delta)$).

Comments: The main idea of the proof is essentially that when finding the limit of $\Lambda_n^{1/n}$ as n tends to infinity, one only needs to consider the nth roots of quotients of the form

$$\frac{\|P_{n,k}\|_{[-1,1]}}{\prod_{j=1, j\neq k}^{n} |x_k - x_j|},$$

and that potential theory almost gives us the entire result. That is to say, potential theory gives us that the nth roots of both $\|P_{n,k}\|_{[-1,1]}$ and $P_1(n, k, \delta)$ converge uniformly in k.

Now the reason for our repetition here in these comments, as well as the reason for our separation of the proof into two steps, is to stress that potential theory almost gives us the whole result. It is only $P_2(n, k, \delta)$ that potential theory does not help us with, and it is in step two that we will deal with this product. In the absence of theorems from potential theory, in step two we will use our separation lemma regarding the Leja points to attain our desired results regarding $P_2(n, k, \delta)$.

4.3 Estimate for The Main Product

In this step we complete the proof that $\Lambda_n^{1/n} \to 1$ by showing that

B) For all $\epsilon > 0$ there exists a δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{|x_j - x_k| < \delta_\epsilon} |x_j - x_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ **whenever** $n > N$.

Now this is not at all easy to prove, and we will need to be clever in our approach.

We begin by assuming that the δ left neighborhood of x_k is contained in $[0, 1]$, that is to say, we begin by assuming that $0 \leq x_k - \delta < x_k$, and by defining the following four sets:

$$\begin{aligned}
X_1 &= \{x_j : j \le n, x_k \le x_j \le \frac{1+x_k}{2}\},\\
X_2 &= \{x_j : j \le n, \frac{1+x_k}{2} < x_j < x_k + \delta\},\\
X_3 &= \{x_j : j \le n, 1 - 2(1-x_k) \le x_j \le x_k, |x_j - x_k| < \delta\},\\
X_4 &= \{x_j : j \le n, |x_j - x_k| < \delta, x_j \notin X_i, i = 1,2,3\}.
\end{aligned}$$

Note that some of these sets may be empty, but this does not bother us. The point is that we now have:

$$\begin{aligned}
P_2(n,k,\delta) = \prod_{|x_j - x_k| < \delta} |x_j - x_k| &= \prod_{x_j \in X_1} |x_j - x_k| \prod_{x_j \in X_2} |x_j - x_k| \times\\
&\times \prod_{x_j \in X_3} |x_j - x_k| \prod_{x_j \in X_4} |x_j - x_k|,
\end{aligned}$$

where if one of these sets X_i is empty, then the corresponding product is considered to be 1. Now rather than showing **B** directly, we will instead divide the proof into four substeps, and show that the following statement holds

> **For all** $\epsilon > 0$ **there exists an** δ_ϵ **and an** N **such that**
>
> $$1 - \epsilon < \left(\prod_{x_j \in X_i} |x_j - x_k| \right)^{1/n}$$
>
> **for** $k = 1, \ldots, n$ **whenever** $n > N$,

for $i = 1, 2, 3, 4$. We claim that this is enough to complete the proof. Indeed, completing these four substeps will clearly show that the inequality in **B** holds for those k satisfying our assumption, and the only reason for assuming that $0 \le x_k - \delta$ is to guarantee that $x_j \ge 0$ for all the x_j's in our sets X_i's. And the only reason for needing $x_j \ge 0$, is so that we can apply Lemma 4.1.5 when estimating $|x_k - x_j|$. But the exact same inequality used in Lemma 4.1.5 applies to negative x_j's, namely that

$$|x_j - x_k| \ge \frac{\sqrt{1 - |x_j|}}{n} + \frac{1}{n^2} + \frac{\sqrt{1 - |x_k|}}{n} + \frac{1}{n^2}.$$

Thus in proving the case when $0 \le x_k - \delta$, we will have also proven the case when $x_k + \delta \le 0$. Now if the δ neighborhood of x_k contains 0, we simply replace x_k with its closest x_{j^*}, where $x_{j^*} < 0$ and form the product

$$\left(\prod_{|x_{j^*} - x_j| < \delta, x_j < 0} |x_{j^*} - x_j| \right)^{1/n} \left(\prod_{|x_k - x_j| < \delta, x_j \ge 0} |x_k - x_j| \right)^{1/n} |x_k - x_{j^*}|^{1/n} \qquad (4.3.3)$$

(here we are assuming $x_k \geq 0$). Now by the above discussion, each one of the first two terms in this product will tend to 1. The last term will tend to 1 since by Lemma 4.1.4 we always have $|x_k - x_j| > 1/n^2$. And since the product of these three terms forms a lower bound for $P_2(n,k,\delta)$ (we do not need to find an upper bound of course), we will have shown that $P_2(n,k,\delta)$ approximates 1 for sufficiently small δ, and sufficiently large N. Thus we will have completed our proof by completing these four substeps.

Before completing these four substeps we make the final comment that we will assume that $x_k \neq 1$. We may do this since if $x_k = 1$, we can simply replace x_k with its closest x_j, and just as in (4.3.3), the product will tend to 1.

We first show that

For all $\epsilon > 0$ there exists an δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{x_j \in X_1} |x_j - x_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

That is, we will find a lower bound for

$$\left(\prod_{x_j \in X_1} |x_j - x_k| \right)^{1/n},$$

and show that this lower bound tends to 1 as δ tends to 0. To do this we begin by noting that the asymptotic distribution of Leja points is the equilibrium distribution, hence, by Lemma 2.4.5, if $m_1 = m_1(n)$ is the total number of Leja points x_j $(j \leq n)$ contained in X_1, then $m_1 < \epsilon_\delta n$, where $\epsilon_\delta \to 0$ as $\delta \to 0$. We now label these points $x_{j_1} < x_{j_2} < \ldots < x_{j_{m_1}}$. Recall that if $X_1 = \emptyset$, that is if $m_1 = 0$, then the product in question is defined to be 1. Next we note that

$$\begin{aligned} |x_{j_{s+1}} - x_{j_s}| \geq \frac{1}{4}\left(\Delta_n(x_{j_{s+1}}) + \Delta_n(x_{j_s})\right) &\geq \frac{1}{4}\left(\Delta_n\left(\frac{1+x_k}{2}\right) + \Delta_n\left(\frac{1+x_k}{2}\right)\right) \\ &\geq \frac{1}{4}\Delta_n(x_k). \end{aligned}$$

Recall that

$$\Delta_n(x) = \frac{\sqrt{1-|x|}}{n} + \frac{1}{n^2}.$$

The first inequality was proven in Lemma 4.1.5. The second inequality follows from the definition of X_1 and the fact that $\Delta_n(x)$ is decreasing on $[0,1]$. The third inequality follows from an easy calculation which we omit. Same is true if $s = 0$ and $x_{j_0} = x_k$. As a result we get

$$|x_{j_s} - x_k| = |x_{j_s} - x_{j_{s-1}}| + |x_{j_{s-1}} - x_{j_{s-2}}| + \ldots + |x_{j_1} - x_k| \geq \frac{s}{4}\Delta_n(x_k).$$

From this it then follows that

$$\frac{1-x_k}{2} = \frac{1+x_k}{2} - x_k \geq |x_{j_{m_1}} - x_k| \geq \frac{m_1}{4}\Delta_n(x_k) \geq \frac{m_1}{4}\frac{\sqrt{1-x_k}}{n}.$$

