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Generalized Derivations of Ternary Lie Algebras and n-BiHom-Lie Algebras

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by

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Dedication

This doctoral dissertation is dedicated to my mother, father, brother, sister and wife.

To my brother *Chihab* may his soul rest in peace.

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All praise is to Allah, the Lord Of The Creation.

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ABSTRACT

We generalize the results of Leger and Luks and other researchers about generalized derivations to the cases of ternary Lie algebras and n-BiHom Lie algebras. We investigate the derivations algebras of ternary Lie algebras induced from Lie algebras, we explore the subalgebra of quasi-derivations and give their properties. Moreover, we give a classification of the derivations algebras for low-dimensional ternary Lie algebras.

For the class of n-BiHom Lie algebras, we study the algebras of generalized derivations and prove that the algebra of quasi-derivations can be embedded in the derivation algebra of a larger n-BiHom Lie algebra.

1 Introduction

In this chapter, we briefly introduce the Hom and BiHom algebraic structures, n-ary algebras and the notion of generalized derivations. We will also state the definitions that we use throughout this dissertation.

For many years the algebras of derivations and generalized derivations have been a subject of study by many researchers. For instance, derivations appear in the study of Hochschild cohomology; in fact, the first cohomology group is the group of derivations modulo the inner derivations and it is called the group of outer derivations. Generalized derivations are particularly important in the study of Lie algebras and their generalizations like Lie superalgebras. Also, generalized derivations have held a central position in the theory of deformations of algebraic structures.

In this dissertation, we mean by a derivation of an algebra $\mathcal{A}$ a linear map D that satisfies the Leibniz rule:

$$D(ab) = D(a)b + aD(b). \tag{1.0.1}$$

Over the years, twisted versions of the Leibniz rule were used to define different generalizations of derivations: δ-derivations [14, 25, 26, 27], σ-derivations, (σ, τ)-derivations, (α, β, γ)-derivations [39], and α^k-derivations are some examples among the generalized derivations that were studied. In these general notions of derivations, the rule (1.0.1) was deformed using one or many parameters as in δ-derivations, (α, β, γ)-derivations and in other cases, endomorphisms are used to twist the standard definition like the σ-derivations and the (σ, τ)-derivations.

Definition 1.0.1 Let $\mathcal{A}$ be an algebra and $\sigma : \mathcal{A} \to \mathcal{A}$ a morphism. A σ-derivation of $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ that satisfies the σ-Leibniz rule

$$D(x.y) = D(x).y + \sigma(x).D(y) \tag{1.0.2}$$

for all $x, y \in \mathcal{A}$.

Among the most important and widely studied σ-derivations are the Jackson q-derivative and the shifted difference operator. These examples lie at the foundations of q-analysis and are extensively investigated in physics and engineering.

Example 1.0.2 *The Jackson q-derivative operator ∂_σ :*

$$\partial_\sigma(a)(t) = \frac{a(qt) - a(t)}{(q-1)t}$$

with $\sigma(f)(t) = f(qt)$,

Example 1.0.3 *The shifted difference operator:*

$$\partial_\sigma(a)(t) = a(t+1) - a(t)$$

with $\sigma(f)(t) = f(t+1)$.

If instead of one morphism σ we use two maps to twist (1.0.1), we get a more general definition:

Definition 1.0.4 Let $\mathcal{A}$ be an algebra and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ two morphisms. A (σ, τ)-derivation of $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ that satisfies the (σ, τ)-Leibniz rule

$$D(x.y) = D(x).\tau(y) + \sigma(x).D(y) \tag{1.0.3}$$

for all $x, y \in \mathcal{A}$.

One particular area where σ-derivations play a crucial role is the deformation theory of mathematical structures as in [16, 30, 32]. In [32] for example, Larsson and Silvestrov used σ-derivations to construct a quasi-deformations of the simple 3-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{F})$. Recall that $\mathfrak{sl}_2(\mathbb{F})$, as a vector space, is generated by the elements H, E, F subject to the brackets:

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

To quasi-deform $\mathfrak{sl}_2(\mathbb{F})$, they first considered a representation $\mathcal{A}$ in terms of first order differential operators acting on some vector space of functions in the variable t:

$$E \mapsto \partial, \qquad H \mapsto -2t\partial, \qquad F \mapsto -t^2\partial.$$

Then they replaced ∂ by a σ-derivation ∂_σ to obtain:

$$[H, E]_\sigma = 2\partial_\sigma(t)\partial_\sigma, \qquad [H, F]_\sigma = 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \qquad [E, F]_\sigma = -(\sigma(t) + t)\partial_\sigma(t)\partial_\sigma,$$

The new bracket $[\cdot, \cdot]_\sigma$ satisfies a twisted Jacobi identity and the obtained algebra is of type quasi-hom Lie. Note that the new quasi-hom Lie algebra structure is not defined on $\mathfrak{sl}_2(\mathbb{F})$ but on $\mathcal{A}.\partial_\sigma$. If a different base algebra $\mathcal{A}$ is chosen (σ also will change since it is dependent on $\mathcal{A}$), we get a different structure on $\mathcal{A}.\partial_\sigma$. In other words, the deformation is based on two parameters, namely $\mathcal{A}$ and σ.

