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Power Graphs of Quasigroups

DayVon L. Walker
University of South Florida, walkerdayvon@gmail.com

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by

DayVon L. Walker

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Major Professor: Brian Curtin, Ph.D.
Lu Lu, Ph.D.
Theodore Molla, Ph.D.

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Dedication

I dedicate this to my parents, Sharon and Yusef, for always pushing me, believing me and always supporting my decisons.

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To Dr. Curtin for constantly being there and guiding me through the entire process even when it seemed that we would never get anywhere.

To the students of the AVID program of Rampello for the emotional support and confidence for me to keep going and expand my knowledge.

To my Coworkers at Mathnasium for creating a joyful work enviroment that made me fall in love with math all over again.

To the University of South Florida for showing me no matter how far you come there is always much more to learn.

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Abstract

We investigate power graphs of quasigroups. The power graph of a quasigroup takes the elements of the quasigroup as its vertices, and there is an edge from one element to a second distinct element when the second is a left power of the first. We first compute the power graphs of small quasigroups (up to four elements). Next we describe quasigroups whose power graphs are directed paths, directed cycles, in-stars, out-stars, and empty. We do so by specifying partial Cayley tables, which cannot always be completed in small examples. We then consider sinks in the power graph of a quasigroup, as subquasigroups give rise to sinks. We show that certain structures cannot occur as sinks in the power graph of a quasigroup. More generally, we show that certain highly connected substructures must have edges leading out of the substructure. We briefly comment on power graphs of Bol loops.

Chapter 1

Introduction

In mathematics one often attempts to gain insight into complicated structures by studying related, simpler objects. This, for example, is the point of using representation or quotients of algebraic structures. In this vein, power graphs were introduced to study groups and semigroups by Kelarev and Quinn [44, 45, 46, 47]. Since that time a number papers studying the power graphs of groups and semigroups have appeared. [1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 25, 31, 32, 33, 35, 36, 38, 39, 40, 41, 42, 43, 48, 50, 51, 52, 53, 54, 55, 56, 58, 59, 60, 61, 62, 66, 67, 69, 71, 72, 73, 74, 76, 77, 78, 80, 81, 82, 83, 87, 91]. Some of these works ask which groups (or group properties) give rise to various graph properties, such as planarity, connectivity, eulerian, hamiltonian, complete, etc. Some works are descriptive/constructive, giving the power graphs of various families of groups, such nilpotent groups, dihedral groups, etc. Other works concern the spectral properties of power graphs. Some works used power graphs of cyclic groups to produce number theoretic results.

The directed power graph of a semigroup takes as its vertices the elements of the semigroup, and has an edge from each element to every power of the element (other than itself). That is to say, there is an edge from each element to every other element in the cyclic subgroup generated by the first element. For example, $(\mathbb{Z}_3, +)$ has element $\{0, 1, 2\}$. We note that $1 + 1 = 2$, $1 + 1 + 1 = 0$, $2 + 2 = 1$, $2 + 2 + 2 = 0$, $0 + 0 = 0$. Thus the directed power graph of $(\mathbb{Z}_3, +)$ has edges from 1 to 0 and 2, from 2 to 0 and 1, and no other edges. A great deal of the group structure is lost in the power graph construction, but certain families and properties of groups can be studied nicely using power graphs.

A key property of group and semigroups is the associativity of the multiplication. This makes the notion of powers of an element unambiguous. In this thesis, we relax associativity. The construction of power graphs can be modified to work for quasigroups, provided one clarifies the meaning of power. Recall that a quasigroup is a set Q with a binary operation $*$ satisfying the rule that for all x, $y \in Q$, there exist unique elements a, $b \in Q$ such that $a * x = b$ and $y * a = b$. We shall define the power graph of a quasigroup to be the graph which takes the elements of the quasigroup as its vertices and which has an edge to every left-power of the element (other than itself). That is to say there is an edge from x to $x * x$, $x * (x * x)$, $x * (x * (x * x))$, ..., except for x itself.

In this thesis we explore power graphs of quasigroups. To our knowledge, power graphs of quasigroups have not been investigated previously (probably for good reason). Quasigroups have far less structure, and their power graphs reflect this. Nonetheless we examine this situation. We find that although there is much less structure than the groups, there is some structure.

In Chapter 2 we briefly recall some background material on graphs, quasigroups, and power graphs. We defer our example of constructing the power graph of a quasigroup to Chapter 2. In Chapter 3, we compute the power graphs of small quasigroups (order at most 4). These provide a number of examples that show that most results about power graphs of groups do not hold for power graphs of quasigroups.

In Chapter 4, we describe when a quasigroup has a power graph among the sparse families of directed paths, directed cycles, in-stars, out-stars, and empty graphs. In each case we give a partial Cayley table which leads to the structure. This begins to reveal the complexity of quaisgroups. Computer searches return thousands Cayley tables (Latin squares) which correspond to such power graphs, but in many instances we have not identified any “nice” construction of a family of such quasigroups. We have been forced to leave the existence in many cases as a conjecture.

In Chapter 5, we observe that sinks in power graphs have a special significance. For instance a subquasigroup of a quasigroup forms a sink in the power graph. We show that certain graphs with many edges cannot be a sink in a quasigroup. Similarly for subgraphs which only have one edge out of it. Despite the large number of quasigroups (orders of magnitude more than digraphs for larger order), there are digraphs that are not power graphs. We also show that any power graph of a quasigroup can be appear as a sink in the power graph of a quasigroup of twice the first's order.

In Chapter 6, we briefly discuss Bol loops. These are quasigroups which are closely related to groups. Although they are not associative in general, the Bol axiom is a slight generalization of associativity. We note that Bol loops are self-associative, so that there is no ambiguity in taking powers of an element. We expect that power graphs of Bol loops will be quite interesting. Here we only scratch the surface of this subject, leaving serious investigation to others.

Finally, in Chapter 7, we discuss the use of *Mathematica* and publicly available databases of small Latin squares, Bol loops, and groups. In this thesis we have generally used examples small enough to study by hand; however, the use of *Mathematica* and the databases has been quite helpful.

Many open questions remain in the subject. Our purpose here has primarily been to initiate a study of power graphs of quasigroups. Given the large number quasigroups and the limited structure of quasigroups, one must temper their expectations of any strong general results. Nonetheless, we do find a few nice properties.

Chapter 2

Background

In this thesis, we consider directed left power graphs from quasigroups. In this chapter we recall some background on graphs (Section 2.1), quasigroups (Section 2.2), and power graphs (Section 2.3).

2.1 Graphs

Graphs provide a convenient language in which to discuss some relationships between objects. There are many flavors of graphs which allow one to model various situations. Their common features are a set of vertices (alternatively referred to as nodes or points in various applications) and a set of edges (alternatively referred to as arcs or lines) indicating a connection or relationship between pairs of vertices. The vertices may be labeled, colored, or have some assigned weight. The edges may be directed or undirected, in addition to having labels, colors, or weights. Some flavors of graphs admit multiple edges between two vertices and edges from a vertex to itself.

In our application the vertices of the graphs will be the elements of a quasigroup (which serve as labels), with a directed edge from one element of the quasigroup to a second element whenever the second is a left power of the first.

There is extensive literature concerning graphs. We will not be using deep results of graph theory, so most introductory references will provide a sufficient background to the subject for our purposes. Basic references include [89, 90]. More advanced references include [10, 11, 30].

In later sections we will introduce some particular families of graphs and some interesting properties. Here we simply give a very basic description.

2.1.1 Directed and undirected graphs

Definition 2.1.1. A *directed graph* (or *digraph*) is ordered pair (V, E), where V is a set and E is a set of ordered pairs of elements of V. We refer to $|V|$, the number of vertices, as the *order* of the digraph and $|E|$, the number of edges, as the *size* of the digraph. A digraph is *finite* when both V and E are finite sets.

Definition 2.1.2. Given an edge $e = (x, y)$ of a digraph, we say that e is *from* its x and *to* y. We refer to x as the *initial vertex* (or *tail*) and y as the *terminal vertex* (or *head*) of the edge. We signify that there is an edge from x to y by writing $x \to y$.

Note 2.1.3. Because E is a set in our definition of digraph (as opposed to a multiset), there is at most one edge from each vertex to any other vertex, that is to say, there are no multiple edges. A *loop* is an edge from a vertex to itself. The digraphs which arise in this thesis have no loops or multiple edges.

We visualize graphs with a drawing when practical rather than giving a formal description as in definition. Figure 2.1.1 shows a small digraph and a visual presentation of the digraph made by placing the vertices and drawing an arrow for each edge from the initial vertex to the terminal vertex.

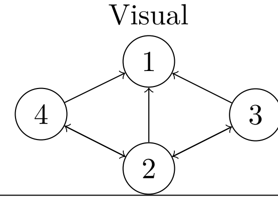
Formal	Visual
$(\{1, 2, 3, 4\}, \{(2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1), (4, 2)\})$	

Figure 2.1: A small digraph

Frequently, neither the formal nor the visual description of a graph provides a practical presentation of a graph. Here we relate properties of the directed left power graph of a quasigroup to those of the quasigroup. The directed left power graph of a finite quasigroup is a finite directed graph without loops or multiple edges. In the case of groups, the directed left power graph can be deduced from its undirected version [16]. This will not be the case for quasigroups. We will at times point out quasigroups which show that theorems about power graphs of groups fail for quasigroups.

Definition 2.1.4. An *undirected graph* (or simply *graph*) is ordered pair (V, E), where V is a set and E is a set of ordered pair of two element subsets of V.

Note 2.1.5. This definition of graph does not admit loops or multiple edges, although there are applications where one should allow these features. An alternate description of an undirected graph is as a symmetric digraph, a digraph in which for each directed edge the directed edge in the opposite direction appears.

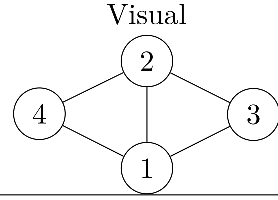
Formal	Visual
$(\{1,2,3,4,5\},$ $\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\})$	

Figure 2.2: A small graph

2.1.2 Graph isomorphism

We will meet a number of distinct quasigroups which have the same directed left power graphs in the following sense.

Definition 2.1.6. A graph *isomorphism* from a digraph $G = (V, E)$ to a digraph $G' = (V', E')$ is bijection $f : V \to V'$ such that $(u, v) \in E$ if and only if $(f(u), f(v)) \in E'$. An isomorphism from a graph to itself is referred to as an *automorphism* of the graph.

In Chapter 4, we show that for a number of graph with relatively few edges, only a small portion of the quasigroup is involved in the construction of the directed left power graph. In particular, there may be large number of distinct quasigroups with the same directed left power graph.

In our study, the algebraic relationships of elements of the quasigroup of rather than the names of the elements matters. Changing the names of the quasigroup elements will lead to isomorphic power graphs. Likewise, automorphisms of the quasigroup induce automorphisms of the power graph. However, there may be graph automorphisms which do not arise in this manner.

2.2 Quasigroups

In this section we recall quasigroups. For practical representation of quasigroups we shall use their Cayley tables. We shall take advantage of the close correspondence between the Cayley tables of quasigroups and Latin squares.

In this thesis we explore directed left power graphs of quasigroups. This exploration does not use any deep properties of quasigroups, although further advances in the subject will surely require more. For more on quasigroups, the reader is referred to [75, 85, 86].

2.2.1 Quasigroups and their Cayley tables

Quasigroups are generalizations of groups; however, unlike groups, quasigroups need not have an identity, and the multiplication in a quasigroup is generally not associative.