Looking at the first and last terms in the above string, we obtain

$$\sqrt{1-x_k} \geq \frac{m_1}{2n}.$$

We then have

$$|x_{j_s} - x_k| \geq \frac{s}{4}\Delta_n(x_k) \geq \frac{s}{4}\frac{\sqrt{1-x_k}}{n} \geq \frac{s}{4}\frac{m_1}{2n}\frac{1}{n} = \frac{sm_1}{8n^2}.$$

From this we obtain

$$\begin{aligned}\prod_{s=1}^{m_1}|x_{j_s} - x_k| \geq \frac{m_1! m_1^{m_1}}{(8n^2)^{m_1}} \geq \frac{m_1^{2m_1}\exp(-m_1)}{(\sqrt{8}n)^{2m_1}} &\geq \frac{m_1^{2m_1}}{(\sqrt{8}en)^{2m_1}} \geq \frac{(\epsilon_\delta n)^{2\epsilon_\delta n}}{(\sqrt{8}en)^{2\epsilon_\delta n}} \\ &= \left(\frac{\epsilon_\delta}{\sqrt{8}e}\right)^{2\epsilon_\delta n}.\end{aligned}$$

Note that in the second inequality above we used that

$$n! \geq \left(\frac{n}{e}\right)^n,$$

which can be easily proven by induction on n (for large n it also easily follows from Stirlings formula, i.e.,

$$\frac{n!}{\sqrt{2\pi}n^n\exp(-n)} \to 1, \quad \text{as } n \to \infty),$$

and in the last inequality, we used that the function

$$f(x) = \left(\frac{x}{\sqrt{8}en}\right)^{2x}$$

is decreasing on $[1, n]$ and that $m_1 \leq \epsilon_\delta n$. This proves that

$$\left(\prod_{s=1}^{m_1}|x_{j_s} - x_k|\right)^{1/n} \geq \left(\frac{\epsilon_\delta}{\sqrt{8}e}\right)^{2\epsilon_\delta},$$

and this is what we wanted to show, because, as $\delta \to 0$, we have $\epsilon_\delta \to 0$, and so

$$\left(\frac{\epsilon_\delta}{\sqrt{8}e}\right)^{2\epsilon_\delta} \to 1.$$

∎

We next show that

For all $\epsilon > 0$ there exists a δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{x_j \in X_2} |x_j - x_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

That is, we will find a lower bound for

$$\left(\prod_{x_j \in X_2} |x_j - x_k| \right)^{1/n},$$

and show that this lower bound tends to 1 as δ tends to 0. To do this we begin by noting that by Lemma 2.4.5, if $m_2 = m_2(n)$ is the total number of Leja points x_j ($j \leq n$) contained in X_2, then $m_2 < \epsilon_\delta n$, where $\epsilon_\delta \to 0$ as $\delta \to 0$. We label these points $x_{j_0}, x_{j_1}, \ldots x_{j_{m_2}}$, according to the rule:

$$\frac{1 + x_k}{2} < x_{j_{m_2}} < x_{j_{m_2 - 1}} < \ldots < x_{j_1} < x_{j_0} \leq \min\{1, x_k + \delta\}.$$

We now use induction to show that

$$1 - x_{j_s} \geq s^2/16n^2. \tag{4.3.4}$$

This is obviously true for $s = 0$. We then have

$$\begin{aligned} 1 - x_{j_{s+1}} = 1 - x_{j_s} + |x_{j_{s+1}} - x_{j_s}| \ &\geq \ \frac{s^2}{16n^2} + \frac{1}{4}\left(\Delta_n(x_{j_s}) + \Delta_n(x_{j_{s+1}})\right) \\ &\geq \ \frac{s^2}{16n^2} + \frac{1}{2}\Delta_n(x_{j_s}) \end{aligned}$$

$$= \frac{s^2}{16n^2} + \frac{1}{2}\left(\frac{\sqrt{1 - x_{j_s}}}{n} + \frac{1}{n^2} \right) \ \geq \ \frac{s^2}{16n^2} + \frac{1}{2}\left(\frac{s}{4n^2} + \frac{1}{n^2} \right) = \frac{s^2}{16n^2} + \frac{s}{8n^2}$$

$$+ \ \frac{1}{2n^2} \geq \frac{(s+1)^2}{16n^2}.$$

Note that in the second inequality above, we used Lemma 4.1.5, and in the first and third inequalities we used our induction hypothesis. In particular, from (4.3.4) we have

$$\frac{m_2^2}{16n^2} \leq 1 - x_{j_{m_2}} \leq 1 - \frac{1 + x_k}{2} = \frac{1 - x_k}{2},$$

and this is what we need to show our claim. Indeed, we now have

$$\prod_{x_{j_s} \in X_2} |x_{j_s} - x_k| = \prod_{s=0}^{m_2} |x_{j_s} - x_k| \geq \left(\frac{1 - x_k}{2} \right)^{m_2 + 1} \geq \left(\frac{m_2^2}{16n^2} \right)^{m_2 + 1}$$

$$= \left(\frac{m_2}{4n}\right)^{2(m_2+1)} \geq \left(\frac{\epsilon_\delta n}{4n}\right)^{2(\epsilon_\delta n+1)} = \left(\frac{\epsilon_\delta}{4}\right)^{2(\epsilon_\delta n+1)} \geq \left(\frac{\epsilon_\delta}{4}\right)^{2(\epsilon_\delta n+\epsilon_\delta n)} = \left(\frac{\epsilon_\delta}{4}\right)^{4\epsilon_\delta n}$$

Note that in the third inequality we used the fact that

$$f(x) = \left(\frac{x}{4n}\right)^{2(x+1)}$$

is monotone decreasing on $[1, n]$ and that $1 \leq m_2 < \epsilon_\delta n$. (Recall that if $m_2 = 0$, then the product in question is defined to be 1.) What we have shown is that

$$\prod_{x_{j_s} \in X_2} |x_{j_s} - x_k| \geq \left(\frac{\epsilon_\delta}{4}\right)^{4\epsilon_\delta n}.$$

By taking nth roots, we obtain

$$\left(\prod_{x_{j_s} \in X_2} |x_{j_s} - x_k|\right)^{1/n} \geq \left(\frac{\epsilon_\delta}{4}\right)^{4\epsilon_\delta},$$

and this is what we wanted to show, since

$$\left(\frac{\epsilon_\delta}{4}\right)^{4\epsilon_\delta} \to 1,$$

as $\epsilon_\delta \to 0$, which is the same as $\delta \to 0$.