In this dissertation we are interested in studying some special generalizations of derivations, mainly, generalized derivations, quasi-derivations, the centroid and the quasi-centroid.

It has been shown that for some special cases, it is possible to induce a post-Lie algebra structure by generalized derivations. In [7], D. Burde and K. Dekimpe proved that under some suitable conditions on a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, a post-Lie algebra structure can be induced by a quasi-derivation of $\mathfrak{n}$. In the following, we recall the definition of a post-Lie algebra and the results introduced by the authors in their article.

Definition 1.0.5 Let $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{n}, \{\cdot, \cdot\})$ be two Lie algebras on a vector space V. A post-Lie algebra structure on the pair $(\mathfrak{g}, \mathfrak{n})$ is a bilinear product $x \cdot y$ such that for all

$x, y, z \in V$, the following identities hold

$$x \cdot y - y \cdot x = [x, y] - \{x, y\};$$

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z);$$

$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}.$$

D. Burde and K. Dekimpe proved that if $\mathfrak{n}$ is semisimple, then the product $x \cdot y$ is given by a linear map $\varphi : \mathfrak{g} \to \mathfrak{n}$ that satisfies some conditions involving the brackets of $\mathfrak{g}$ and $\mathfrak{n}$. But if in addition, the Lie bracket of $\mathfrak{g}$ is given as a linear function of the Lie bracket of $\mathfrak{n}$, then φ is indeed a quasi-derivation of $\mathfrak{n}$.

Proposition 1.0.6 *[7] Let $x \cdot y$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{n}$ semisimple, and $x \cdot y = \{\varphi(x), y\}$ for some $\varphi \in End(V)$. Assume that $[x, y] = \tau(\{x, y\})$ for some $\tau \in End(V)$, then φ is a quasi-derivation of the Lie algebra $\mathfrak{n}$.*

In [33], it was shown that if $(L, [\cdot, \cdot])$ is a simple Lie algebra of a rank at least 2, then $QDer(L) = ad(L) \oplus \mathbb{C} \cdot id$, where $QDer(L)$ is the space of quasi-derivations, and $ad(L)$ is the set of adjoint maps of L. That is, for $u \in L$, $ad(u) = [u, \cdot]$. Using this fact with the previous result, one can conclude a complete description of a post-Lie structure on special pairs of Lie algebras.

Proposition 1.0.7 *[7] Suppose that $x \cdot y$ is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where n is simple of a rank at least 2. If $[x, y] = \tau(\{x, y\})$ for some $\tau \in End(V)$, then*

$$x \cdot y = \{\{z, x\}, y\} + \lambda\{x, y\}$$

for some $z \in \mathfrak{n}$ and some $\lambda \in \mathbb{C}$.

1.1 Generalized Derivations

In this section, we introduce the types of generalized derivations that we will be studying in this dissertation.

Definition 1.1.1 Let $(\mathcal{A}, \mu)$ be an algebra. A linear map $D : \mathcal{A} \to \mathcal{A}$ is called a *derivation* of $\mathcal{A}$ if for any $a, b \in \mathcal{A}$, the following identity holds

$$\mu(D(a), b) + \mu(a, D(b)) = D(\mu(a, b)).$$

The space of the derivations of $(\mathcal{A}, \mu)$ is denoted by $Der(\mathcal{A})$.

Remark: As the space $End(\mathcal{A})$ of linear maps of $\mathcal{A}$, equipped with the commutator $[f_1, f_2] = f_1 f_2 - f_2 f_1$ is a Lie algebra denoted by $\mathfrak{gl}(\mathcal{A})$, the set $\mathrm{Der}(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{A})$ since $[Der(\mathcal{A}), Der(\mathcal{A})] \subseteq Der(\mathcal{A})$.