Definition 2.2.1. A *quasigroup* is a set Q with a binary operation $*$ satisfying the rule that for all a, $b \in Q$, there exist unique elements x, $y \in Q$ such that $a * x = b$ and $y * a = b$.

We can describe a quasigroup by its multiplication or Cayley table.

Definition 2.2.2. Given a finite quasigroup Q, its *Cayley table* is the array whose rows and columns are indexed by elements of Q where the (x, y)-entry is $x * y$.

Example 2.2.3.

$*$	1	2
1	1	2
2	2	1

For each element x of a quasigroup Q, right multiplication by x and left multiplication by x both induce permutations on Q. In particular, each row and each column contains each entry of the Cayley table exactly once. We refer to this as the *Latin square property* for this array.

Definition 2.2.4. A *Latin square* is an $n \times n$ array containing n copies of each of the n symbols, arranged in such a way that no symbol is repeated in any row or column.

Example 2.2.5. A Latin square of order 4:

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 2 & 3 & 1 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}.$$

For further introduction to Latin squares see [5, 28, 49].

The Cayley table of a quasigroup includes in its first row and column a list of the elements, as in Example 2.2.3, so that one may read the multiplication. Removing this decoration leaves a pure Latin square, as in Example 2.2.5. To be more concise, we typically describe quasigroups by giving a Latin square L with entries 1, 2, ..., n and understand that the ij- entry of L is $i * j$. In this way, any Latin square can be viewed as the Cayley table of a quasigroup.

Example 2.2.6. The Latin square of Example 2.2.5 as a Cayley table:

$$\begin{array}{c|cccc} * & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 4 & 3 \\ 2 & 3 & 4 & 2 & 1 \\ 3 & 2 & 3 & 1 & 4 \\ 4 & 4 & 1 & 3 & 2 \end{array}.$$

2.2.2 Isomorphism

We describe how two Latin squares can be Cayley tables of isomorphic quasigroups by changing the names/order of the elements.

Definition 2.2.7. Let Q, Q' be quasigroups. By an *isomorphism* from Q to Q' we mean a bijection $\phi : Q \to Q'$ such that $\phi(xy) = \phi(x)\phi(y)$ for all x, $y \in Q$. An *automorphism* of Q is an isomorphism from Q to itself.

Changing names and ordering leads to a somewhat trivial sort of isomorphism which can lead to a different Cayley table.

Lemma 2.2.8. *Let Q be a quasigroup, and let π be a permutation of the elements of Q. Suppose T is a Cayley table of Q. The array formed by applying π to the rows, columns, and entries of T is the Cayley table T' of a quasigroup Q' which is isomorphic to Q via the map π. Moreover, if $T' = T$ then π is an automorphism of Q.*

Example 2.2.9. Take $\pi = (12)$ and $T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$. Applying π to permute the first two rows, the first two columns, and the entry values 1, 2 yields $T' = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

From the multiplication we confirm the isomorphism.

T	T'
$1 * 1 = 1$	$2 \cdot 2 = 2$
$1 * 2 = 2$	$2 \cdot 1 = 1$
$1 * 3 = 3$	$2 \cdot 3 = 3$
$2 * 1 = 3$	$1 \cdot 2 = 3$
$2 * 2 = 1$	$1 \cdot 1 = 2$
$2 * 3 = 2$	$1 \cdot 3 = 1$
$3 * 1 = 3$	$3 \cdot 2 = 3$
$3 * 2 = 3$	$3 \cdot 1 = 3$
$3 * 3 = 1$	$3 \cdot 3 = 2$

If one starts from a list of Latin squares and attempts to find a list of quasigroups, one must take into account that many Latin squares may represent the same quasigroup. The actual automorphisms of a quasigroup arise from a name change as above which results in an identical Cayley table.

Example 2.2.10. The group $\mathbb{Z}_3$ has an automorphism in which the two generators are swapped. For consistency with the above label the elements 1, 2, 3 and make 1 the identity. The Cayley table is: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$. Applying $\pi = (23)$ to the rows, columns and entries yields the same Cayley table.

Thus, although all permutations of 1, 2, ..., n transform the a Cayley table of a quasigroup Q to the Cayley table of an isomorphic quasigroup, some permutations (the automorphisms of Q) do not yield distinct Cayley tables.

Example 2.2.11. There are 12 Latin squares of order 3. They can easily be enumerated, as each is uniquely determined by the (1,1)-, (1,2)- and (2,1)-entries; there are 3 choices for the (1,1)-entry, and 2 choices each for the (1,2)- and (2,1)-entries.

Up to isomorphism there are 5 quasigroups of order 3. Each of the 12 Latin squares of order 3 arises as their Cayley table of one of them. Three isomorphism classes of quasigroups have three distinct Cayley tables, one class has two distinct Cayley tables, and one has just one Cayley table.

Q_1 :	$T_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$	$(12)T_1 = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$	$(13)T_1 = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$
Q_2 :	$T_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$	$(12)T_2 = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$	$(13)T_2 = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$
Q_3 :	$T_3 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$	$(12)T_3 = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$	$(13)T_3 = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$
Q_4 :	$T_4 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$	$(12)T_4 = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$	
Q_5 :	$T_5 = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$		

2.3 Power graphs

Directed power graphs were first introduced to study groups and semigroups in [44, 45, 46, 47]. Recall that a *semigroup* is a set S with an associative binary operation. A *group* is a semigroup that has a two-sided identity and two-sided inverses. The associativity allows for an unambiguous construction of power graphs.

Definition 2.3.1. Let S be a semigroup. The *directed power graph* of S takes S as its vertex set with an edge from a to a distinct b if $b = a^n$ for some positive integer n.

Undirected power graphs of groups were introduced in [16]. There it was shown that the directed power graph of a group can be reconstructed, up to isomorphism, from the undirected power graph. Thus many papers on power graphs of groups focus on undirected power graphs.

Definition 2.3.2. Let G be a group. The *undirected power graph* of G takes G as its vertex set with an edge between distinct a and b if $b = a^n$ or $b = a^n$ for some positive integer n.

For more information about power graphs of groups and semigroups, the reader is referred to the survey [2], which contains a full review of the literature at time of its publication.

Here we carry the definition of the directed power graph to quasigroups. The fact that quasigroups are not associative in general means that we must disambiguate what we mean by a power. Two natural candidates present themselves, namely, right and left powers.

Definition 2.3.3. Let Q be a quasigroup, and let $x \in Q$. For each positive integer n, the *n-th right* and *left powers of* x are respectively $x^{\rho_n} = ((x * x) * x) * \cdots * x$ and $x^{\lambda_n} = x * \cdots * (x * (x * x))$, where n factors of x appear in each expression.

Definition 2.3.4. Let Q be a quasigroup.

- The *directed right power graph of* Q takes Q as its vertex set with an edge from a to a distinct b if $b = a^{\rho_n}$ for some positive integer n.
- The *directed left power graph of* Q takes Q as its vertex set with an edge from a to a distinct b if $b = a^{\lambda_n}$ for some positive integer n.

We shall favor directed left power graphs of quasigroups. We will not consider the undirected version of power graphs for quasigroups, as there are examples of nonisomorphic directed left power graphs with the same undirected graph. Thus we shall often use the term *power graph* of a quasigroup to refer to its directed left power graph.

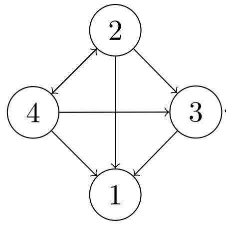
Example 2.3.5. Consider the quasigroup Q with Cayley table/Latin square

$$T = \begin{array}{c|cccc} * & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 & 1 \\ 3 & 3 & 4 & 1 & 2 \\ 4 & 4 & 1 & 2 & 3 \end{array}$$

(in fact, Q is isomorphic to $\mathbb{Z}_4$).

- The edges from 1 in the directed left power graph are computed as follows:
  - The $(1,1)$-entry of T gives $1 * 1 = 1$. There are no edges leaving 1.
- The edges from 2 in the directed left power graph are computed as follows:
  - The $(2,2)$-entry of T gives $2 * 2 = 3$. There is an edge from 2 to 3.
  - The $(2,3)$-entry of T is $2 * (2 * 2) = 2 * 3 = 4$. There is an edge from 2 to 4.
  - The $(2,4)$-entry of T is $2 * (2 * (2 * 2)) = 2 * 4 = 1$. There is an edge from 2 to 1.
  - The $(2,1)$-entry of T is $2 * (2 * (2 * (2 * 2))) = 2 * 1 = 2$. The list is complete.
- The edges from 3 in the directed left power graph are computed as follows:
  - The $(3,3)$-entry of T gives $3 * 3 = 1$. There is an edge from 3 to 1.
  - The $(3,1)$-entry of T gives $3 * (3 * 3) = 3 * 1 = 3$. The list is complete.
- The edges from 4 in the directed left power graph are computed as follows:
  - The $(4,4)$-entry of T gives $4 * 4 = 3$. There is an edge from 4 to 3.
  - The $(4,3)$-entry of T is $4 * (4 * 4) = 4 * 3 = 2$. There is an edge from 4 to 2.
  - The $(4,2)$-entry of T is $4 * (4 * (4 * 4)) = 4 * 2 = 1$. There is an edge from 4 to 1.

– The $(4,1)$-entry of T is $4*(4*(4*(4*4))) = 4*1 = 4$. The list is complete.

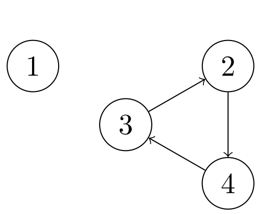
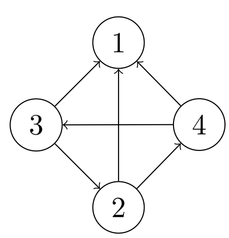
The power graph of Q is .

Note 2.3.6. To compute the edges in a directed left power graph we can view row i as the permutation whose bottom row in two-row notation is row i. The neighbors of i are precisely the elements of the cycle of the permutation containing i, other than i itself.

Note 2.3.7. The directed right power graph is computed similarly, using columns rather than rows. It can also be computed by applying the process of Example 2.3.5 to the transpose of a Cayley table. The right power graph of a quasigroup will arise as a left power graph for the quasigroup whose Cayley table is the transpose of the first.

The Cayley table in Example 2.3.5 is symmetric (the multiplication is abelian), so it has the same left and right power graph. In fact, since it is the Cayley table of a group, associatively would also imply that the left and right power graphs coincide. In general quasigroups are not associative, let alone abelian. Thus the left and right power graphs will differ.

Example 2.3.8. We give an undecorated Cayley table T for which the directed left and right power graphs differ:

T	Left power graph	Right power graph
$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}$		

In general, there appears to be little relationship between left and right power graphs of a quasigroup. A few properties will be shared.

- i is idempotent iff the (i, i)-entry is i iff there are no edges in the left power graph with initial vertex i iff there are no edges in the right power graph with initial vertex i. See also Section 4.5.

- Subquasigroups give rise to sinks (see Section 5.1), which will also be shared by both left and right power graphs.

In this thesis we shall explore other aspects, and leave further investigation of the relationship between left and right power graphs to other work.

Chapter 3

Small Examples

We begin this chapter by discussing the number of quasigroups and digraphs in Section 3.1. We then describe the directed power graphs of small quasigroups in Sections 3.2, 3.3, and 3.4.