■

We next show that

For all $\epsilon > 0$ there exists an δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{x_j \in X_3} |x_j - x_k|\right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

That is, we will find a lower bound for

$$\left(\prod_{x_j \in X_3} |x_j - x_k|\right)^{1/n},$$

and show that this lower bound tends to 1 as δ tends to 0. To do this we begin by noting that by Lemma 2.4.5, if $m_3 = m_3(n)$ is the total number of Leja points x_j ($j \leq n$) contained in X_3, then $m_3 < \epsilon_\delta n$, where $\epsilon_\delta \to 0$ as $\delta \to 0$. We label these points according to the rule

$$1 - 2(1 - x_k) < x_{j_{m_3}} \ldots < x_{j_1} < x_k.$$

We then have

$$|x_{j_{s+1}} - x_{j_s}| \geq \frac{1}{4}\left(\Delta_n(x_{j_{s+1}}) + \Delta_n(x_{j_s})\right) \geq \frac{1}{4}2\Delta_n(x_{j_s}) \geq \frac{1}{2}\Delta_n(x_k).$$

We note that in the first inequality above we used Lemma 4.1.5. The second and third inequalities follow from the fact that $\Delta_n(x)$ is decreasing on $[0,1]$. So we have

$$|x_{j_{s+1}} - x_{j_s}| \geq \frac{1}{2}\Delta_n(x_k),$$

and thus

$$|x_{j_s} - x_k| = |x_{j_s} - x_{j_{s-1}}| + |x_{j_{s-1}} - x_{j_{s-2}}| + \ldots + |x_{j_1} - x_k| \geq \frac{s}{2}\Delta_n(x_k).$$

We now recall that we are assuming $x_k < 1$. Since

$$1 - 2(1 - x_k) < x_{j_{m_3}} < x_k < 1,$$

we have the following

$$2(1 - x_k) = 1 - (1 - 2(1 - x_k)) \geq |x_{j_{m_3}} - x_k| \geq \frac{m_3\Delta_n(x_k)}{2}$$

from where it follows that

$$1 - x_k \geq \frac{m_3\sqrt{1 - x_k}}{4n},$$

and thus that

$$\frac{m_3}{4n} \leq \sqrt{1 - x_k}.$$

Now taking our inequality

$$|x_{j_s} - x_k| \geq \frac{s\Delta_n(x_k)}{2},$$

and replacing

$$\frac{\sqrt{1 - x_k}}{n} + \frac{1}{n^2} \quad \text{with} \quad \frac{m_3}{4n^2},$$

we get

$$|x_{j_s} - x_k| \geq \frac{sm_3}{8n^2},$$

and this is just what we need. Indeed, we have

$$\begin{aligned}\prod_{x_j \in X_3} |x_j - x_k| = \prod_{s=1}^{m_3} |x_{j_s} - x_k| \quad &\geq \quad \prod_{s=1}^{m_3}\left(\frac{sm_3}{8n^2}\right) = \frac{m_3! m_3^{m_3}}{(\sqrt{8}n)^{2m_3}} \geq \left(\frac{m_3}{\sqrt{8}en}\right)^{2m_3} \\ &\geq \quad \left(\frac{\epsilon_\delta n}{\sqrt{8}en}\right)^{2\epsilon_\delta n} = \left(\frac{\epsilon_\delta}{\sqrt{8}e}\right)^{2\epsilon_\delta}.\end{aligned}$$

Note that in the above calculations, we used Stirlings formula in the second inequality, and we used that

$$f(x) = \left(\frac{x}{2en}\right)^{2x}$$

is monotone decreasing on $[1, n]$ in the third inequality. If we take the nth roots of the first and last terms of the above, we obtain that

$$\left(\prod_{x_j \in X_3} |x_j - x_k|\right)^{1/n} \geq \left(\frac{\epsilon_\delta}{2e}\right)^{2\epsilon_\delta},$$

and since

$$\left(\frac{\epsilon_\delta}{2e}\right)^{2\epsilon_\delta} \to 1$$

as $\epsilon_\delta \to 0$, which is the same as $\delta \to 0$, this is what we wanted to show.

■

Finally, we show that

For all $\epsilon > 0$ there exists an δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{x_j \in X_4} |x_j - x_k|\right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

That is, we will find a lower bound for

$$\left(\prod_{x_j \in X_4} |x_j - x_k|\right)^{1/n},$$

and show that this lower bound tends to 1 as δ tends to 0.

This will be the most difficult of our four substeps. Recalling that

$$X_4 = \{x_j : x_j \in [x_k - \delta, 1 - 2(1 - x_k)]\},$$

we begin by assuming that $\delta = 2^{R+1}(1 - x_k)$, and by dividing $[x_k - \delta, 1 - 2(1 - x_k)]$ into the R intervals

$$[1 - 2^{r+1}(1 - x_k), 1 - 2^r(1 - x_k)], \quad r = 1, 2, \ldots, R.$$

Again citing Lemma 2.4.5, if we denote the number of Leja points in the rth interval by k_r, we have $\sum k_r \leq \epsilon_\delta n$. We will denote the rth interval by I_r, and for those intervals with $k_r > 0$, we will label the Leja points in increasing order: $x_{j_1} < x_{j_2} < \ldots < x_{j_{k_r}}$. Our strategy will be to first show that

$$\prod_{x_j \in I_r} |x_j - x_k| \geq \left(\frac{k_r}{4n}\right)^{2k_r},$$

(for those intervals I_r with $k_r > 0$) and that the nth root of this product tends to 1 as $\delta \to 0$. But what we really want to show is that (here we consider $\prod_{x_j \in I_r} |x_j - x_k|$ to be 1 for those k_r equal to 0)

$$\left(\prod_{r=1}^{R} \prod_{x_j \in I_r} |x_j - x_k| \right)^{1/n} \to 1$$

uniformly in k as $n \to \infty$ and as $\delta \to 0$. And since x_k may be close to 1, R may be very large, it is not enough to simply show that

$$\left(\prod_{x_j \in I_r} |x_j - x_k| \right)^{1/n} \to 1$$

for finitely many values of r. Thus after obtaining the result that

$$\prod_{x_j \in I_r} |x_j - x_k| \geq \left(\frac{k_r}{4n} \right)^{2k_r},$$

we will work to obtain a result comparing the size of all k_r to a fixed k_{r^*}. We will then estimate our product

$$\prod_{r=1}^{R} \prod_{x_j \in I_r} |x_j - x_k|$$

in terms of k_{r^*}. We will complete the proof by showing that the nth root of this lower bound tends to 1.

We begin to show that

$$\prod_{x_j \in I_r} |x_j - x_k| \geq \left(\frac{k_r}{4n} \right)^{2k_r}.$$

If $x_{j_1}, x_{j_2}, \ldots, x_{j_{k_r}}$ are the Leja points in

$$I_r = [1 - 2^{r+1}(1 - x_k), 1 - 2^r(1 - x_k)],$$

we then have the following

$$\begin{aligned} |x_{j_{s+1}} - x_{j_s}| &\geq \frac{1}{4} \left(\Delta_n(x_{j_{s+1}}) + \Delta_n(x_{j_s}) \right) \geq \frac{1}{2} \Delta_n(x_{j_s}) \\ &\geq \frac{1}{2} \frac{\sqrt{1 - (1 - 2^r(1 - x_k))}}{n} = \frac{1}{2} \frac{2^{r/2}\sqrt{1 - x_k}}{n}. \end{aligned}$$

In the first inequality in the above equations we used Lemma 4.1.5. In the second and third inequalities we used the fact that $\Delta_n(x)$ is decreasing on $[0, 1]$. Now since

$$|x_{j_{s+1}} - x_{j_s}| \geq \frac{2^{r/2}\sqrt{1 - x_k}}{2n},$$

it follows that

$$2^r(1-x_k) = 1 - 2^{r+1}(1-x_k) - (1 - 2^r(1-x_k)) \geq$$

$$|x_{j_{k_r}} - x_{j_{k_r-1}}| + |x_{j_{k_r-1}} - x_{j_{k_r-2}}| + \ldots + |x_{j_2} - x_{j1}| \geq \frac{k_r}{4}\frac{2^{r/2}\sqrt{1-x_k}}{n}.$$