Now let $\Delta(\mathcal{A})$ be the subset of $End(\mathcal{A}) \times End(\mathcal{A}) \times End(\mathcal{A})$ defined by:

$$\Delta(\mathcal{A}) = \{(f, f', f'') \in End(\mathcal{A})^3 : \mu(f(a), b) + \mu(a, f'(b)) = f''(\mu(a, b))\}.$$

Definition 1.1.2 A linear map D of $(\mathcal{A}, \mu)$ is called a *generalized derivation* if there exists two maps D' and D'' such that $(D, D', D'') \in \Delta(\mathcal{A})$. We denote by $GDer(\mathcal{A})$ the space of generalized derivations of $\mathcal{A}$.

Definition 1.1.3 A linear map D of $(\mathcal{A}, \mu)$ is called a *quasi-derivation* if there exists a map D' such that $(D, D, D') \in \Delta(\mathcal{A})$. We denote by $QDer(\mathcal{A})$ the space of quasi derivations of $\mathcal{A}$.

Remark: The terms *generalized derivation* and *quasi-derivation* are not new and have been used with different defining conditions in many articles [17, 42, 18].

Remark: It is clear that any quasi-derivation is a generalized derivation, moreover, if D is a derivation, then (D, D, D) is in $\Delta(\mathcal{A})$. Therefore $Der(\mathcal{A})$ is a subset of $\mathrm{QDer}(\mathcal{A})$ which is itself a subset of GDer($\mathcal{A}$). We have therefore the following inclusions

$$Der(\mathcal{A}) \subset QDer(\mathcal{A}) \subset GDer(\mathcal{A}).$$

Example 1.1.4 *Let* $(\mathfrak{g}, [\cdot,\cdot])$ *be the 2-dimensional Lie algebra with* $\mathfrak{g} = Span\{X, Y\}$ *and such that* $[X, Y] = Y$. *Let a map* $D : \mathfrak{g} \to \mathfrak{g}$ *be defined by* $D(X) = 0$ *and* $D(Y) = X$. *A direct computation shows that* D *is a quasi-derivation of* $\mathfrak{g}$.

Example 1.1.5 *Let* V *be a vector space with a basis* $\mathcal{B} = \{e_0, e_1, e_2, e_3, e_4, e_5\}$. *Define the bracket* $[\cdot,\cdot]$ *by:*

$$[e_0, e_1] = e_1;\ [e_0, e_3] = e_3;\ [e_0, e_5] = e_5;\ [e_1, e_2] = e_5;\ [e_3, e_4] = e_5,$$

the omitted brackets are zeros. The map $f \in End(V)$ *given by* $f(e_1) = -e_4$, $f(e_3) = e_2$ *and* $f(e_i) = 0$ *for* $i \neq 1, 3$ *is a quasi-derivation of* $(V, [\cdot,\cdot])$.

Definition 1.1.6 The *centroid* $C(\mathcal{A})$ of $(\mathcal{A}, \mu)$ is the space of linear maps $D \in End(\mathcal{A})$ such that

$$D(\mu(a, b)) = \mu(D(a), b) = \mu(a, D(b))$$

for all $a, b \in \mathcal{A}$.

Definition 1.1.7 The *quasi-centroid* $QC(\mathcal{A})$ of $(\mathcal{A}, \mu)$ is the space of linear maps $D \in End(\mathcal{A})$ such that

$$\mu(D(a), b) = \mu(a, D(b))$$

for all $a, b \in \mathcal{A}$.

The most important research on the algebras of generalized derivations of Lie algebras and their subalgebras is the article of G.F. Leger and E. Luks [33], where the authors studied the structure and properties of the algebras of generalized derivations, quasi-derivations, the centroid and quasi-centroid of finite-dimensional Lie algebras. For a Lie algebra L with toral Cartan subalgebras, they established the equality $QDer(L) = Der(L) + C(L)$. They gave a characterization on Lie algebras for which $QC(L) = C(L)$ and $GDer(L) = QDer(L)$. Moreover, the authors described sufficient conditions so that $GDer(L) = \mathfrak{gl}(L)$ and $QDer(L) = \mathfrak{gl}(L)$. The results of Leger and Luks were generalized by many researchers to other classes of algebras. For example, Chen, Ma and Li [12] studied the generalized derivations of color Lie algebras; Zhou and Fan considered the cases of Hom-Lie color algebras [51] and n-Hom Lie superalgebras [50]; Generalized derivations of Hom-Lie algebras were investigated in [49]. Kaygorodov and Popov explored generalized derivations of color n-ary Ω-algebras in [21]. For more on the generalized derivation algebras, the reader will be referred to [48, 29, 46, 47, 11].

In this thesis we extend some of the results of Leger and Luks to the cases of ternary Lie algebras and n-BiHom-Lie algebras.