3.1 Number of quasigroups and digraphs

We begin our investigation by describing the power graphs of quasigroups of small order. By [79], the number $\mathcal{Q}_n$ of quasigroups of order n is bounded by

$$2^{(n-3)(n-1)/4} < \mathcal{Q}_n \leq 2^{\left(\frac{\log_2 n!}{n-2} + \frac{n}{n-4}\right)(n-2)^2}.$$

The number $\mathcal{Q}_n$ is given in Table 3.1. See [64] or sequence A057991 in the OEIS [84]. It grows extremely rapidly.

Table 3.1: The number of quasigroups of small orders

n	$\mathcal{Q}_n$
0	1
1	1
2	1
3	5
4	35
5	1,411
6	1,130,531
7	12,198,455,835
8	2,697,818,331,680,661
9	15,224,734,061,438,247,321,497
10	2,750,892,211,809,150,446,995,735,533,513
11	19,464,657,391,668,924,966,791,023,043,937,578,299,025

Then number $\mathcal{L}_n$ of Latin squares of order n, which are the possible Cayley tables of quasigroups, grows more rapidly. It is bounded by [88]

$$\frac{(n!)^{2n}}{n^{n^2}} \leq \mathcal{L}_n \leq \prod_{k=1}^{n} (k!)^{n/k}.$$

Table 3.2: The number of Latin squares of small orders

n	$\mathcal{L}_n$
1	1
2	2
3	12
4	576
5	161,280
6	812,851,200
7	61,479,419,904,000
8	108,776,032,459,082,956,800
9	5,524,751,496,156,892,842,531,225,600
10	9,982,437,658,213,039,871,725,064,756,920,320,000
11	776,966,836,171,770,144,107,444,346,734,230,682,311,065,600,000

We compare the number of quasigroups to the number of digraphs $\mathcal{D}_n$ of order n, sequence A000273 in the OEIS [84]. We recall the first few values of $\mathcal{D}_n$ in Table 3.3. The number is on the order of $2^{n*(n-1)}/n!$ as a function of the order n [65].

The number of digraphs grows far less rapidly than the number of quasigroups, with the latter overtaking the former at size 7. There must be digraphs of small order that do not arise as the directed power graph of any quasigroup.

3.2 Power graphs of small quasigroups

The quasigroups of order 1 and 2 have Cayley tables and power graphs:

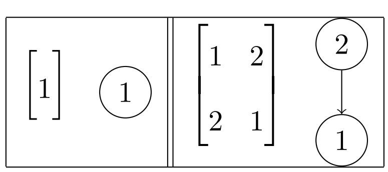


Table 3.3: The number of digraphs of small sizes

n	$\mathcal{D}_n$
0	1
1	1
2	3
3	16
4	218
5	9,608
6	1,540,944
7	882,033,440
8	1,793,359,192,848
9	13,027,956,824,399,552
10	341,260,431,952,972,580,352
11	32,522,909,385,055,886,111,197,440

In the next two sections we list Cayley tables and the corresponding power graphs for quasigroups of orders 3 and 4. Beyond that the numbers grow too large to treat here.

3.3 Power graphs of quasigroups of order 3

In Table 3.4, we list a Cayley table of each quasigroup of order 3 and their power graphs. See Example 2.2.11. These examples are small enough and few enough in number that one may compute and inspect manually.

Observe that the nonisomorphic quasigroups Q_1 and Q_2 have isomorphic directed left power graphs.

3.4 Power graphs of quasigroups of order 4

We now list the Cayley tables of all 35 quasigroups of order 4 in Tables 3.5, 3.6, and 3.7. Using *Mathematica* (see Chapter 7), we are able to verify that there are:

- 31 distinct power graphs in this list. Sets of equal graphs: $\{1,8\}$, $\{2,9\}$, $\{3,10\}$, $\{4,11\}$.
- 23 isomorphism classes. Sets of isomorphic graphs $\{1,8\}$, $\{2,3,9,10\}$, $\{4,11\}$, $\{5,6\}$, $\{12,13\}$, $\{15,17\}$, $\{19,20\}$, $\{21,22\}$, $\{23,24\}$, $\{27,28\}$, $\{30,33\}$, $\{31,32\}$.

Table 3.4: Quasigroups of order 3 and their power graphs

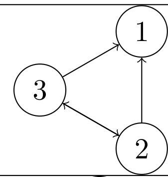
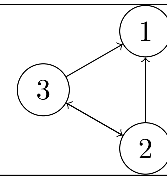
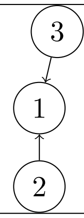
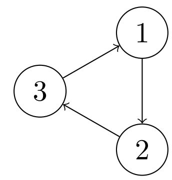
Q_1:	$\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}$		Q_2:	$\begin{bmatrix}1 & 2 & 3\\3 & 1 & 2\\2 & 3 & 1\end{bmatrix}$	
Q_3:	$\begin{bmatrix}1 & 3 & 2\\2 & 1 & 3\\3 & 2 & 1\end{bmatrix}$		Q_4:	$\begin{bmatrix}2 & 1 & 3\\1 & 3 & 2\\3 & 2 & 1\end{bmatrix}$	
Q_5:	$\begin{bmatrix}1 & 3 & 2\\3 & 2 & 1\\2 & 1 & 3\end{bmatrix}$				

Table 3.5: Quasigroups of order 4 and their power graphs, part I

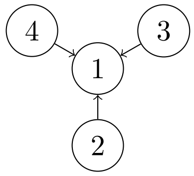
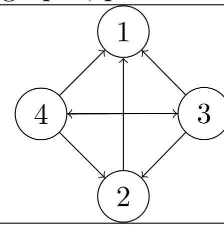
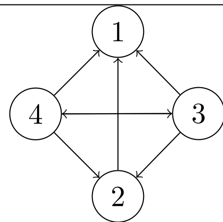
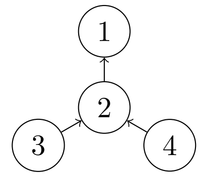
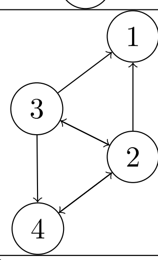
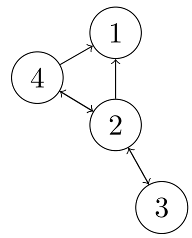
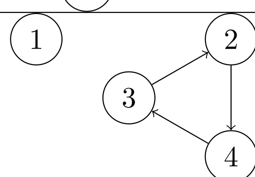
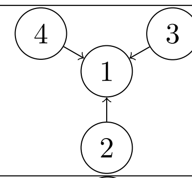
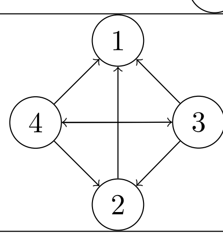
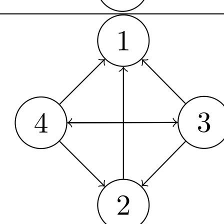
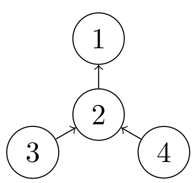
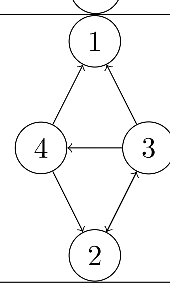
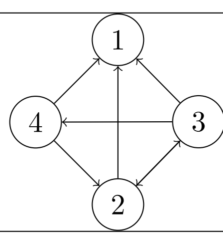
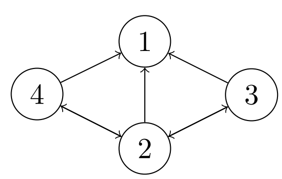
1.	$\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&4&1&2\\4&3&2&1\end{bmatrix}$		2.	$\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&4&2&1\\4&3&1&2\end{bmatrix}$	
3.	$\begin{bmatrix}1&2&3&4\\2&1&4&3\\4&3&1&2\\3&4&2&1\end{bmatrix}$		4.	$\begin{bmatrix}1&2&3&4\\2&1&4&3\\4&3&2&1\\3&4&1&2\end{bmatrix}$	
5.	$\begin{bmatrix}1&2&3&4\\2&3&4&1\\4&1&2&3\\3&4&1&2\end{bmatrix}$		6.	$\begin{bmatrix}1&2&3&4\\3&1&4&2\\4&3&2&1\\2&4&1&3\end{bmatrix}$	
7.	$\begin{bmatrix}1&2&3&4\\3&4&1&2\\4&3&2&1\\2&1&4&3\end{bmatrix}$		8.	$\begin{bmatrix}1&2&4&3\\2&1&3&4\\3&4&1&2\\4&3&2&1\end{bmatrix}$	
9.	$\begin{bmatrix}1&2&4&3\\2&1&3&4\\3&4&2&1\\4&3&1&2\end{bmatrix}$		10.	$\begin{bmatrix}1&2&4&3\\2&1&3&4\\4&3&1&2\\3&4&2&1\end{bmatrix}$	
11.	$\begin{bmatrix}1&2&4&3\\2&1&3&4\\4&3&2&1\\3&4&1&2\end{bmatrix}$		12.	$\begin{bmatrix}1&2&4&3\\2&3&1&4\\3&4&2&1\\4&1&3&2\end{bmatrix}$	
13.	$\begin{bmatrix}1&2&4&3\\3&1&2&4\\4&3&1&2\\2&4&3&1\end{bmatrix}$		14.	$\begin{bmatrix}1&2&4&3\\3&4&2&1\\2&3&1&4\\4&1&3&2\end{bmatrix}$	

Table 3.6: Quasigroups of order 4 and their power graphs, part II

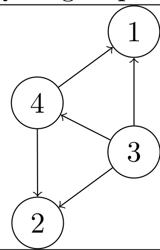
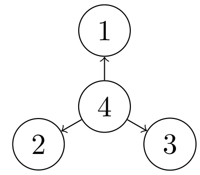
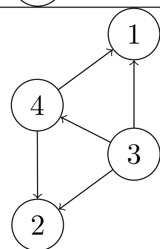
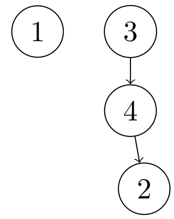
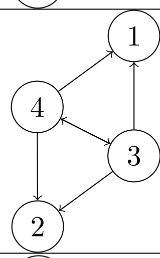
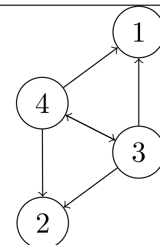
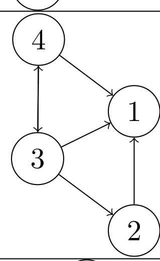
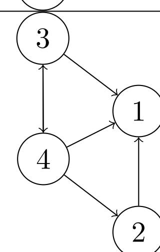
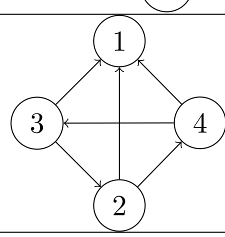
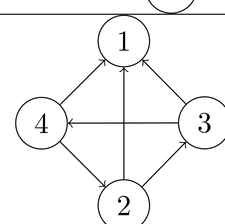
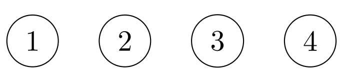
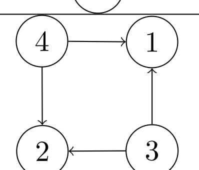
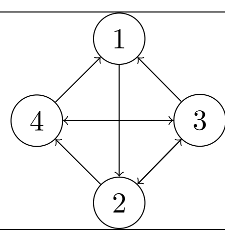
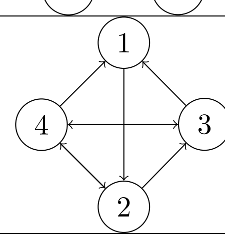
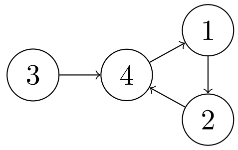
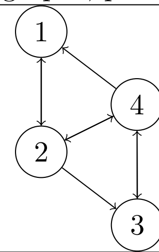
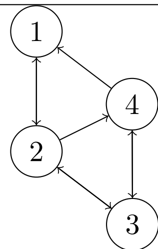
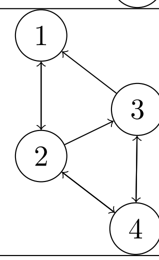
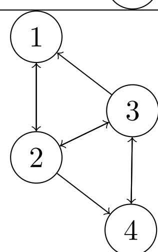
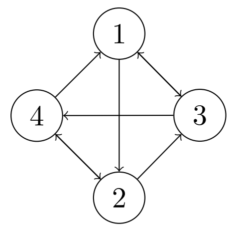
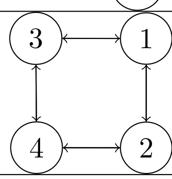
15.	$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 2 \end{bmatrix}$		16.	$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 4 & 1 & 3 \end{bmatrix}$	
17.	$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 3 & 1 \end{bmatrix}$		18.	$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 2 & 3 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$	
19.	$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 2 & 3 & 1 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \end{bmatrix}$		20.	$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \end{bmatrix}$	
21.	$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 4 & 2 & 1 & 3 \end{bmatrix}$		22.	$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & 3 & 4 \\ 4 & 2 & 1 & 3 \\ 3 & 4 & 2 & 1 \end{bmatrix}$	
23.	$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix}$		24.	$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 3 & 1 & 2 & 4 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \end{bmatrix}$	
25.	$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$		26.	$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 2 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix}$	
27.	$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 3 & 4 & 2 \\ 3 & 4 & 2 & 1 \\ 4 & 2 & 1 & 3 \end{bmatrix}$		28.	$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 3 & 4 & 2 & 1 \end{bmatrix}$	