It is easy to see that the same inequality is true also for $k_r = 1$. Dividing the first and last terms in the above string of inequalities by

$$2^r\sqrt{1-x_k}$$

(again recall that we are assuming $x_k \neq 1$), we obtain

$$\frac{k_r 2^{-r/2}}{4n} \leq \sqrt{1-x_k}. \tag{4.3.5}$$

By squaring both sides we obtain

$$\frac{k_r^2 2^{-r}}{16n^2} \leq 1 - x_k.$$

If we now take this and multiply both sides by 2^r we obtain

$$2^r(1-x_k) \geq \frac{k_r^2}{16n^2}.$$

We are now ready to consider the product of our $|x_{j_s} - x_k|$'s for those intervals I_r with $k_r \neq 0$. We have

$$\prod_{s=1}^{k_r} |x_{j_s} - x_k| \geq (2^r(1-x_k))^{k_r} \geq \left(\frac{k_r^2}{16n^2}\right)^{k_r} = \left(\frac{k_r}{4n}\right)^{2k_r}. \tag{4.3.6}$$

Since $k_r \leq \epsilon_\delta n$, we have

$$\left(\prod_{s=1}^{k_r} |x_{j_s} - x_k|\right)^{1/n} \geq \left(\frac{\epsilon_\delta}{4}\right)^{2\epsilon_\delta} \to 1, \quad \text{as } \delta \to 0.$$

At this point we would have completed our proof had we been dealing with a finite fixed R. But to repeat what was stated above, what we really want to show is that

$$\left(\prod_{r=1}^{r=R} \prod_{x_j \in I_r} |x_j - x_k|\right)^{1/n} \to 1$$

uniformly in k as $n \to \infty$, $\delta \to 0$ (and where $\prod_{x_j \in I_r} |x_j - x_k|$ is considered to be 1 when $k_r = 0$).

With the help of (4.3.5) we will now find a k_{r^*}, C such that $k_r \leq C k_{r^*}$, and with the help of (4.3.6) we will then complete our proof. To find k_{r^*}, C, we begin by recalling that if γ_n is the probability measure concentrated on the first n Leja points, with the value of $1/n$ at each point, then $\gamma_n \xrightarrow{*} \mu$ (see preliminaries). From this (by Lemma 2.4.9) it follows that there exists N such that $N < n$ implies

$$\sum_{r=1}^{R} \frac{k_r}{n} > \frac{1}{4\pi} \int_{1-2^R(1-x_k)}^{1-2(1-x_k)} \frac{1}{\sqrt{1-t^2}} dt. \tag{4.3.7}$$

Now since

$$\frac{1}{\sqrt{1-t^2}} \geq \frac{1}{2} \frac{1}{\sqrt{1-t}},$$

we can rewrite (4.3.7) as

$$\sum_{r=1}^{R} \frac{k_r}{n} > \frac{1}{4\pi} \int_{1-2^R(1-x_k)}^{1-2(1-x_k)} \frac{1}{\sqrt{1-t}} dt.$$

By integrating we obtain

$$\sum_{r=1}^{R} \frac{k_r}{n} > \frac{1}{2\pi}\sqrt{2^R(1-x_k)} - \frac{1}{2\pi}\sqrt{2(1-x_k)} = \frac{1}{2\pi}\sqrt{2^R}\sqrt{1-x_k}\left(1 - \frac{\sqrt{2}}{\sqrt{2^R}}\right).$$

After this rough estimate we rewrite this inequality as

$$\sum_{r=1}^{R} \frac{k_r}{n} > \frac{1}{2\pi}\sqrt{2^R}\sqrt{1-x_k}\frac{2}{7} = \frac{1}{7\pi}\sqrt{2^R}\sqrt{1-x_k}. \tag{4.3.8}$$

We are now ready to find our k_{r^*}. With the help of (4.3.5) and (4.3.8) we will find a large k_r, and we will label this large k_r as k_{r^*}. To do this we note that for large M and for $R > M$, we have the following

$$4\left(\frac{\sqrt{2} - \sqrt{2}^{R-M}}{1-\sqrt{2}}\right) \leq \left(\sqrt{2}\right)^R \frac{1}{3}\frac{1}{7\pi} \tag{4.3.9}$$

We fix such an M (actually, $M = 20$ would suffice). We now claim that we must have

$$\max\left\{\frac{k_R}{n}, \frac{k_{R-1}}{n}, \ldots, \frac{k_{R-M}}{n}\right\} \geq \frac{1}{M}\sqrt{2}^R \frac{1}{3}\frac{1}{7\pi}\sqrt{1-x_k}.$$

Indeed, if this is not the case then by (4.3.5) and (4.3.9), we have the following

$$\sum_{r=1}^{R} \frac{k_r}{n} = \sum_{r=1}^{R-M-1} \frac{k_r}{n} + \sum_{r=R-M}^{R} \frac{k_r}{n} \leq$$

$$\sum_{r=1}^{R-M-1} 4(\sqrt{2})^r\sqrt{1-x_k} + \sum_{r=R-M}^{R} \frac{k_r}{n} =$$

$$4\left(\frac{\sqrt{2}-\sqrt{2}^{R-M-1}}{1-\sqrt{2}}\right)\sqrt{1-x_k}+\sum_{r=R-M}^{R}\frac{k_r}{n}\leq$$

$$\sqrt{2}^{R}\frac{1}{3}\frac{1}{7\pi}\sqrt{1-x_k}+\sum_{r=R-M}^{R}\frac{k_r}{n}<$$

$$\sqrt{2}^{R}\frac{1}{3}\frac{1}{7\pi}\sqrt{1-x_k}+M\frac{1}{M}\sqrt{2}^{R}\frac{1}{3}\frac{1}{7\pi}\sqrt{1-x_k}=$$

$$\frac{2}{3}\frac{1}{7\pi}\sqrt{2^R}\sqrt{1-x_k}$$

Looking at the first and last terms in the above calculations, we find a contradiction to (4.3.8). It thus follows that we must have

$$\max\{\frac{k_R}{n},\frac{k_{R-1}}{n},\ldots,\frac{k_{R-M}}{n}\}\geq\frac{1}{M}\sqrt{2}^{R}\frac{1}{3}\frac{1}{7\pi}\sqrt{1-x_k}.$$

We label this max as $\frac{k_{r^*}}{n}$. Now since

$$\frac{k_{r^*}}{n}\geq\frac{1}{M}\sqrt{2}^{R}\frac{1}{3}\frac{1}{7\pi}\sqrt{1-x_k}, \tag{4.3.10}$$

comparison of (4.3.5) and (4.3.10) now gives

$$84\pi M k_{r^*}^{\frac{r-R}{2}}\geq k_r.$$

We let $\theta=84\pi M$ and we rewrite this inequality as

$$k_r\leq\theta 2^{\frac{r-R}{2}}k_{r^*} \tag{4.3.11}$$

and this is the inequality we shall use to finish our proof.