1.2 The Class of BiHom-Lie Algebras

The deformation theory of algebraic, analytic and geometric structures has been a field of active research for mathematicians and physicists for many decades. The main purpose of deforming an object is to construct a more general structure which in most of the cases belongs to the same category. This is especially true in the case of Lie algebras. In this spirit and motivated by the study of the quantum deformations (also called q-deformations) of the Witt and Virasoro algebras, Hartwig, Larsson and Silvestrov developed an approach of deformation based on σ-derivations in [16]. They introduced the class of Hom-Lie algebras as deformed Lie algebras, where the defining Jacobi identity is twisted by a single map.

Definition 1.2.1 [16] A *Hom-Lie* algebra is a triple $(L, [\cdot,\cdot], \alpha)$ where L is a vector space, $[\cdot,\cdot]$ a bilinear skew symmetric mapping on $L \times L$ and $\alpha : L \to L$ a linear map such that the following deformed Jacobi identity is satisfied

$$[\alpha(x), [y, z]] + [\alpha(z), [x, y]] + [\alpha(y), [z, x]] = 0,$$

for all $x, y, z \in L$. Note that this version of the Jacobi identity is called the *hom-Jacobi* identity.

The reason behind the use of the term "Hom-algebras" for this new class of algebras is, as Silvestrov explained, the fact that the map used to deform the original algebra is a homomorphism.

Example 1.2.2 *(Jackson $\mathfrak{sl}_2$) The Jackson $\mathfrak{sl}_2$ is a classical example of a Hom-Lie algebra that is a q-deformation of the Lie algebra $\mathfrak{sl}_2$ by a Jackson derivation. Let $\{x_1, x_2, x_3\}$ be the basis of $\mathfrak{sl}_2$ and q be a parameter in $\mathbb{K}$. The bracket and the map α are given by:*

$$\begin{aligned}
[x_1, x_2] &= -2qx_2, & \alpha(x_1) &= qx_1,\\
[x_1, x_3] &= x_3, & \alpha(x_1) &= q^2x_2,\\
[x_2, x_3] &= -\frac{1}{2}(1+q)x_1, & \alpha(x_1) &= qx_3.
\end{aligned}$$

When $q = 1$, we recover the classical $\mathfrak{sl}_2$.

The appearance of Hom-Lie algebras opened the doors to a significant research activity on Hom-type algebras over the past few years and it is still a growing field. A more general class named *quasi-Lie* algebras and a subclass called *quasi-Hom-Lie* algebras were introduced in [31, 30]. Let us recall the definition and show how quasi-Hom-Lie algebras and Hom-Lie algebras can be recovered from quasi-Lie algebra.

Definition 1.2.3 [31] Let L be a vector space. A *quasi-Lie* algebra structure on L is a tuple $(L, \langle\cdot,\cdot\rangle, \alpha, \beta, \theta, \omega)$ where $\langle\cdot,\cdot\rangle : L \times L$ is a bilinear bracket on L; α, β, two linear maps; and $\theta, \omega : D_\theta, D_\omega \subseteq L \times L \to \mathscr{L}(L)$ such that

$$\langle x, y\rangle = \omega(x, y)\langle y, x\rangle \quad \text{for } (x, y) \in D_\omega,$$

$$\circlearrowleft_{x,y,z} \theta(z, x)(\langle\alpha(x), \langle y, x\rangle\rangle + \beta\langle x, \langle y, x\rangle\rangle) = 0$$

for $(z, x), (y, z), (x, y) \in D_\theta$.

Taking $\theta = \omega$ in the previous definition gives the class of *quasi-hom-Lie* algebras and if in addition one takes $\theta = \omega = -id$ and $\beta = id$, then one gets the class of hom-Lie algebras, finally, a hom-Lie algebra with $\alpha = id$ is a Lie algebra. In other words, quasi-hom-Lie algebras, hom-Lie algebras and Lie algebras are all subclasses of quasi-Lie algebras. In fact, the class of quasi-Lie algebras includes also a subclass of color Lie algebras and super-Lie algebras. This can be summarized in the following scheme

$$\textit{Lie} \subset \textit{Hom-Lie} \subset \textit{Quasi-Hom-Lie} \subset \textit{Quasi-Lie}$$

$$\textit{Color-Lie} \subset \textit{Quasi-Lie}$$

$$\textit{Super-Lie} \subset \textit{Quasi-Lie}$$

If the Jacobi identity is twisted by two morphisms instead of one, then we get a BiHom-Lie algebra. In fact, BiHom algebras are generalizations of Hom-algebras.