Table 3.7: Quasigroups of order 4 and their power graphs, part III

29. $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 1 & 2 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ 	30. $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 4 & 3 \\ 4 & 3 & 1 & 2 \end{bmatrix}$ 
31. $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \end{bmatrix}$ 	32. $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 4 & 2 & 1 & 3 \\ 1 & 3 & 4 & 2 \end{bmatrix}$ 
33. $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \\ 1 & 2 & 4 & 3 \end{bmatrix}$ 	34. $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \end{bmatrix}$ 
35. $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 4 & 1 & 3 & 2 \\ 3 & 2 & 4 & 1 \\ 1 & 4 & 2 & 3 \end{bmatrix}$ 	

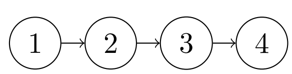
Chapter 4

Families of Power Graphs of Small Size

In this chapter we consider how some families of digraphs can arise as power graphs of quasigroups. We focus on paths, cycles, in-stars, out-stars, and empty graphs which have small size (few edges) relative to the number of possible edges for a digraph of their order (number of vertices). These families have roughly the same number of edges as vertices. Forcing n edges in the power graph of a quasigroup of order n vertices requires specifying $2n$ out of the n^2 entries of the Cayley table of the quasigroup. For small n, there may be no such quasigroup; however, we expect that there are such quasigroups once n grows large enough, as only a very small proportion of entries are determined.

4.1 Paths

Definition 4.1.1. A graph on n vertices is a *directed path* whenever its vertices $\{v_i\}_{i=1}^{n}$ can be ordered so that $v_1 \to v_2 \to \cdots \to v_{n-1} \to v_n$ with no other edges.

Example 4.1.2. A path of order 4 can be drawn 

Lemma 4.1.3. *Let G be the path on vertices $\{i\}_{i=1}^{n}$ with $1 \to 2 \to \cdots \to n-1 \to n$. Then G is* *the power graph of any quasigroup with partial Cayley table*

$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & \cdots & & n-1 & n \\
\hline
1 & 2 & 1 & & & & & \\
2 & & 3 & 2 & & & & \\
3 & & & 4 & 3 & & & \\
\vdots & & & & \ddots & \ddots & & \\
n-1 & & & & & & n & n-1 \\
n & & & & & & & n
\end{array}
.
$$

Proof. Since $i \to i+1$ for $1 \leq i \leq n-1$, and nothing else, must have $i \cdot i = i+1$ and $i \cdot (i \cdot i) = i \cdot (i+1) = i$ for $1 \leq i \leq n-1$. Also n has no out-edges, so $n \cdot n = n$. We conclude that any partial Latin square as in the statement is the Cayley table of a quasigroup whose left power graph is the given directed path. □

Lemma 4.1.4. *No quasigroups of order 3, 4, or 5 have a directed path as their directed power graph.*

Proof. For order 3, the result follows from Table 3.4. For order 4, we order the vertices so that we must have partial Cayley table as in Lemma 4.1.3. However, there is no admissible entry for the (1,4)-entry of the Cayley table. For order 5, first note that the Latin square property forces the (1,4)- and (1,5)- entries to be 4 and 3, respectively. Then the (2,4)-entry is forced to be 1, leaving no admissible value for the (2,5)-entry. □

Example 4.1.5. We list four Cayley tables of quasigroups of order 6 whose directed power graphs are the path of Lemma 4.1.3.

$$\begin{bmatrix} 2 & 1 & 6 & 4 & 5 & 3 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 6 & 3 & 5 & 4 & 2 \\ 3 & 4 & 1 & 2 & 6 & 5 \\ 4 & 2 & 5 & 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 4 & 5 & 3 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 3 & 6 & 1 & 5 & 4 & 2 \\ 1 & 4 & 3 & 2 & 6 & 5 \\ 4 & 2 & 5 & 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 4 & 5 & 3 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 3 & 6 & 1 & 5 & 4 & 2 \\ 4 & 2 & 3 & 1 & 6 & 5 \\ 1 & 4 & 5 & 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 & 6 & 3 & 4 \\ 6 & 3 & 2 & 4 & 5 & 1 \\ 5 & 6 & 4 & 3 & 1 & 2 \\ 1 & 2 & 6 & 5 & 4 & 3 \\ 3 & 4 & 1 & 2 & 6 & 5 \\ 4 & 5 & 3 & 1 & 2 & 6 \end{bmatrix}$$

Conjecture 4.1.6. *For all $n \geq 6$ there are quasigroups whose directed power graph is a directed path.*

We describe a case when it is possible to extend a Latin square satisfying Lemma 4.1.3.

Definition 4.1.7. A *transversal* of a Latin square L of order n is a collection of n-many triples $\{(r, c, e)\}$ with $L(r, c) = e$ such that no two triples have common value in any coordinate.

The existence of a transversal is not guaranteed. In general, there need not be a transversal in an arbitrary Latin square, let alone a transversal that satisfies the additional conditions which we will impose in Lemmas 4.1.9 and 4.1.11.

Example 4.1.8. The fourth Latin square in Example 4.1.5 has a nice transversal:

$$\begin{bmatrix} 2 & 1 & \boxed{5} & 6 & 3 & 4 \\ 6 & 3 & 2 & \boxed{4} & 5 & 1 \\ 5 & 6 & 4 & 3 & \boxed{1} & 2 \\ 1 & \boxed{2} & 6 & 5 & 4 & 3 \\ \boxed{3} & 4 & 1 & 2 & 6 & 5 \\ 4 & 5 & 3 & 1 & 2 & \boxed{6} \end{bmatrix}.$$

Lemma 4.1.9. *Suppose L is a Latin square of order n with partial entries as in Lemma 4.1.3 with a transversal which intersects the specified partial entries in the (n, n)-position and no others. Then we may extend L to a Latin square $\hat{L}$ of order $n + 1$ which has the partial Cayley table of*

Lemma 4.1.3 by replacing each entry on the transversal with $n+1$ and filling the last row and column to fulfill the Latin square property. Namely, the ith entry of column $n+1$ is the entry on the transversal in row i, and the jth entry of row $n+1$ is the entry on the transversal in column j. The $(n+1, n+1)$-entry is $n+1$.

Proof. The construction results in a Latin square with the same diagonal and super-diagonal except that in $\hat{L}$ the $(n-1, n-1)$-entry is n, the $(n-1, n)$-entry is $n-1$, and the (n, n)-entry is n, as required. $\square$

Example 4.1.10. We illustrate Lemma 4.1.9 using with the transversal of Example 4.1.8.

$$\begin{bmatrix} 2 & 1 & \boxed{5} & 6 & 3 & 4 \\ 6 & 3 & 2 & \boxed{4} & 5 & 1 \\ 5 & 6 & 4 & 3 & \boxed{1} & 2 \\ 1 & \boxed{2} & 6 & 5 & 4 & 3 \\ \boxed{3} & 4 & 1 & 2 & 6 & 5 \\ 4 & 5 & 3 & 1 & 2 & \boxed{6} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 7 & 6 & 3 & 4 & 5 \\ 6 & 3 & 2 & 7 & 5 & 1 & 4 \\ 5 & 6 & 4 & 3 & 7 & 2 & 1 \\ 1 & 7 & 6 & 5 & 4 & 3 & 2 \\ 7 & 4 & 1 & 2 & 6 & 5 & 3 \\ 4 & 5 & 3 & 1 & 2 & 7 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 & 7 \end{bmatrix}.$$

We may further extend this result, as it has an appropriate transversal:

$$\begin{bmatrix} 2 & 1 & 7 & 6 & 3 & \boxed{4} & 5 \\ 6 & 3 & 2 & 7 & \boxed{5} & 1 & 4 \\ 5 & \boxed{6} & 4 & 3 & 7 & 2 & 1 \\ \boxed{1} & 7 & 6 & 5 & 4 & 3 & 2 \\ 7 & 4 & 1 & \boxed{2} & 6 & 5 & 3 \\ 4 & 5 & \boxed{3} & 1 & 2 & 7 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 & \boxed{7} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 7 & 6 & 3 & 8 & 5 & 4 \\ 6 & 3 & 2 & 7 & 8 & 1 & 4 & 5 \\ 5 & 8 & 4 & 3 & 7 & 2 & 1 & 6 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 4 & 1 & 8 & 6 & 5 & 3 & 2 \\ 4 & 5 & 8 & 1 & 2 & 7 & 6 & 3 \\ 3 & 2 & 5 & 4 & 1 & 6 & 8 & 7 \\ 1 & 6 & 3 & 2 & 5 & 4 & 7 & 8 \end{bmatrix}.$$

This final Latin square does not have a transversal suitable for Lemma 4.1.9. (It has a transversal with the 3, 7, 1, 6, 5, 4, 2, 8 from sucessive rows starting at the top, but the 5 is not permitted in Lemma 4.1.9). The lack of a nice transversal does not mean that there is no quasigroup with the desired directed power graph.

A similar construction extends a path at the initial end when a suitable transversal exists.

Lemma 4.1.11. *Suppose L is a Latin square of order n with partial entries as in Lemma 4.1.3 with a transversal which includes the position within the first column with entry 1 while avoiding the diagonal and super-diagonal. Replace all entries on the transversal except the 1 with 0 and complete the new first row and column using the Latin square property. Add 1 to every entry to make the set of entries 1, 2, ..., $n+1$. The resulting Latin square of order $n+1$ satisfies Lemma 4.1.3, that is to say, it is the Cayley table of a quasigroup whose power graph is a path on $n+1$ vertices.*

Example 4.1.12. We illustrate Lemma 4.1.11.