Recall that in this substep we meant to show that

$$\left(\prod_{x_j\in X_4}|x_j-x_k|\right)^{1/n}$$

approximates 1 for sufficiently small δ and $n>N$, by showing that

$$\left(\prod_{r=1}^{R}\prod_{x_j\in I_r}|x_j-x_k|\right)^{1/n}$$

approximates 1 for sufficiently small δ and $n > N$. Now to show the latter, as in the other substeps, we will find a lower bound for

$$\left(\prod_{r=1}^{R}\prod_{x_j\in I_r}|x_j-x_k|\right)^{1/n},$$

and show that this lower bound tends to 1 for large N. We proceed as follows

$$\begin{aligned}\prod_{r=1}^{R}\prod_{x_j\in I_r}|x_j-x_k| \geq \prod_{r=1}^{R}\left(\frac{k_r}{4n}\right)^{2k_r} &\geq \prod_{r=1}^{R}\left(\frac{k_r}{4en\theta}\right)^{2k_r}\\ &\geq \prod_{r=1}^{R}\left(\frac{k_{r^*}\theta 2^{\frac{r-R}{2}}}{4en\theta}\right)^{2\left(2^{\frac{r-R}{2}}k_{r^*}\theta\right)}\end{aligned}$$

We note that in the first inequality we used (4.3.6), and in the second inequality we used that

$$f(x)=\left(\frac{k}{x}\right)^{k}$$

is decreasing on $(0,\infty)$. In the third inequality we used (4.3.11), and that

$$f(x)=\left(\frac{x}{4en\theta}\right)^{x}$$

is decreasing on $[1,2n\theta]$. (Note that $k_{r^*}\theta 2^{\frac{r-R}{2}} < 2n\theta$ since $k_{r^*} < n$). We now continue with

$$\prod_{r=1}^{R}\left(\frac{k_{r^*}\theta 2^{\frac{r-R}{2}}}{4en\theta}\right)^{2\left(2^{\frac{r-R}{2}}k_{r^*}\theta\right)} = \prod_{r=1}^{R}\left(\frac{k_{r^*}}{4en}\right)^{2\left(2^{\frac{r-R}{2}}k_{r^*}\theta\right)}\prod_{r=1}^{R}\left(2^{\frac{r-R}{2}}\right)^{2\left(2^{\frac{r-R}{2}}k_{r^*}\theta\right)} \tag{4.3.12}$$

Now since $\sum_{k=1}^{\infty}-k2^{-k} > -\infty$, the second product in the right hand side of (4.3.12) is greater than the k_{r^*}-th power of some positive constant Q. We can thus continue with

$$\begin{aligned}&\prod_{r=1}^{R}\left(\frac{k_{r^*}\theta 2^{\frac{r-R}{2}}}{4en\theta}\right)^{2\left(2^{\frac{r-R}{2}}k_{r^*}\theta\right)} \geq \prod_{r=1}^{R}\left(\frac{k_{r^*}}{4en}\right)^{2\left(2^{\frac{r-R}{2}}k_{r^*}\theta\right)}Q^{k_{r^*}} =\\ &\prod_{r=1}^{R}\left(\left(\frac{k_{r^*}}{4en}\right)^{2k_{r^*}\theta}\right)^{(\sqrt{2})^{r-R}}Q^{k_{r^*}} = \left(\left(\frac{k_{r^*}}{4en}\right)^{2k_{r^*}\theta}\right)^{\frac{\sqrt{2}-(\sqrt{2})^{1-R}}{\sqrt{2}-1}}Q^{k_{r^*}} \geq\\ &\left(\left(\frac{\epsilon_\delta n}{4en}\right)^{2\epsilon_\delta n\theta}\right)^{\frac{\sqrt{2}-(\sqrt{2})^{1-R}}{\sqrt{2}-1}}Q^{\epsilon_\delta n}.\end{aligned}$$

Taking nth roots of the first and last terms in our string of inequalities, we obtain

$$\left(\prod_{r=1}^{R} \prod_{x_j \in I_r} |x_j - x_k| \right)^{1/n} \geq \left(\left(\frac{\epsilon_\delta}{4e} \right)^{2\epsilon_\delta \theta} \right)^{\frac{\sqrt{2}}{\sqrt{2}-1}} Q^{\epsilon_\delta}.$$

Since this last term tends to 1 as $\epsilon_\delta \to 0$, which is the same as $\delta \to 0$, this completes the fourth substep and thus the entire proof.

■

Actually, in the preceding proof, we did more than prove that the Lebesgue constants associated with Leja sequences on $[-1, 1]$ are subexponential. Since in our proof we only needed that the measures associated with Leja sequences converge weak star to the equilibrium distribution for $[-1, 1]$, and that any two points x_j, x_n in our sequence satisfy the distancing rule

$$|x_j - x_n| \geq \frac{1}{4} \left(\frac{\sqrt{1 - |x_j|}}{n} + \frac{1}{n^2} \right) + \frac{1}{4} \left(\frac{\sqrt{1 - |x_j|}}{n} + \frac{1}{n^2} \right), \quad j < n,$$

we actually proved the following Theorem.

Theorem 4.3.1 *Let* $\{x_j\}$ *be a sequence on* $[-1, 1]$*, and let*

$$\gamma_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j}.$$

Then if

$$\gamma_n \xrightarrow{*} \mu_{[-1,1]},$$

and if each pair of points in our sequence also satisfies the distancing rule

$$|x_j - x_n| \geq \frac{1}{4} \left(\frac{\sqrt{1 - |x_j|}}{n} + \frac{1}{n^2} \right) + \frac{1}{4} \left(\frac{\sqrt{1 - |x_j|}}{n} + \frac{1}{n^2} \right), \quad j < n,$$

then the Lebesgue constants associated with this sequence are subexponential.

5 Leja Points on More General Sets

We now consider Leja points on more general sets. We will first extend our result concerning the Lebesgue constants associated with Leja points on $[-1, 1]$ (Theorem 4.0.1) to a curve in the complex plane. Motivated by this extension, we will then proceed to give a result concerning the union of finitely many *nice* compact sets.

5.1 Leja Points on an Arc

Recall that a Jordan arc is the homeomorphic image of $[-1, 1]$.

In extending Theorem 4.0.1 to arcs in the complex plane, we shall use as a guide the proof of Theorem 4.0.1. The reader should recall that potential theory almost gave us the whole proof of Theorem 4.0.1. That is, we reduced our problem greatly with the help of Theorem 2.4.11 and Theorem 2.4.12. But Theorems 2.4.11 and 2.4.12 apply to curves in the plane as well as the interval $[-1, 1]$, and thus we will again be able to reduce our problem greatly with the aid of Potential theory.

The reader should also recall that in the section of Theorem 4.0.1's proof entitled **estimate for the main product**, the main idea used was a distancing rule for Leja points on $[-1, 1]$. We will also need a distancing rule for our proof regarding Leja points on a curve. Now, in the case of $[-1, 1]$ we obtained our distancing rule through applications of Bernstein's inequality and Markov's inequality. Similarly, in our proof regarding Leja points on a curve, we shall apply a complex version of Bernstein's inequality as well as Markov's inequality. However, in obtaining our version of Bernstein's inequality, we shall need the Bernstein-Walsh lemma. Our application of the Bernstein-Walsh Lemma will require us to restrict ourselves to curves which are C^1 and which have nonzero derivatives. The reason for needing the assumption that our curve is C^1, is so that we will have at our disposal the following result of Dzjadyk [4]. It speaks about the level sets of the Green function associated with a smooth Jordan arc. In general, the Green's function associate with a compact set K of positive capacity is defined as

$$g(z) = \int \log|z - t| d\mu_K(t) - \log \mathrm{Cap}(K),$$

where μ_K is the equilibrium measure of K.