Definition 1.2.4 [15] A *BiHom-Lie* algebra is a 4-tuple $(L, [\cdot,\cdot], \alpha, \beta)$ where L is a vector space, $[\cdot,\cdot]$ a bilinear bracket and α, β two linear maps such that:

1. $\alpha \circ \beta = \beta \circ \alpha$
2. $\alpha([x, y]) = [\alpha(x), \alpha(y)], \qquad \beta([x, y]) = [\beta(x), \beta(y)]$
3. $[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)]$
4. $[\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(z), [\beta(x), \alpha(y)]] + [\beta^2(y), [\beta(z), \alpha(x)]] = 0$
   (BiHom Jacobi identity),

for all $x, y \in L$.

Remark: The second condition in the definition is sometimes dropped and a BiHom-Lie algebra where (2) holds is called multiplicative. Moreover, if α and β are bijective then $(L, [\cdot,\cdot], \alpha, \beta)$ is called a regular BiHom-Lie algebra.

Remark: A Hom-Lie algebra $(L, [\cdot,\cdot], \alpha)$ can induce a BiHom-Lie algebra by taking $\beta = \alpha$. Conversely, if $(L, [\cdot,\cdot], \alpha, \alpha)$ is a regular BiHom-Lie algebra, then $(L, [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra.

The concept of BiHon structures was introduced for the first time in [15] from a categorical approach. The authors considered the classes of BiHom-associative algebras, Bihom-Lie algebras, BiHom-bialgebras, BiHom-coassociative coalgebras and studied their structures. Their goal was to generalize the construction of the Hom-structures using the "twisting principle". At this point a natural question arises: Could this method be generalized to a higher order? According to the authors, the answer is no! They claim that it wouldn't be possible to construct a TriHom-associative algebra. However, BiHom-Lie algebra can be induced from a BiHom-associative algebra or also from a Lie algebra.

Definition 1.2.5 [15] A BiHom associative algebra is 4-tuple (V, μ, α, β) where V is a vector space, $\alpha : V \to V$, $\beta : V \to V$ and $\mu : V \otimes V \to V$ are linear maps such that for all $u, v, w \in V$, the following hold

$$\alpha \circ \beta = \beta \circ \alpha$$

$$\alpha(\mu(u,v)) = \mu(\alpha(u), \alpha(v)), \quad \textit{and} \quad \beta(\mu(u,v)) = \mu(\beta(u), \beta(v)), \qquad \text{Multiplicativity}$$

$$\mu(\alpha(u), \mu(v,w)) = \mu(\mu(u,v), \beta(w)) \qquad \text{BiHom} - \text{associativity}$$

Proposition 1.2.6 *[15] If (V, μ, α, β) is a BiHom-associative algebra such that α and β are bijective, then with the bracket $[\cdot,\cdot]$ defined on V by*

$$[u, v] = uv - (\alpha^{-1}\beta(v))(\alpha\beta^{-1}(u))$$

for any $u, v \in V$, and by writing uv instead of $\mu(u,v)$, $(V, [\cdot,\cdot], \alpha, \beta)$ is a BiHom-Lie algebra.

Proposition 1.2.7 *[15] Let $(L, [\cdot,\cdot])$ be a Lie algebra and let α and β be two commuting Lie algebra morphisms on L. Define $\{\cdot,\cdot\} : L \otimes L$, by $\{a, b\} = [\alpha(a), \beta(b)]$. Then $(L, \{\cdot,\cdot\}, \alpha, \beta)$ is a BiHom-Lie algebra called the* Yau twist *of $(L, [\cdot,\cdot])$ and denoted by $L_{\alpha,\beta}$.*

1.3 n-ary Algebras

We observe that n-ary operations and multilinear structures arise in many contexts in theoretical physics such as statistical mechanics, String theory, M-branes and Quark models. For example, ternary algebras can be used to provide solutions to the Yang-Baxter equation. Also, Nambu mechanics (which is a generalization of Hamiltonian mechanics that was introduced by Y. Nambu, where he considered two Hamiltonians instead of one) involves an n-ary bracket that satisfy an n-ary version of the Jacobi identity. The study of supergravity solutions describing M2-branes ending on M5-branes led Basu and Harvey [8] to the conclusion that the Lie algebra appearing in the Nahm equations has to be replaced by a 3-Lie algebra. With this growing interest, the algebraic structure of n-Lie algebras (also called Nambu algebras or Filippov algebras) was introduced and studied by Filippov in [13] as a generalization of Lie algebras

Definition 1.3.1 [13] An *n-Lie algebra* $(L, [\cdot, ..., \cdot])$ is a vector space L equipped with a skew-symmetric n-linear map $[\cdot, ..., \cdot] : L^n \to L$ such that for every $x_1, ..., x_{n-1}, y_1, ..., y_n \in L$, the following identity holds

$$[x_1, .., x_{n-1}, [y_1, .., y_n]] = \sum_{i=1}^{n} [y_1, .., y_{i-1}, [x_1, .., x_{n-1}, y_i], y_{i+1}, .., y_n].$$

This identity is called the *fundamental identity* or *Filippov identity.* As one would expect, the case $n = 2$ is just the Jacobi identity.