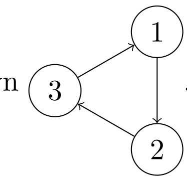
$$\begin{bmatrix} 2 & 1 & 5 & 6 & 3 & \boxed{4} \\ 6 & 3 & 2 & 4 & \boxed{5} & 1 \\ 5 & \boxed{6} & 4 & 3 & 1 & 2 \\ \boxed{1} & 2 & 6 & 5 & 4 & 3 \\ 3 & 4 & 1 & \boxed{2} & 6 & 5 \\ 4 & 5 & \boxed{3} & 1 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 & 3 & 2 & 5 & 4 \\ 4 & 2 & 1 & 5 & 6 & 3 & 0 \\ 5 & 6 & 3 & 2 & 4 & 0 & 1 \\ 6 & 5 & 0 & 4 & 3 & 1 & 2 \\ 0 & 1 & 2 & 6 & 5 & 4 & 3 \\ 2 & 3 & 4 & 1 & 0 & 6 & 5 \\ 3 & 4 & 5 & 0 & 1 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 7 & 4 & 3 & 6 & 5 \\ 5 & 3 & 2 & 6 & 7 & 4 & 1 \\ 6 & 7 & 4 & 3 & 5 & 1 & 2 \\ 7 & 6 & 1 & 5 & 4 & 2 & 3 \\ 1 & 2 & 3 & 7 & 6 & 5 & 4 \\ 3 & 4 & 5 & 2 & 1 & 7 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 & 7 \end{bmatrix}$$

Lemma 4.1.13. *No group of order greater than 2 has a path as its directed power graph.*

Proof. Ordering the vertices as in Lemma 4.1.3, we see the only possibility for the identity is n. However, the $(n-2, n)$-entry is not $n-2$ (which is in the $(n-2, n-1)$-position). $\square$

4.2 Cycles

Definition 4.2.1. A graph on n vertices is a *directed cycle* whenever its vertices $\{v_i\}_{i=1}^n$ can be ordered so that $v_1 \to v_2 \to \cdots \to v_{n-1} \to v_n \to v_1$ and it has no other edges.

Example 4.2.2. A directed cycle of size 3 can be drawn  .

Lemma 4.2.3. *Let G be the directed cycle on vertices $\{i\}_{i=1}^n$ with $1 \to 2 \to \cdots \to n-1 \to n \to 1$. Then G is the power graph of any quasigroup with partial Cayley table*

	1	2	3	$\cdots$		$n-1$	n
1	2	1					
2		3	2				
3			4	3			
$\vdots$				$\ddots$	$\ddots$		
$n-1$						n	$n-1$
n	n						1

.

Proof. Here $i \to i+1$ and nothing else for $i \leq n-1$, so $i \cdot i = i+1$ and $i \cdot (i \cdot i) = i$ for $i \leq n-1$. In addition, $n \to 1$, so $n \cdot n = 1$ and $n \cdot (n \cdot n) = n$. The result follows. □

Lemma 4.2.4. *No quasigroups of order 4 have a directed 4-cycle as their directed power graph.*

Proof. For order 4, observe that the Latin square property implies that the partial Cayley table of Lemma 4.2.3 has (1,4)-entry 3, allowing no admissible (2,4)-entry. □

Note that Q4 of Table 3.4 is of order 3 whose directed power graph is a directed cycle. The unique quasigroup of order 5 having the graph described in Lemma 4.2.3 is given in Example 4.2.5.

Example 4.2.5. The quasigroup of order 5 with partial Cayley table as in Lemma 4.2.3 whose directed power graph is a cycle has the following Cayley table:

$$\begin{bmatrix} 2 & 1 & 5 & 4 & 3 \\ 4 & 3 & 2 & 1 & 5 \\ 1 & 5 & 4 & 3 & 2 \\ 3 & 2 & 1 & 5 & 4 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

The cyclic nature of Lemma 4.2.3 and of Example 4.2.5 suggest the following.

Lemma 4.2.6. *For all odd integers n, there exists a quasigroup of order n whose power graph is a directed cycle.*

Proof. We take each row to be a cyclicly shifted form of $R = [n\ n-1\ n-2\ \cdots\ 3\ 2\ 1]$. The first row is R shifted 2 places right. As we move down from one row to the next, we shift two more places to the right. Because n is odd, shifting by 2 never puts 1 (or any other entry) in the same column twice. Thus we get a Latin square which satisfies the description of Lemma 4.2.3. $\square$

Conjecture 4.2.7. *There are quasigroups of all orders greater than 5 whose power graph is a directed cycle, not just odd order.*

Example 4.2.8. A quasigroup of order 6 whose directed power graph is a cycle:

$$\begin{bmatrix} 2 & 1 & 6 & 4 & 5 & 3 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 1 & 5 & 4 & 3 & 2 & 6 \\ 3 & 6 & 1 & 5 & 4 & 2 \\ 4 & 2 & 3 & 1 & 6 & 5 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{bmatrix}$$

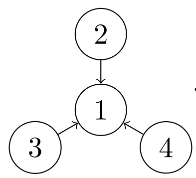
Lemma 4.2.9. *There is no group whose directed power graph is a directed cycle.*

Proof. When the vertices are ordered as in Lemma 4.2.3, there is no idempotent element (an entry i in the (i,i)-position) to serve as the identity. $\square$

4.3 In-stars

Definition 4.3.1. A graph on n vertices is an *in-star* whenever its vertices $\{v_i\}_{i=1}^n$ can be ordered so that there is an edge from v_i to v_1 for $2 \le i \le n$ and there are no other edges.

Example 4.3.2. An in-star can be drawn as



.

Lemma 4.3.3. *Let G be the in-star on vertices $\{i\}_{i=1}^n$ with $i \to 1$ for $2 \le i \le n$. Then G is the power graph of any quasigroup with partial Cayley table*

	1	2	3	$\cdots$	$n-1$	n
1	1					
2	2	1				
3	3		1			
$\vdots$	$\vdots$			$\ddots$		
$n-1$	$n-1$				1	
n	n					1

.

Proof. Here $i \cdot i = 1$ and $i \cdot (i \cdot i) = i \cdot 1 = i$ $(1 \le i \le n)$. $\square$

Example 4.3.4. Consider the group $G = Z_2 \times Z_2 \times \cdots \times Z_2$. For all $x \in G$, $x \cdot x = e$ with $e \cdot e = e$. In particular, the power graph of G is an in-star.

Lemma 4.3.5. *There is a quasigroup of every order $n \ge 2$ whose directed power graph is an in-star.*

Proof. We take each row to be a cyclicly shifted form of $R = [n\ n-1\ n-2\ \cdots\ 3\ 2\ 1]$. The first row is R shifted forward 1. As we move down from one row to the next, we shift one more place to the right. We get a Latin square which satisfies the description of Lemma 4.3.3. □

Note that Example 4.3.4 arises from a group, but the construction in the proof of Lemma 4.3.5 is never a group for $n > 2$ as there is no identity.

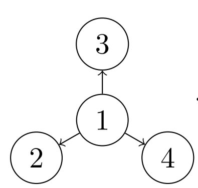
Lemma 4.3.6. *If G is a finite group whose directed power graph is an in-star, then G is an elementary abelian 2-group, and thus as in Example 4.3.4.*

Proof. The center c of the in-star satisfies $c \cdot c = c$ and and every other element i satisfies $i \cdot i = c$, $i \cdot c = i$. □

4.4 Out-Stars

Definition 4.4.1. A graph on n vertices is an *out-star* whenever its vertices $\{v_i\}_{i=1}^n$ can be ordered so that there is an edge from v_1 to v_i for $2 \leq i \leq n$ and there are no other edges.

Example 4.4.2. An out-star can be drawn as



.

Lemma 4.4.3. *Let G be the out-star on vertices $\{i\}_{i=1}^n$ with $1 \to i$ for $2 \leq i \leq n$. Then G is the* *power graph of any quasigroup with partial Cayley table*

$$\begin{array}{c|cccccc}
 & 1 & 2 & 3 & \cdots & n-1 & n \\
\hline
1 & 2 & 3 & & \cdots & n & 1 \\
2 & & 2 & & & & \\
3 & & & 3 & & & \\
\vdots & & & & & & \\
n-1 & & & & & n-1 & \\
n & & & & & & n
\end{array}.$$

Proof. For $i > 1$, $i \cdot i = i$ and $1 \cdot (1 \cdots 1)$ defines an n-cycle. $\square$

Lemma 4.4.4. *There is no quasigroup of order 3 whose directed power graph is an out-star.*

Proof. One can make the observation from Table 3.4. Alternatively, observe that the $(1, 3)$-entry is 1, the $(2, 2)$-entry is 2, and the $(3, 3)$-entry is 3, leaving no valid values for the $(2, 3)$-entry. $\square$

Example 4.4.5. Here are Cayley tables of quasigroups of order 4 and 5 whose directed power graph is an out-star: $\begin{bmatrix} 2 & 3 & 4 & 1 \\ 4 & 2 & 1 & 3 \\ 1 & 4 & 3 & 2 \\ 3 & 1 & 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 & 4 & 5 & 1 \\ 1 & 2 & 5 & 3 & 4 \\ 4 & 5 & 3 & 1 & 2 \\ 5 & 1 & 2 & 4 & 3 \\ 3 & 4 & 1 & 2 & 5 \end{bmatrix}$.

Conjecture 4.4.6. *There is a quasigroup of every order other than 1 and 3 whose directed power graph is an out-star.*

Lemma 4.4.7. *There is no group of order greater than 2 whose directed power graph is an out-star.*

Proof. The center generates the group, as its powers run over every element. But there must be $\phi(n) > 1$ generator in a cyclic group of order $n > 2$, which would have the same out neighbors as the center. $\square$

4.5 Empty graphs

Definition 4.5.1. A graph is an *empty graph* whenever it has no edges.

Lemma 4.5.2. *Let Q be a quasigroup with elements $\{i\}_{i=1}^{n}$. Then following are equivalent.*

1. *The power graph of Q is the empty graph.*

2. *Every element of Q is an idempotent.*

3. *Q has partial Cayley table*

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & \cdots & n-1 & n \\ \hline 1 & 1 & & & & & \\ 2 & & 2 & & & & \\ 3 & & & 3 & & & \\ \vdots & & & & \ddots & & \\ n-1 & & & & & n-1 & \\ n & & & & & & n \end{array}.$$

Proof. Observe that the power graph of Q is the empty graph if and only if $i \cdot i = i$. $\square$

Lemma 4.5.3. *There is a quasigroup of every odd order $n \geq 3$ whose directed power graph is an empty graph.*

Proof. We take each row to be a cyclic shift of the form of $R = [n\ n-1\ \cdots\ 3\ 2\ 1]$. The first row is R shifted right by 1. As we move down from one row to the next, we shift two more places to the right. Thus satisfying the the description of Lemma 4.5.2. $\square$

Example 4.5.4. There are other quasigroups of odd order with empty power graphs when the order is greater than 5. Here are some which are also commutative (having symmetric Cayley tables).

$$\begin{bmatrix} 1 & 5 & 2 & 3 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 2 & 4 & 3 & 5 & 1 \\ 3 & 1 & 5 & 4 & 2 \\ 4 & 3 & 1 & 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 & 5 & 4 & 7 & 6 \\ 3 & 2 & 1 & 6 & 7 & 4 & 5 \\ 2 & 1 & 3 & 7 & 6 & 5 & 4 \\ 5 & 6 & 7 & 4 & 1 & 2 & 3 \\ 4 & 7 & 6 & 1 & 5 & 3 & 2 \\ 7 & 4 & 5 & 2 & 3 & 6 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 & 7 \end{bmatrix}$$

Example 4.5.5. The restriction to odd order in Lemma 4.5.3 appears to be only due to the limitation of the proof idea, not the nonexistence of appropriate quasigroups.