Theorem 5.1.1 *Let Γ be a Jordan arc in the plane with continuous curvature, and with endpoints at $z_0 \neq 1$ and at 1, and let g be the Green function associated with Γ.*

Let z be a point on Γ which is closer to 1 than to z_0. Then the distance, $\sigma(z)$, from z to the level curve $g = 1/n$ satisfies the following inequality

$$\sigma(z) \geq \frac{C}{n}\sqrt{|1-z| + 1/n^2}$$

where C is a constant depending only on the curve.

Comment: In Dzjadyk's paper [4], his result was more general. His result concerned piecewise smooth curves and their junction points, rather than merely curves with endpoint at 1. We have however, restricted ourselves to the very special case of his result which we shall need.

As stated above, we need this in our application of the Bernstein-Walsh lemma, which we now state (for a proof see [12]).

Theorem 5.1.2 (Bernstein-Walsh Lemma) *Let K be a compact subset of the plane and g its associate Green function. Then*

$$|P_n(z)| \leq \|P_n\|_K \exp(ng(z)), \quad z \in C.$$

With the Bernstein-Walsh lemma we are now ready to give our complex version of the Bernstein inequality.

Lemma 5.1.3 *Let P_n be a polynomial of degree n and let Γ be an open curve in the complex plane with endpoints at $z_0 \neq 1$ and at 1. Then for all points $z \in \Gamma$ which lie closer to 1 than to z_0, the following inequality holds*

$$|P_n'(z)| \leq \frac{nC\|P_n\|_\Gamma}{\sqrt{|1-z|}}.$$

Proof. As in Theorem 5.1.1, let $\sigma(z)$ be the distance from z to the level curve $g = 1/n$, where g is the Green function associated with Γ. Then since z lies on Γ, all points t within $\sigma(z)$ of z are also within the level curve $g = 1/n$. Thus for all such t we have $g(t) \leq 1/n$. By Cauchy's Theorem, the Bernstein-Walsh Lemma, and Theorem 5.1.1 we then have

$$\begin{aligned}
|P_n'(z)| &= \left|\frac{1}{2\pi i}\int_{|t-z|=\sigma(z)} \frac{P_n(t)}{(t-z)^2}dt\right| \leq \frac{1}{\sigma(z)} \sup_{|t-z|=\sigma(z)} |P_n(t)| \\
&\leq \frac{1}{\sigma(z)}\|P_n\|_\Gamma \exp(n \sup_{|t-z|=\sigma(z)} g(t)) \leq \frac{1}{\sigma(z)}\|P_n\|_\Gamma \exp(n(1/n)) \\
&\leq \frac{nC\|P_n\|_\Gamma \exp(1)}{\sqrt{|1-z| + 1/n^2}} \leq \frac{nC\|P_n\|_\Gamma}{\sqrt{|1-z|}},
\end{aligned}$$

as we desired. (we note that in the last inequality C absorbed $\exp(1)$)

■

While Bernstein's inequality for curves requires us to introduce Dzjadyk's result on the level lines of Green functions and is slightly more involved than in the $[-1,1]$ case, Markov's inequality for an arc (or actually for any connected compact set) in the complex plane is almost identical to the $[-1,1]$ case.

Theorem 5.1.4 (Markov's inequality) *Let P_n be a polynomial of degree n, and let Γ be a Jordan arc in the complex plane. Then there exists a constant C such that the following inequality holds*

$$|P_n'(z)| \leq Cn^2 \|P_n\|_\Gamma.$$

For a proof see [8].

Comment Actually, this is Markov's inequality for compact subsets of the real line. We earlier however, considered the case where $K = [-1,1]$, for which the constant in Markov's inequality is 1.

We are now closer to being able to find our distancing rule for Leja points on a curve. There is still however, one last property of curves which we wish to discuss. The proof of our distancing rule will go more smoothly if we have a certain *monotonicity* rule to our curve. This *monotonicity* rule follows easily from our assumptions that Γ is C^1 with nonzero derivative.

Lemma 5.1.5 *Let $\Gamma = x(t) + iy(t)$ $(t \in [0,1])$ be C^1 arc with non-zero derivative. Then there exists δ such that every disc D of radius δ centered at some $z = x(t_z)+iy(t_z)$ has the property that any two points $z_1, z_2 \in (\Gamma \cap D)$ can be connected by a curve $\Gamma_{[t_{z_1}, t_{z_2}]} \subset \Gamma$ such that either $|x'(t)| > 1/2|y'(t)|$ for all $t \in [t_{z_1}, t_{z_2}]$, or $|y'(t)| > 1/2|x'(t)|$ for all $t \in [t_{z_1}, t_{z_2}]$.*

Proof. Let $z = x(t_z)+iy(t_z) \in \Gamma$. Then either $|x'(t_z)| \geq |y'(t_z)|$ or $|y'(t_z)| \geq |x'(t_z)|$. Assume w.l.o.g. that $|x'(t_z)| \geq |y'(t_z)|$. Since Γ is C^1, there exists a disc D_z centered at z such that for all $w = x(t_w) + iy(t_w) \in (D_z \cap \Gamma)$, $|x'(t_w)| > 1/2|y'(t_w)|$. We may assume that D_z is also small enough so that any two points $w_1, w_2 \in (D_z \cap \Gamma)$ can be connected by a path $\Gamma_{[t_{w_1}, t_{w_2}]} \subset D$. Since Γ is compact there exists $D_{z_1}, \ldots, D_{z_n}$ such that the union of these discs cover Γ. For this finite subcover there exists δ such that all discs of radius δ with centers on Γ lie in one of the D_{z_i}. This is our required δ.

■

We are now ready to find our distancing rule for Leja points on a smooth arc in the complex plane. Three simple lemmas are to following. The third one is our distancing rule for Leja points on a curve.

Lemma 5.1.6 *Let $\Gamma = x(t) + iy(t)$ be a C^1 arc with non-zero derivative and with endpoints at $z_0 \neq 1$ and at 1. Let δ be as in Lemma 5.1.5, and let $\{z_j\}$ be a sequence of Leja points on Γ. Then if D is a disc of radius δ whose points lie closer to 1 than to z_0, and if $|x'(t)| > 1/2|y'(t)|$ for all $z = x(t) + iy(t)$ in D , we have*

$$\frac{\sqrt{|1-x_j|} + \sqrt{|1-x_n|}}{3Cn} \leq |x_j - x_n|,$$

whenever $j < n$ and $z_j = x_j + iy_j, z_n = x_n + iy_n \in D$.

Comment: This Lemma basically says that $|x_j - x_n|$ will be *large* when the points z_j, z_n lie close to 1 and when $|x'|$ is *large*. We want the reader to note that the conditions in this lemma present no restrictions. First, this is because one of either $|x'|$ or $|y'|$ must be greater than or equal to the other one. Second, this is because the selection of Leja points from a compact set in the complex plane is invariant with respect to translations in the plane, so that the positions of the endpoints of our arc are irrelevant. Also, the δ in Lemma 5.1.5 was chosen without regard to the endpoints of Γ.