Example 1.3.2 *Let A be an $(n+1)$-dimensional vector space with the basis $\{v_1, ..., v_{n+1}\}$. The bracket*

$$[v_1, ..., \hat{v}_i ..., v_{n+1}] = (-1)^{n+1+i} v_i$$

for $i \in \{1, ..., n+1\}$ provides A with an n-Lie algebra structure.

Example 1.3.3 *Let $A = \mathbb{K}[X_1, ..., X_n]$ be the algebra of n indeterminate polynomials. Define*

$$[P_1, ..., P_n] = Jac(P_1, ..., P_n),$$

where $Jac(P_1, ..., P_n)$ is the determinant of the Jacobian matrix of $P_1, ..., P_n$. Thus $(A, [\cdot, ..., \cdot])$ is an n-Lie algebra.

Definition 1.3.4 Let $(L, [\cdot, ..., \cdot])$ and $(L', \{\cdot, ..., \cdot\})$ be two n-Lie algebras. An n-Lie algebra morphism is a linear map $f : L \to L'$ satisfying

$$f([x_1, ..., x_n]) = \{f(x_1), ..., f(x_n)\},$$

for every $x_1, ..., x_n \in L$.

Definition 1.3.5 Let $(L, [\cdot, ..., \cdot])$ be an n-Lie algebra. Let S be a subspace of L. We say that S is a *subalgebra* of L if $[S, ..., S] \subset S$. And we say that S is an *ideal* of L if $[S, L, ..., L] \subset S$.

Definition 1.3.6 Let $(L, [\cdot, ..., \cdot])$ be an n-Lie algebra. The *center* of L is the ideal defined by

$$Z(L) = \{z \in L : [z, x_1, ...x_{n-1}] = 0, \text{ for every } x_1, ...x_{n-1} \in L\}.$$

Even though Lie algebras and n-Lie algebras have many similarities when it comes to their identities and algebraic properties, it has become clear that some constructions and generalizations like the quantum deformation require extra structure. The construction of $(n+1)$-ary algebras from n-ary algebras is also a well-studied problem, in this dissertation we use some results given in the study of this type of algebras. The case of ternary Lie algebras induced by Lie algebras was discussed in [3], ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras were introduced in [4]. In [5], the authors presented a procedure to construct $(n+1)$-ary Hom-Nambu-Lie algebras from n-ary Hom-Nambu-Lie algebras and as a generalization, $(n+1)$-ary BiHom-Lie algebras induced by n-ary BiHom-Lie algebras were considered in [28].

In the second chapter of this dissertation, we study the algebras of generalized derivations of ternary Lie algebras. First, we briefly review the basic definitions and examples. Then we introduce the algebras of generalized derivations, quasi-derivations, central derivations and other subalgebras. We close the chapter by a classification for low dimensional algebras. The third chapter is dedicated to the generalized derivations of n-BiHom-Lie algebras we study their properties and investigate the algebras of derivations of $(n+1)$-BiHom-Lie algebras induced by n-BiHom-Lie algebras.

2 Derivations of Ternary Lie Algebras *

In this chapter we study derivations of ternary Lie algebras. Precisely, we investigate the relation between derivations of Lie algebras and the induced ternary Lie algebras. We also explore the spaces of quasi-derivations, the centroid and the quasi-centroid and give some properties. Finally, we compute these spaces for low-dimensional ternary Lie algebras $\mathfrak{g}$.

In the first section, we review the basics of ternary Lie algebras, give some examples and recall the construction given in [4] that allows one to induce ternary Lie algebras by a Lie algebra and a trace function. Section 2 deals with derivations of ternary Lie algebras and some generalizations. We discuss the space $Der(\mathfrak{g})$ and other subspaces, we give their properties and study the connection between derivations of Lie algebras and induced ternary Lie algebras. We show that if $\mathfrak{g}$ is a ternary Lie algebra with trivial center which can be decomposed to a sum of ideals then we can reduce the study of its derivations to those of the components. Moreover we discuss centroids, quasi-derivations, quasi-centroids, $(\alpha, \beta, \gamma, \theta)$-derivations and $(\alpha, \beta, \gamma, \theta)$-quasiderivations. In Section 3, we compute the set of derivations and other generalized derivations of low-dimensional ternary Lie Algebras.