$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

Conjecture 4.5.6. *There are quasigroups of all orders greater than 2 whose power graph is empty, not just odd orders.*

Chapter 5

Sinks in Power Graphs

In this chapter we comment on sinks in power graphs. In Section 5.1, we note that sinks arise naturally in connection with subquasigroups, although this is not the only way they arise. In Sections 5.2 and 5.3, we show that certain structures cannot arise as sinks in power graphs, and in Section 5.4 we show that any already realized power graph can arise as a sink in the power graph of a quasigroup of twice its size.

5.1 Sinks and subquasigroups

Definition 5.1.1. An induced subgraph S of a directed graph G is a *sink* whenever all edges of G with initial vertex in S have terminal vertex in S.

Definition 5.1.2. A *subquasigroup* of a quasigroup Q is a subset $S \subseteq Q$ which is a quasigroup with the operation induced by that of Q.

It is clear from definition that the following holds.

Lemma 5.1.3. *If S is a subquasigroup of a quasigroup Q, then the subgraph of the power graph of Q induced on S is a sink.*

Not all sinks in power graphs arise from subquasigroups. For example the union of two subgroups is not a subgroup in general, but the union of two sinks is a sink.

Definition 5.1.4. Given an element x of a quasigroup, let us refer to the left-powers x, $x \cdot x$, $x \cdot (x \cdot x)$, ... as the *left-cycle* of x. (In an associative quasigroup, this is the subquasigroup generated by x).

Lemma 5.1.5. *A sink in the power graph of a quasigroup is the union of the left-cycles generated by the elements of the sink.*

Proof. Suppose S is a sink, and let $x \in S$. Then the left-cycle of x is in S as there is an edge from x to each of its left-powers. Now S is a sink and therefore contains the terminal vertices (the left-powers of x). Also, each element is in its left-cycle and S is comprised of its elements. The result follows. $\square$

Lemma 5.1.6. *In an associative quasigroup, the union of left-cycles generated by any subset of elements is a sink in its power graphs.*

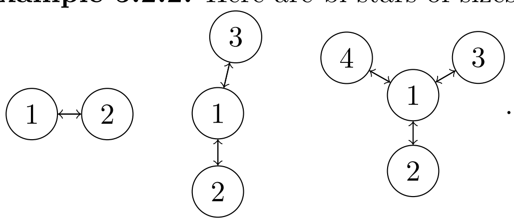
Proof. Under associativity, the left-cycle of any element is a subquasigroup, and thus it forms a sink in the (left) power graph. The union of sinks is a sink, so the result follows. $\square$

5.2 Forbidden Sinks

In this section, we observe that certain digraphs cannot arise as sinks in the power graph of any quasigroup. We begin with a fairly straight forward case. We refer to such a digraph as a *forbidden sink.* The Latin square property is somewhat restrictive and leads to several forbidden sinks.

Definition 5.2.1. A graph on n vertices is a *bi-star* whenever its vertices $\{v_i\}_{i=1}^{n}$ can be ordered so that there is an edge from v_i to v_1 and from v_i to v_1 for $2 \leq i \leq n$ and there are no other edges.

Example 5.2.2. Here are bi-stars of sizes 2, 3, and 4:



Lemma 5.2.3. *No power graph of a quasigroup contains a sink which is isomorphic to a bi-star.*

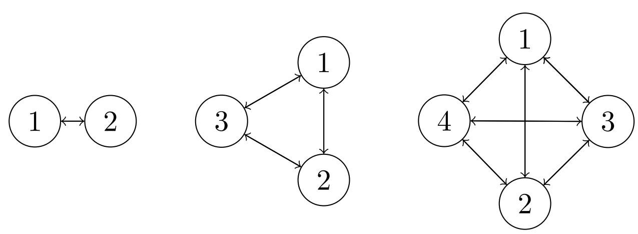
Proof. For $i > 1$, entry (i, i) is 1 and entry $(i, 1)$ is i, where as entry $(1, 1)$ cannot be 1, leaving no valid entries for the $(1, 1)$ position. $\square$

In fact, this proof gives a stronger result.

Corollary 5.2.4. *No power graph of a quasigroup contains a sink on vertices v_1, ..., v_n having edges $v_i \to v_1$ $(2 \leq i \leq n)$, at least one edge from v_1 to one of v_2, ..., v_n, and no edges $v_i \to v_j$ for $2 \leq i, j \leq n$ (a star).*

Definition 5.2.5. By a *complete digraph*, we mean a digraph in which there is an edge from every vertex to every other vertex.

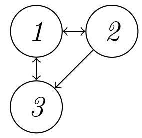
Example 5.2.6. The complete digraphs of orders 2, 3, and 4. The complete digraph of order 2 is a bi-star.



Theorem 5.2.7. *No quasigroup has a power graph containing a complete digraph of order greater than 1 as a sink.*

Proof. We label the elements of the complete graph of order n by 1, 2, ..., n. There is an edge from each vertex in the complete graph to every other and no others. Thus the first n entries of rows 1 through n are each the bottom row of an n-cycle on 1 through n writing the permutation in two-row form. In particular, i never appears in column i since this would imply a fixed point of the permutation. In all columns, there are only $n - 1$ valid entries for the the n rows, which is impossible. □

Complete digraphs do occurs as subgraphs, even though they do not arise as sinks. For example in a cyclic group, The set of generators induce a complete digraph, but the identity is adjacent to every other element.

Lemma 5.2.8. *The following is a forbidden sink:* 

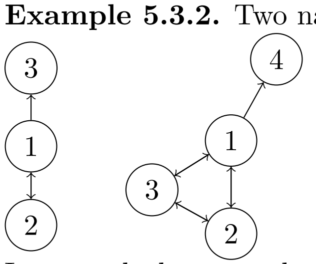
Proof. By exploring all possiblilties for the second example we observe that the $(1, 3)$ and the $(3, 3)$ are the same or the $(1, 1)$ entry and the $(3, 1)$ are the same. $\square$

5.3 Forbidden narrow escapes

Certain "near sinks" are also forbidden. Let us introduce a term to speak about this situtation.

Definition 5.3.1. By a *narrow escape* of a digraph G we mean a subset U such that with exactly one exception, all edges of G with initial vertex in U have terminal vertex in U. Suppose (v, g) is the unique edge with $v \in U$ and $g \notin U$. We say that the narrow escape is *via* v.

Example 5.3.2. Two narrow escapes from complete diagraphs via 1:



In general, there may be edges into the narrow escape, and the vertex to which it escapes may have other connections in the larger graph.

Checking the small examples, we find no complete digraphs appearing as narrow escapes.

Theorem 5.3.3. *No complete digraph of order greater than 1 appears as a narrow escape in the power graph of any quasigroup.*

Proof. We label the elements of the complete digraph of order n by 1, 2, ..., n. By symmetry, we assume that the narrow escape is via 1, and that there is an edge from 1 to $n+1$ (that is, the narrow escape is via 1). This is consistent with the graphs in Example 5.3.2 (ignoring other vertices, edges into the narrow escape and other connections to $n + 1$).

There is an edge from each vertex in the narrow escape to every other. Thus the first $n+1$ entries of the first row are the bottom row of an $(n+1)$-cycle on 1 through $n+1$ writing the permutation in two-row form, and the first n entries of rows 2 through n are each the bottom row of an n-cycle on 1 through n writing the permutation in two-row form. In particular, i never appears in column i since this would imply a fixed point of the permutation. In all but the escape column, there are only $n-1$ valid entries for the the n rows, which is impossible. $\square$

The proof idea can be pushed further, as even one column with no valid entry in the last row would suffice to block the configuration. Describing criteria to force this condition can become rather cumbersome. However, here is one that is fairly straightforward.

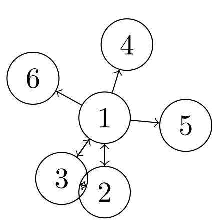
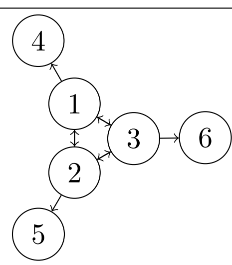
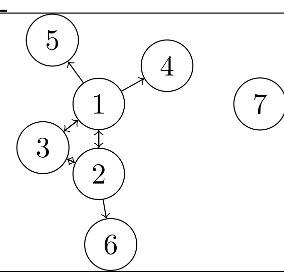
Theorem 5.3.4. *Let G be the power graph of a quasigroup. If K is complete directed subgraph of G of order greater than 1, then there are at least as many edges of G with an initial vertex in K and terminal vertex not in K as there are vertices in K.*

Proof. Suppose K has order $n > 1$. Label the elements of K by 1, 2, ..., n. Furthermore suppose there are at most $n-1$ edges of G with an initial vertex in K and terminal vertex not in K. Consider the entries of its Cayley table of the quasigroup. Among the first n columns, there are at most $n-1$ columns which contain an element other than 1 through n in the first n rows; in particular, there is a column i whose first n rows contains only entries 1 through n. However, column i cannot contain i by the argument in the proof of theorem 5.3.3. Since we cannot fill the n entries of column i with n distinct entries, we conclude that this situation is impossible. The result follows. $\square$

We view Theorem 5.3.4 as one of the major results of this thesis. It demonstrates how the Latin square property of the Cayley table of a quasigroup places restrictions of the structure of the power graph of the quasigorup. Theorem 5.3.4 does not directly forbid the existence of complete directed subgraphs; rather, it forbids complete directed subgraphs that do not have sufficiently many connections to the rest of the graph. We note that the theme of forbidden substructures is very important in modern graph theory.

Example 5.3.5. Theorem 5.3.4 does not specify how the edges exit the complete subdigraph. The complete subdigraph of the 6th entry of Table 3.5 has a complete subdigraph on vertices 2, 3. There are two vertices exiting at 2. The first quasigroup of Table 3.4 has a complete subdigraph on vertices 2, 3. There are two vertices exiting, one at each end.

Example 5.3.6. Here are quasigroups whose power graphs are complete digraph of order 3 with three configurations of three edges out.

$\begin{bmatrix} 4 & 5 & 6 & 2 & 3 & 1 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 1 & 6 & 5 & 4 & 2 & 3 \\ 6 & 4 & 3 & 1 & 5 & 2 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{bmatrix}$	$\begin{bmatrix} 2 & 4 & 1 & 3 & 6 & 5 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 3 & 1 & 6 & 5 & 4 & 2 \\ 6 & 2 & 5 & 4 & 3 & 1 \\ 1 & 6 & 4 & 2 & 5 & 3 \\ 4 & 5 & 3 & 1 & 2 & 6 \end{bmatrix}$	$\begin{bmatrix} 4 & 5 & 1 & 2 & 3 & 7 & 6 \\ 2 & 3 & 6 & 5 & 7 & 1 & 4 \\ 3 & 1 & 2 & 7 & 6 & 4 & 5 \\ 1 & 6 & 7 & 4 & 2 & 5 & 3 \\ 6 & 7 & 4 & 1 & 5 & 3 & 2 \\ 7 & 2 & 5 & 3 & 4 & 6 & 1 \\ 5 & 4 & 3 & 6 & 1 & 2 & 7 \end{bmatrix}$
		

It is unlikely that the directed graphs which can occur as the power graph of a quasigroup can be characterized by a list of (families of) forbidden substructures. We would expect that the order grows, new structures in the digraphs would arise as forbidden substructures.