Proof. In what follows we will use the notation $z_i = x(t_i) + iy(t_i)$, and $P_n = \prod_{j=1}^{n} |z - z_j|$. Let $M_n = \|P_n\|$, then by the definition of the nth Leja point, we have

$$M_n = \left| \int_{z_j}^{z_n} P_n'(t)dt \right| = \left| \int_{t_j}^{t_n} P_n'(\Gamma(t))\Gamma'(t)dt \right| =$$

$$\left| \int_{t_j}^{t_n} P_n'(\Gamma(t)) \left(x'(t) + iy'(t)\right) dt \right| \leq \int_{t_j}^{t_n} |P_n'(\Gamma(t))| \, |x'(t)| \, dt$$

$$+ \int_{t_j}^{t_n} |P_n'(\Gamma(t))| \, |y'(t)| \, dt \leq$$

(here we are using Lemma 5.1.5 and the assumption that $x' > 1/2y'$)

$$3 \int_{t_j}^{t_n} |P_n'(\Gamma(t))| \, |x'(t)| \, dt \leq$$

(here we are using Lemma 5.1.3)

$$3 \int_{t_j}^{t_n} \left| \frac{CnM_n x'(t)}{\sqrt{\sqrt{(1 - x(t))^2 + y^2(t)}}} \right| dt \leq 3 \int_{t_j}^{t_n} \left| \frac{CnM_n x'(t)}{\sqrt{\sqrt{(1 - x(t)^2}}} \right| dt$$

$$= 3 \int_{t_j}^{t_n} \left| \frac{CnM_n x'(t)}{\sqrt{1 - x(t)}} \right| dt$$

$$= 3CnM_n \left| \sqrt{|1 - x_j|} - \sqrt{|1 - x_n|} \right|,$$

where in the last step we used that either $x'(t) > 0$ or $x'(t) < 0$ on the interval $[t_j, t_n]$. Now from

$$M_n \leq 3CnM_n \left| \sqrt{|1 - x_j|} - \sqrt{|1 - x_n|} \right| \leq 3CnM_n \frac{|x_j - x_n|}{\sqrt{|1 - x_j|} + \sqrt{|1 - x_n|}},$$

we obtain

$$\frac{\sqrt{|1-x_j|}+\sqrt{|1-x_n|}}{3Cn} \leq |x_j - x_n|,$$

as we desired.

■

Lemma 5.1.7 *Let $\Gamma = x(t) + iy(t)$ be a C^1 arc with non-zero derivative, and let $\{z_j\}$ be a sequence of Leja points on Γ, where $z_j = x_j + iy_j$. Further, let δ be as in Lemma 5.1.5, and let D be a disc of radius δ for which $|x'(t)| > 1/2|y'(t)|$, for all t such that $x(t) + iy(t) \in D$. Then there exists C_0 such that for $j < n$, we have for $x_j, x_n \in D$*

$$\frac{C_0}{n^2} \leq |x_j - x_n|.$$

Proof.

As usual, let $P_n = \prod_{j=1}^{n} |z - z_j|$, and let $M_n = \|P_n\|$. Then by Theorem 5.1.4, we have

$$|P_n(z_n) - P_n(z_j)| = \left| \int_{t_{z_j}}^{t_{z_n}} P'(\Gamma(t))\Gamma'(t)dt \right| \leq Cn^2 M_n \int_{t_{z_j}}^{t_{z_n}} |\Gamma'(t)|\, dt =$$

$$Cn^2 M_n \int_{t_{z_j}}^{t_{z_n}} \sqrt{|x'(t)|^2 + |y'(t)|^2} dt \leq Cn^2 M_n \int_{t_{z_j}}^{t_{z_n}} 3|x'(t)| dt =$$

$$3Cn^2 M_n |x_j - x_n|,$$

where in the last equality we used that $x'(t)$ is either strictly positive or strictly negative on $[t_{z_j}, t_{z_n}]$. From the definition of P_n and the definition of the nth Leja point, this implies

$$M_n \leq 3Cn^2 M_n |x_j - x_n|.$$

Letting $C_0 = \frac{1}{3C}$, we obtain our desired result.

■

Lemma 5.1.8 *Let $\Gamma = x(t) + iy(t)$ be a C^1 arc with non-zero derivative and with endpoints at $z_0 \neq 1$ and at 1. Let δ be as in Lemma 5.1.5, and let $\{z_j\}$ be a sequence of Leja points on Γ. Then there exists C such that if D is a disc of radius δ whose points lie closer to 1 than to z_0, and if $|x'(t)| > 1/2|y'(t)|$ for all $z = x(t) + iy(t)$ in D, then*

$$|x_j - x_i| \geq C \left(\frac{\sqrt{|x_j|} + \sqrt{|x_i|}}{n} + \frac{1}{n^2} \right).$$

Similarly, if D is a disc of radius δ whose points lie closer to 1 than to z_0, and if $|y'(t)| > 1/2|x'(t)|$ for all $z = x(t) + iy(t)$ in D, then

$$|y_j - y_i| \geq C \left(\frac{\sqrt{|y_j|} + \sqrt{|y_i|}}{n} + \frac{1}{n^2} \right).$$

Proof. Combining Lemmas 5.1.6 and 5.1.7, and assuming $i < j$, we have

$$|x_j - x_i| \geq C \left(\frac{\left|\sqrt{|x_j|} + \sqrt{|x_i|}\right|}{j} + \frac{1}{j^2} \right).$$

But since $j < n$, the desired result is easily seen to follow from this.

∎

With our distancing rule (Lemma 5.1.8) we are now ready to prove the main result of this section. To repeat what was stated in the introduction to this section, we will use as a guide for our proof the proof of Theorem 4.0.1.

Theorem 5.1.9 *Let Γ be a C^1 arc in the plane with nonzero derivative. Then the Lebesgue constants associated with a sequence of Leja points on Γ are subexponential.*

Proof. We must show that

$$\lim_{n\to\infty} \Lambda^{1/n} = 1,$$

where

$$\Lambda_n = \sup_{z\in\Gamma} \left(\sum_{j=1}^{n} \frac{\prod_{j=1}^{n} |(z - z_j)|}{\prod_{j=1}^{n} |(z_k - z_j)|} \right)$$

As in Theorem 4.0.1, it suffices to show that

$$\left(n \left(\max_{k=1,\ldots,n} \left(\frac{\|P_{n,k}\|_{[-1,1]}}{\prod(n,k)} \right) \right) \right)^{1/n} \to 1,$$

and by Theorem 2.4.11, this in turn shows that it actually suffices to show only that

$$\left(\prod(n,k) \right)^{1/n} \to \mathrm{Cap}(\Gamma)$$

uniformly in k, where

$$\prod(n,k) = \prod_{j=1, j\neq k}^{n} |z_j - z_k|.$$

At this point, just as in Theorem 4.0.1, we will split $\prod(n,k)$ into two products, $P_1(n,k,\delta)$ and $P_2(n,k,\delta)$, where

$$P_1(n,k,\delta) = \prod_{|z_j - z_k| \geq \delta} |z_j - z_k|,$$

and

$$P_2(n,k,\delta) = \prod_{|z_j - z_k| < \delta} |z_j - z_k|.$$

By Theorem 2.4.12, we know that

$$(P_1(n,k,\delta))^{1/n}$$

approximates Cap(Γ) for large n and small δ. Thus, just as in the proof of Theorem 4.0.1, our proof has been reduced to showing that for sufficiently small δ,

$$(P_2(n,k,\delta))^{1/n}$$

approximates 1 uniformly in k as $n \to \infty$. To be precise, we have reduced our proof to showing the following

* **For all $\epsilon > 0$ there exists an δ_ϵ and an N such that**

$$1 - \epsilon < \left(\prod_{|z_j - z_k| \le \delta_\epsilon} |z_j - z_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

At this point we make the assumption that our δ is sufficiently small as to satisfy the conclusion of Lemma 5.1.5. We further make the assumptions, w.l.o.g., that Γ has an endpoint at 1, that z_k lies closer to 1 than to Γ's other endpoint, and that in this δ neighborhood of z_k we have $|x'(t)| > 1/2|y'(t)|$. Indeed, we do not lose any generality with these assumptions because the selection of Leja points from a compact set is invariant with respect to translations and rotations, and because our assumptions about $\Gamma(t) = x(t) + iy(t)$ guarantee that either $|x'(t)|$ is *large* or $|y'(t)|$ is *large*.