2.1 Ternary Lie algebras

Definition 2.1.1 Let $[\cdot,\cdot,\cdot]$ be a skew-symmetric trilinear map on a $\mathbb{K}$-vector space $\mathfrak{g}$. We say that $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ is a *ternary Lie algebra* or a 3-*Lie algebra* if the map $[\cdot,\cdot,\cdot]$ satisfies for

*Sections of this chapter are taken from [1], which has been published in the journal "Int. Electron. J. Algebra", Vol.21, no.4, 2017.

all $x_1, .., x_5 \in \mathfrak{g}$ the identity

$$[x_1, x_2, [x_3, x_4, x_5]] = [[x_1, x_2, x_3], x_4, x_5] + [x_3, [x_1, x_2, x_4], x_5] + [x_3, x_4, [x_1, x_2, x_5]]. \quad (2.1.1)$$

This identity is called the *Nambu identity* or sometimes the *fundamental identity* or *Filippov identity*.

Example 2.1.2 *Let V be a three-dimensional vector space with basis $\{e_1, e_2, e_3\}$. Any skew-symmetric trilinear map $[\cdot, \cdot, \cdot] : V \longrightarrow V$ satisfies the identity* (2.1.1)*. To verify this we let $i, j = 1, 2, 3$ with $i < j$ and by an easy computation we obtain*

$$[[e_i, e_j, e_1], e_2, e_3] + [e_1, [e_i, e_j, e_2], e_3] + [e_1, e_2, [e_i, e_j, e_3]] = [e_i, e_j, [e_1, e_2, e_3]].$$

Example 2.1.3 *Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices over the field of complex numbers. The bracket $[A, B, C] = \circlearrowleft Tr(A)\Gamma(B, C)$, where Tr is the trace function and Γ is the commutator operator defined by $\Gamma(A, B) = AB - BA$, this bracket gives $M_n(\mathbb{C})$ a ternary Lie algebra structure. The symbol $\circlearrowleft$ means that we are taking a cyclic summation on A, B, C.*

Example 2.1.4 *The algebra of polynomials in 3 variables x_1, x_2, x_3, with the bracket defined by the functional jacobian:*

$$[f_1, f_2, f_3] = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{vmatrix} \quad (2.1.2)$$

is a ternary Lie algebra.

Example 2.1.5 *The following ternary Lie algebra is the only 4-dimensional simple ternary*

Lie algebra. The bracket are defined with respect to the basis $\{e_1, e_2, e_3, e_4\}$ *by*

$$[e_1, e_2, e_3] = -e_4,$$
$$[e_1, e_2, e_4] = e_3,$$
$$[e_1, e_3, e_4] = -e_2,$$
$$[e_2, e_3, e_4] = e_1.$$

Definition 2.1.6 Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a ternary Lie algebra and let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$.

- We say that $\mathfrak{h}$ is a *ternary Lie sub-algebra* of $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ if it is closed under the bracket, that is if $[\mathfrak{h}, \mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.
- A subspace $\mathcal{I}$ of $\mathfrak{g}$ is called an *ideal* if $[\mathcal{I}, \mathfrak{g}, \mathfrak{g}] \subset \mathcal{I}$.
- A ternary Lie algebra is said to be *simple* if it has no proper ideal.
- The *center* of $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ is the set

  $$Z(\mathfrak{g}) = \{u \in \mathfrak{g}; [u, x_1, x_2] = 0 \ for\ all\ x_1, x_2 \in \mathfrak{g}\}.$$

  $Z(\mathfrak{g})$ is an abelian ideal of $\mathfrak{g}$.
  An easy fact is that the center of a non-abelian simple ternary Lie algebra is trivial.
- The subspace $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]$ is a ternary Lie sub-algebra of $\mathfrak{g}$ called the *derived algebra* of $\mathfrak{g}$.
- A *morphism* of ternary Lie algebra is a linear map $\varphi : (\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}}) \longrightarrow (\eta, [\cdot,\cdot,\cdot]_{\eta})$ such that for any $x, y, z \in \mathfrak{g}$ we have

  $$\varphi([x, y, z]_{\mathfrak{g}}) = [\varphi(x), \varphi(y), \varphi(z)]_{\eta}.$$

Remark 2.1.7 *As in the case of Lie algebras, the kernel of a ternary Lie algebras morphism is an ideal of* $\mathfrak{g}$. *In fact, if* u *is in* $\ker(\varphi)$ *then for any* $v, w \in \mathfrak{g}$, $\varphi([u, v, w]_{\mathfrak{g}}) =$

$[\varphi(u), \varphi(v), \varphi(w)]_\eta = 0$.