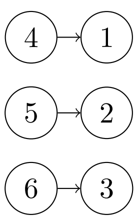
5.4 Realized Sinks

In this section we show that any power graph is realized as a sink in a power graph of twice its order.

Lemma 5.4.1. *Given 4 Latin squares L_1, L_2, L_3, L_4 of order n, form a Latin square of order $2n$ by adding n to every entry of L_2 and L_3 and forming a block array:* $L = \left[\begin{array}{c|c} L_1 & L_2(\textit{entry}+n) \\ \hline L_3(\textit{entry}+n) & L_4 \end{array}\right]$. *Then L_1 defines a sink of the directed power graph of L which is identical to that of L_1. Moreover, there must be an edge from every vertex in $\{n+1, n+2, \ldots, 2n\}$ to at least one vertex in $\{1, 2, \ldots, n\}$.*

Each of entries 1–4 and 8–11 of the table of quasigroups of order 4 can be decomposed into 4 latin squares of order 2.

Example 5.4.2. Given a quasigroup whose directed power graph is empty (Lemma 4.5.2), we can take four copies as follows: $\left[\begin{array}{c|c} \text{empty} & \text{empty(entry}+n) \\ \hline \text{empty(entry}+n) & \text{empty} \end{array}\right]$. The power graph of this quasigroup is n disjoint copies of an edge.



Example 5.4.3. The power graph of $\left[\begin{array}{c|c} \Gamma & * \\ \hline \text{empty(entry}+n) & \text{in-star center } k \end{array}\right]$ is a copy of Γ with n edges from vertices $n+1$, $n+2$, ..., $2n$ to vertex k.

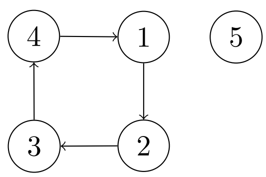
5.5 Sums of graphs

Definition 5.5.1. The *sum* of two graphs G and H, denoted $G + H$, is the graph whose edges and vertices are the disjoint unions of those of G and H.

Note that each of the summands is a sink. However, it is not obvious how to realize the sum of two graphs as a power graph. That is to say, this graph construction does not give any insight into the quasigroup.

Example 5.5.2. We can realize a directed 4-cycle and a single isolated vertex, even though the directed 4-cycle alone is not the power graph of any quasigroup.

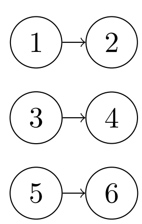
$$\begin{bmatrix} 2 & 1 & 5 & 4 & 3 \\ 1 & 3 & 2 & 5 & 4 \\ 5 & 2 & 4 & 3 & 1 \\ 4 & 5 & 3 & 1 & 2 \\ 3 & 4 & 1 & 2 & 5 \end{bmatrix}$$



We note that the substructure identified for sparse graphs in Chapter 3 can be combined (with appropriate shifts of entries). It then remains to check if the partial Latin squares can be completed.

Example 5.5.3. There is no quasigroup whose directed power graph consists of two disjoint directed edges. We may refer to section 3.4 or argue directly to see this. However, the sum of three paths of length 1 can be realized.

	1	2	3	4	5	6
1	2	1	3	6	5	4
2	1	2	6	5	4	3
3	5	6	4	3	1	2
4	6	3	5	4	2	1
5	3	4	1	2	6	5
6	4	5	2	1	3	6



5.6 Graphs with large size

In this section, we comment on when the power graph of a quasigroup has many edges (large size). We do so at this point as it fits well with the theme of realized sinks and graph sums. We begin with a simple result for undirected power graphs of groups.

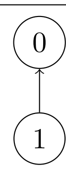
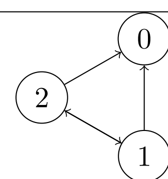
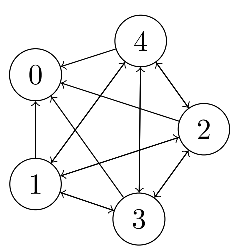
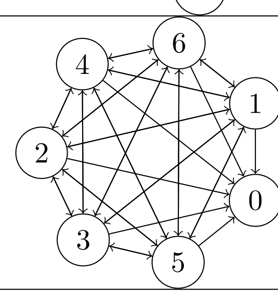
Theorem 5.6.1. *[16, Theorem 2.12] A finite group has a complete (undirected) power graph if and only if it is cyclic and has prime power order.*

The directed versions of the groups of prime order are not much more complicated.

Example 5.6.2. Consider the power graph of the additive group of Z_p for a prime p. The Cayley table of Z_p is a cyclotomic matrix of the form:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \dots & p-2 & p-1 \\ 1 & 2 & 3 & 4 & \dots & p-2 & p-1 & 0 \\ \\ p-1 & \dots & 0 & 1 & 2 & 3 & \dots & p-2 \end{bmatrix}$$

Table 5.8: Quasigroups of Prime Order and their power graphs

$p=2$		$p=3$	
$p=5$		$p=7$	

Lemma 5.6.3. *For any prime p, the power graph of the additive group on Z_p consists of a complete digraph on $\{1, 2, \dots, p-1\}$ with a directed edge from each of these vertices to the identity 0.*

Observe that there are $p^2 - 2p + 1$ many edges in the directed power graph of Z_p, whereas a complete digraph of size p has $p^2 - p$ many edges.

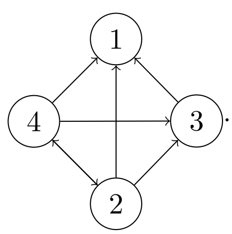
Cyclic groups of prime power order p^n also have a fairly straightforward directed power graph. There is a complete digraph on the generators (of which there are $\varphi(p^n) = (p-1)p^{n-1}$ many) which also have out-edges to every other element. There is a unique maximal subgroup of order p^{n-1}; all of its generators form a complete digraph and have out edges to every element other than the generators of the whole group. This continues recursively. See [27].

Theorem 5.6.4. *[26, Theorem 1.2] Among all finite groups of a given order, the cyclic group of that order has the maximum number of edges in its directed power graph. Moreover, the cyclic group of order n has $\left(\prod_{h=1}^{k} \frac{p_h^{2\alpha_h+1}+1}{p_h+1}\right) - n$ many edges, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.*

Among quasigroups, the cyclic groups do not always obtain the maximum number of edges in their directed power graphs. Nor is there necessarily a unique quasigroup obtaining the maximum.

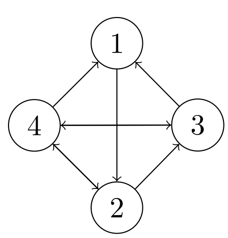
Example 5.6.5. The cyclic group of order 4 has 7 edges in its power graph:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

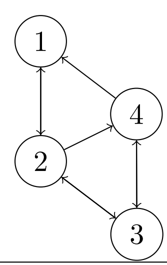


However, there are quasigroups order 4 with 8 edges in their power graphs.

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \\ 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \end{bmatrix}$$



Conjecture 5.6.6. *Among quasigroups of prime order, the cyclic group attains the maximum number of edges among the directed power graphs. Note: In general it will not be the unique quasigroup which attains this number. For example, the quasigroups Q_1 and Q_2 of order 3 Table 3.4 both have 4 edges. It is Q_1 that is a $\mathbb{Z}_3$.*

Examining Lemma 4.4.3, we note that we could make vertex k a source in the power graph by placing into row k the second row of the 2-row permutation representation of an n-cycle with the property that the kth entry is not k.

Chapter 6

Bol Loops

In this chapter, we briefly turn our attention to more group-like quasigroups, namely to loops and Bol loops.

6.1 Loops

Definition 6.1.1. A *loop* is a quasigroup with an identity.

Lemma 6.1.2. *Let Q be a quasigroup with elements $\{i\}_{i=1}^{n}$. Then following are equivalent.*

1. *Q is a loop and 1 is the identity of Q.*

2. *Q has partial Cayley table*

$$\begin{array}{c|cccccc}
 & 1 & 2 & 3 & \cdots & n-1 & n \\
\hline
1 & 1 & 2 & 3 & \cdots & n-1 & n \\
2 & 2 & & & & & \\
3 & 3 & & & & & \\
\vdots & & & & & & \\
n-1 & n-1 & & & & & \\
n & n & & & & &
\end{array}.$$

Proof. Clear from the definition of identity. $\square$

Lemma 6.1.3. *In the directed power graph of a finite loop, there is an edge from every non-identity* *element to the identity and there are no edges from the identity to any other element.*

Proof. The products x, $x \cdot x$, $x \cdot (x \cdot x)$ will eventually have a repeated x, which must have been preceded by the identity in light of Lemma 6.1.2. □

Given a Latin square we may permute the rows, columns, and entry to produce another Latin squares. Latin squares related by these operations are said to be *isotopic*. Every Latin square is isotopic to a Latin square satisfying the partial description of Lemma 6.1.2. Note that the isotopy operations do not preserve the algebraic structure of associated quasigroups.

6.2 Bol Loops

We will turn our attention to Bol loops, where we have self-associativity.

Definition 6.2.1. Let Q be a quasigroup.

1. Q is said to be *right Bol* whenever $((ca)b)a = c((ab)a)$ for all a,b,$c \in Q$.

2. Q is said to be *left Bol* whenever $a(b(ac)) = (a(ba))c$ for all a,b,$c \in Q$.

3. Q is said to be *Bol* whenever it is left or right Bol.

4. Q is said to be *Moufang* whenever is it both left and right Bol.

These properties can be seen as a generalization of the associativity property. An associative Moufang loop is a group.

Lemma 6.2.2. *A Bol loop Q is self-associative, that is to say $(aa)a = a(aa)$ for all $a \in Q$.*

Proof. Set $b = 1$ $c = a$ in $((ca)b)a = c((ab)a)$ to get $((aa)1)a = a((a1)a)$, so $(aa)a = a(aa)$. □

In Bol loops, the self-associativity removes ambiguity in powers, so there is a unique power graph constructed from the Bol loop.

Theorem 6.2.3. *A left Bol loop has the property that the subloop generated by any element is a subgroup.*

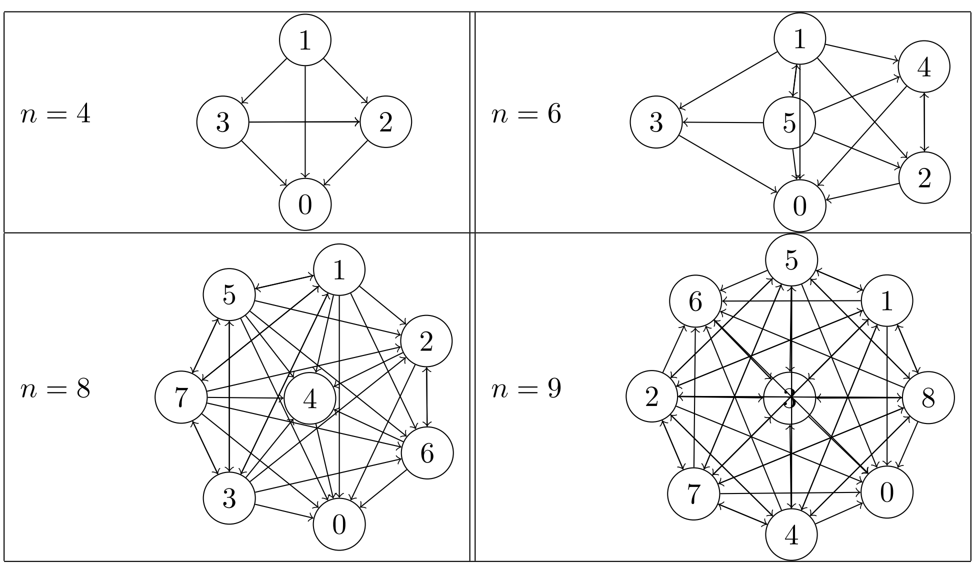
In light of Theorem 6.2.3, each element has left and right inverse in the Bol loop, namely its inverse in the subgroup it generates. The size of the subgroup it generates is its order.