Since

$$\left(\prod_{|x_j - x_k| \le \delta_\epsilon} |x_j - x_k| \right)^{1/n}$$

forms a lower bound for

$$\left(\prod_{|z_j - z_k| \le \delta_\epsilon} |z_j - z_k| \right)^{1/n},$$

it follows that we will have completed our proof if we can show the following statement:

** **For all $\epsilon > 0$ there exists an δ_ϵ and an N such that**

$$1 - \epsilon < \left(\prod_{|x_j - x_k| \le \delta_\epsilon} |x_j - x_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

If we now make one final assumption w.l.o.g. that Γ has tangent line $y = 0$ at 1, and that our points approach 1 from the left, then our $|x_j - x_k|$'s satisfy the inequality given in Lemma 5.1.8, and our x_j's are distributed according to the equilibrium distribution for $[-1, 1]$ near 1 (see [12]). Thus ** is the exact statement proved in Theorem 4.0.1. Thus we have already shown **. This completes the proof.

■

5.2 Leja Points on More General Sets

Now we wish to consider the union of piecewise smooth arcs, curves, and closed domains with piecewise smooth boundary. Lemma 5.1.6 will satisfy our needs in this section, as we will use it to estimate the distance between Leja points which lie on the same smooth arc. However, we will also need an estimate for the distance between Leja points which lie on separate arcs. Since Lemma 5.1.7 only applies to Leja points lying on the same smooth arc we will need the following result, and the distancing estimate which follows.

Lemma 5.2.1 *([14])Let K be a connected compact set. Then for every $D > 0$ there is a C_D such that if P_n is a polynomial of degree at most n then*

$$|P_n'(z)| \leq C_D n^2 \|P_n\|_K, \quad \text{dist}(z, K) \leq D/n^2$$

Lemma 5.2.2 *Let K be the union of finitely many smooth arcs, and let $\{z_j\}$ be a sequence of Leja points on K. Then there exists C such that*

$$C/n^2 < |z_j - z_n|, \quad j, k \leq n.$$

Proof. We can certainly choose C small enough so that the desired inequality holds for all points z_j, z_n lying on different components of K. So we may assume w.l.o.g. that K is connected. Let $P_{n-1} = \prod_{j=1}^{n-1}(z - z_j)$, and let $M = \|P_{n-1}\|_K$. Applying Lemma 5.2.1 with $D = 1$, there exists C_1 such that

$$|P_n'(z)| \leq C_1 n^2 \|P_n\|_K, \quad \text{dist}(z, K) \leq 1/n^2. \tag{5.2.1}$$

We may now assume $|z_j - z_n| \leq 1/n^2$ (otherwise there is nothing to prove). Then every point on the line segment l that connects z_j and z_n is of distance $\leq 1/n^2$ from K. By the definition of the nth Leja point, and by (5.2.1) we then have

$$M = \left| \int_{z_j}^{z_n} P_{n-1}'(z) dz \right| \leq \text{length}(l) \|P_n'\|_l \leq C_1 n^2 M |z_j - z_n|.$$

from where the claim follows (by letting $C = 1/C^1$). □.

Theorem 5.2.3 *Let K be a compact set on the plane such that its outer boundary can be written as a finite union of (not necessarily disjoint) C^1 arcs with non-zero derivative. Then the Lebesgue constants associated with any Leja sequence on K are subexponential.*

Proof. Since the outer boundary of K can be represented as a finite union of C^1 arcs, and since Leja points lie on the outer boundary, we may assume that K itself can be represented as a finite union of C^1 arcs.

So let K be the union of finitely many C^1 arcs with non-zero derivatives, $K = \cup_{i=1}^m \Gamma_i$. The proof of this Theorem will again follow Theorem 4.0.1's proof. Since K is the union of finitely many arcs, K is regular and Theorems 2.4.11 and 2.4.12 apply. Thus, as before, our problem has been reduced to proving the following statement:

(*) For all $\epsilon > 0$ there exists an δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{|z_j - z_k| < \delta_\epsilon} |z_j - z_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

In order to prove **(*)**, it suffices to prove the following for $i = 1, \ldots, m$:

(**) For all $\epsilon > 0$ there exists an δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{|z_j - z_k| < \delta_\epsilon, z_j \in \Gamma_i} |z_j - z_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

Now to prove **(**)**, there are two cases. The first case is if $z_k \in \Gamma_i$. In this case, Lemma 5.1.8 holds when estimating the distance from z_k to any $z_j \in \Gamma_i$. Further, the distribution of points $z_j \in \Gamma_i$ will be less dense than if K were a single arc. That is, if Γ_i has endpoint at 1 and tangent line $y = 0$ approaching 1 from the left, then the distribution will be less dense than the arcsine distribution. Thus in the case of $z_k \in \Gamma_i$, **(**)** follows from Theorem 5.1.9.

The second case comes when $z_k \notin \Gamma_i$. In this case, replace z_k in **(**)** with z_{j^*}, where z_{j^*} is the Leja point on Γ_i which is closest to z_k. Then for any point $z_j \in \Gamma_i$, we will have

$$|z_k - z_j| \geq \frac{1}{2}|z_j - z_{j^*}|.$$

Now with z_{j^*} in place of z_k, **(**)** will hold precisely because all the points on the curve Γ_i satisfy an inequality of the form

$$|z_{j^*} - z_j| \geq Cd(z_{j^*}, z_j),$$

where $d(z, w)$ is a distancing function which depends on the nearest endpoint along Γ_i to z_{j^*}. But we have

$$|z_k - z_j| \geq \frac{1}{2}|z_j - z_{j^*}|,$$

and thus that

$$|z_k - z_j| \geq \frac{1}{2}Cd(z_{j^*}, z_j),$$

for all z_j where $j \neq j^*$. It then follows that we have :

For all $\epsilon > 0$ there exists an δ_ϵ and an N such that

$$1 - \epsilon < \left(\prod_{|z_j - z_k| < \delta_\epsilon, z_j \in \Gamma_i, j \neq j^*} |z_j - z_k| \right)^{1/n}$$

for $k = 1, \ldots, n$ whenever $n > N$.

Since we always have

$$|z_k - z_{j^*}| \geq C_2/n^2,$$

where C_2 is a constant which depends on K (we have this by Lemma 5.2.2), it follows that **(**)** holds with z_k as well. This completes the proof.

■

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