Definition 6.2.4. Given a digraph G and an element $g \in G$, the *out-neighbors* of g is the set $N^+(g) = \{h \in G \mid g \to h\}$ and and the *neighborhood* of g is the set $\bar{N}^+(g) = \{g\} \cup N^+(g)$.

Corollary 6.2.5. *Let Q be a Bol loop. For all $g \in Q$, the neighborhood of g is the directed power graph of a cyclic group with order that of g in Q.*

The induced graph on every neighborhood in the power graph of a Bol loop is the power graph of some cyclic group. In Table 6.9, we draw the power graphs of cyclic groups of small order. In light of Example 5.6.2, we don't need to draw those of prime order. The identity is the unique element whose neighborhood is a single vertex.

Table 6.9: Cyclic groups and their power graphs



Chapter 7

Computation

Because the numbers of quasigroups and digraphs grow rapidly, we discuss some publicly available databases and some *Mathematica* code that we developed to draw and study power graphs of quasigroups.

7.1 Databases of Latin Squares

We take advantage of the following publicly available databases:

Brendan McKay Small Latin Squares	https://users.cecs.anu.edu.au/~bdm/data/latin.html
Eric Moorhouse Small Bol Loops	http://ericmoorhouse.org/pub/bol/
GAP Small groups	https://www.gap-system.org/Manuals/pkg/ SmallGrp-1.3/doc/chap1.html#X8398F2577B719D99

McKay's database consist of lists of representatives of various classes of Latin squares, such as representatives of reduced, isotopy classes, main classes. For many combinatorial purposes, this is sufficient, but some work is required to obtain a Cayley table for each isomorphism class of quasigroup. Here are the first few reduced Latin squares of order 6 from the appropriate database.

```
012345 103254 234501 325410 450123 541032

012345 103254 234501 325410 450132 541023

012345 103254 234510 325401 450123 541032

012345 103254 234510 325401 450132 541023
```

⋮

Each line is a latin square, with rows separated by spaces. Some processing is applied to the loaded line to convert the data into an appropriate data structure.

Moorhouse's data is delimited differently, but can still readily usable. For example, one of the Bol loops of order 8 is the following.

```
0 1 2 3 4 5 6 7

1 0 3 2 5 4 7 6

2 3 1 0 6 7 5 4

3 2 0 1 7 6 4 5

4 5 7 6 1 0 2 3

5 4 6 7 0 1 3 2

6 7 4 5 3 2 1 0

7 6 5 4 2 3 0 1
```

7.2 Data structures

To implement procedures in *Mathematica* or other computer algebra system, we make a few conventions.

Definition 7.2.1. By an IIE (integer index and entry) Latin square we mean a Latin square of order n for which the set $\{1, 2, \ldots, n\}$ serve as the indices for the rows and columns and as the entries.

Recall that we view Latin squares as Cayley tables of quasigroup. So with the IIE convention the elements of the quasigroup are $\{1, 2, \ldots, n\}$ and the product of $i * j$ is the ij-entry of L. The Python-based computer algebra system *Sage* begins indexing from 0, so had we chosen to use it, we would have appropriately modified this convention.

In *Mathematica*, we represent the Cayley tables as lists of lists, a natural structure in this program. Figure 7.3 gives a Latin square and its *Mathematica* list of lists presentation.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 3 & 7 & 5 & 6 \\ 3 & 7 & 5 & 6 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 & 6 & 7 & 5 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \\ 6 & 5 & 7 & 2 & 4 & 1 & 3 \\ 7 & 4 & 6 & 5 & 2 & 3 & 1 \end{bmatrix} \quad \begin{array}{|l} \{\{1, 2, 3, 4, 5, 6, 7\}, \\ \{2, 1, 4, 3, 7, 5, 6\}, \\ \{3, 7, 5, 6, 1, 4, 2\}, \\ \{4, 3, 2, 1, 6, 7, 5\}, \\ \{5, 6, 1, 7, 3, 2, 4\}, \\ \{6, 5, 7, 2, 4, 1, 3\}, \\ \{7, 4, 6, 5, 2, 3, 1\}\} \end{array}$$

Figure 7.3: A Cayley table and its list of lists representation

When the elements of a quasigroup have natural names, we may present them with these names rather than in an IIE form. For example, the additive group of a the integers mod n more naturally named by 0, 1, ..., $n-1$.

7.3 *Mathematica* Code for directed power graphs of quasigroups

We find the out-neighbors via iterated left multiplication with the following simple *Mathematica* code. It takes as input a IIE Latin square L and an index x. It successively computes $x * x$, $(x * x) * x$, etc. until it once again reaches x, storing each of the powers in a list.

```
leftneighbors[L_, x_]:=Module[{current = L[[x, x]], powers = {}},
 While[current ≠ x,
  AppendTo[powers, current];
  current = L[[x, current]]];
Return[powers]]
```

This code is readily modified to return directed edges. Here **DirectedEdge** is the *Mathematica* data type that can be used to construct a displayable graph.

```
leftoutedges[L_, x_]:=Module[{current = L[[x, x]], edgelist = {}},
While[current ≠ x,
 AppendTo[edgelist, DirectedEdge[x, current]];
```

```
current = L[[x, current]]];
Return[edgelist]]
```

The left power graph is constructed by combining the lists of left out-edges for all elements. The input is simply an IIE Latin square L.

```
leftgraph[L_]:=Module[{elist = {}, b},
For[b = 1, b ≤ Length[L], b++,
 PrependTo[elist, leftoutedges[L, b]]];
Return[Flatten[elist]]]
```

7.4 *Mathematica* code for filling partial Latin squares

In general completing a partial Latin square is and NP-complete problem. However, it is fairly elementary to write a backtracking routine to run through all possibilities to fill a partial Latin square.

We assume the entries of a Latin square of order n are 1,2, ..., n. We designate a distingushed symbol for empty entries in the partial Latin square (here we use 0):

```
EmptyEntry = 0;
```

Given a partial Latin square L and coordinates i,j, return a list of entries not yet used in row i and column j of L

```
FeasibleEntry[L_, i_, j_]:=Module[{n = Length[L]},
If[L[[i, j]] ≠ EmptyEntry, L[[i, j]], Complement[Range[n], L[[i]], L[[All, j]]]]]
```

We give a routine that will scan through all positions and 1) leave determined entries alone, 2) fill in any entries uniquely determined by row/column entries. It repeats until it can't proceed further.

```
FillDeterminedEntries[L_]:=
Module[{F = L, n = Length[L], current, stable = False, numdone = 0},
While[Not[stable],
```

```
stable = True;
Do[ Do[
current = FeasibleEntry[F, i, j];
If[current == {}, stable = True; Break[]];
If[Length[current] == 0, numdone++];
If[Length[current] == 1, F[[i, j]] = current[[1]];
 numdone++; stable = False] ,
{j, 1, n}], {i, 1, n}];
If[numdone == n^2, stable = True];
];
Return[F]]
```

We are ready to give a backtrack procedure to try to fill a partial Latin square. We will use a global parameter to stop the backtrack after finding a specified number of fills.

```
count = 0;
StopAfter = 4;
BacktrackFill[L_]:=Module[{F = FillDeterminedEntries[L], currentpos, feaslist},
currentpos = FirstPosition[F, EmptyEntry];
If[count ≥ StopAfter, Return[Null]];
(* If enough are found, don't proceed *)
If[currentpos==Missing["NotFound"], Print[MatrixForm[F]]; count++;
Return[F]];
(* If no entries are empty, increase counter and return filled Latin square *)
feaslist =  FeasibleEntry[F, currentpos[[1]], currentpos[[2]]];
Do[F[[currentpos[[1]], currentpos[[2]]]] = x;
 BacktrackFill[F];  If[count ≥ StopAfter, Break[]], {x, feaslist}]]
(* Loop through feasible entries for current position, fill it and try to extend recursively *)
```

We can run the backtrack routine on some of the partial Latin squares that we will meet in the next section. For now, these can be taken as partial Latin squares with few entries filled in.

Example 7.4.1. The following function returns a partial Latin square with the property that any quasigroup with a Cayley table with these entries has directed power graph which is a directed path (see section 4.1).

```
path[k_]:=Module[{F=DiagonalMatrix[Range[k] + 1]+DiagonalMatrix[Range[k−1], 1]},
F[[k, k]]=k; F]
MatrixForm[path[3]]
```

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Lifting the restriction of StopAfter, we can use count to find the number of ways to complete a given partial Latin square. For example there are 1057 completions of **path[7]**. In general the number of completions of the sparse partial Latin squares considered here grow too large to work with.

Example 7.4.2. The following function returns a partial Latin square with the property that any quasigroup with a Cayley table with these entries has directed power graph which is a directed cycle (see section 4.2).

```
cycle[k_]:=Module[{F=DiagonalMatrix[Range[k] + 1]+DiagonalMatrix[Range[k − 1], 1]},
F[[k, 1]] = k; F[[k, k]] = 1; F]
cycle[4]
```

{{2, 1, 0, 0}, {0, 3, 2, 0}, {0, 0, 4, 3}, {4, 0, 0, 1}}

It turns out that the partial Latin square **cycle[4]** cannot be extended to a complete Latin square. This is reported as follows.

```
count = 0; MatrixForm[BacktrackFill[cycle[4]]]
```

Null

Example 7.4.3. The following function returns a partial Latin square with the property that any quasigroup with a Cayley table with these entries has directed power graph which is an out-star (see section 4.4).

Outstar[k_]:=Module[{F = DiagonalMatrix[Range[*k*]]}, F[[1]] = RotateLeft[Range[*k*]]; F]
MatrixForm[Outstar[6]]

$$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Example 7.4.4. The following function returns a partial Latin square with the property that any quasigroup with a Cayley table with these entries has directed power graph which is empty (see section 4.5).

empty[k_]:=DiagonalMatrix[Range[*k*]]
count = 0; MatrixForm[BacktrackFill[empty[4]]];

$$\begin{pmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

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About the Author

DayVon Lamont Walker, born June 18th 1994 is the second of six kids from Sharon and Yusef Walker. DayVon was raised in Bronx, New York were he attended PS 175 for most of his elementary and all of middle school. He graduated from High School of Math, Science and Engineering at New York City College, one of the nine specialized high schools in New York City. DayVon then moved down to Tampa, Florida to attend the University of Tampa. During his time there Dayvon took an active role in many organizations such as President of the Math Club, President of Pi Mu Episilon, President of the Career Honor Society, and Recruitment chair for Tau Kappa Epsilon fraternity. After DayVon received his Bachelors of Science in Mathematics, he chose to continue his studies in pursuit of his Masters of Art in pure and applied mathematics, all while working as an AVID tutor at Rampello Magnet School and Lead instructor at Mathnasium of South Tampa.