However, its image $\Im(\varphi)$ *is not always an ideal but a ternary Lie sub-algebra of* η*: For* $v_1, v_2, v_3 \in \Im(\varphi)$ *we have* $[v_1, v_2, v_3]_\eta = [\varphi(u_1), \varphi(u_2), \varphi(u_3)]_\eta = \varphi([u_1, u_2, u_3]_\mathfrak{g})$ *for some* $u_1, u_2, u_3 \in \mathfrak{g}$.

The two following propositions are given in [40] in a context of Hom-Lie algebras, here we state them in the case of ternary Lie algebras.

Proposition 2.1.8 *Given two ternary Lie algebras* $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ *and* $(\eta, [\cdot,\cdot,\cdot]_\eta)$*, the space* $\mathfrak{g}\oplus\eta$ *with the bracket defined by*

$$[(u_1, v_1), (u_2, v_2), (u_3, v_3)]_{\mathfrak{g}\oplus\eta} = ([u_1, u_2, u_3]_\mathfrak{g}, [v_1, v_2, v_3]_\eta)$$

is a ternary Lie algebra.

Proposition 2.1.9 *A linear map* $\varphi : (\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g}) \longrightarrow (\eta, [\cdot,\cdot,\cdot]_\eta)$ *is morphism of ternary Lie algebras if and only if its graph* $\mathcal{G}_\varphi$ *is a ternary Lie sub-algebra of* $(\mathfrak{g} \oplus \eta, [\cdot,\cdot,\cdot]_{\mathfrak{g}\oplus\eta})$.

Proof. Suppose that $\varphi : (\mathfrak{g}, [\cdot,\cdot,\cdot]_\mathfrak{g}) \longrightarrow (\eta, [\cdot,\cdot,\cdot]_\eta)$ is morphism of ternary Lie algebras and let $u, v, w \in \mathfrak{g}$. We have

$$\begin{aligned}[(u, \varphi(u)), (v, \varphi(v)), (w, \varphi(w))]_{\mathfrak{g}\oplus\eta} &= ([u, v, w]_\mathfrak{g}, [\varphi(u), \varphi(v), \varphi(w)]_\eta) \\ &= ([u, v, w]_\mathfrak{g}, \varphi([u, v, w]_\mathfrak{g})) \in \mathcal{G}_\varphi.\end{aligned}$$

Then $\mathcal{G}_\varphi$ is closed under the bracket $[\cdot,\cdot,\cdot]_{\mathfrak{g}\oplus\eta}$.

Conversely, if $\mathcal{G}_\varphi$ is a ternary Lie sub-algebra of $(\mathfrak{g} \oplus \eta, [\cdot,\cdot,\cdot]_{\mathfrak{g}\oplus\eta})$, then

$$\mathcal{G}_\varphi \ni [(u, \varphi(u)), (v, \varphi(v)), (w, \varphi(w))]_{\mathfrak{g}\oplus\eta} = ([u, v, w]_\mathfrak{g}, [\varphi(u), \varphi(v), \varphi(w)]_\eta).$$

Thus $[\varphi(u), \varphi(v), \varphi(w)]_\eta = \varphi([u, v, w]_\mathfrak{g})$.

∎

2.1.1 Ternary Lie algebras induced by Lie algebras

In [4], the authors gave a procedure to construct a ternary Lie algebra structure from a Lie bracket over the same vector space using a trace map. Precisely, we have the following.

Proposition 2.1.10 *[4] Let* $(\mathfrak{g}, [\cdot, \cdot])$ *be a Lie algebra and* $\tau : \mathfrak{g} \longrightarrow \mathbb{K}$ *be a trace map on* $\mathfrak{g}$*, then* $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\tau)$ *is a ternary Lie algebra, where*

$$[x, y, z]_\tau = \tau(x)[y, z] + \tau(z)[x, y] + \tau(y)[z, x].$$

The ternary Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\tau)$ *is called the ternary Lie algebra induced by the Lie algebra* $(\mathfrak{g}, [\cdot, \cdot])$ *and the trace map* τ*.*

Remark 2.1.11 *We recall that a trace function* $\tau : \mathfrak{g} \longrightarrow \mathbb{K}$ *is a linear map such that* $\tau([x, y]) = 0$ *for all* $x, y \in \mathfrak{g}$*.*

We give an example to illustrate this construction.

Example 2.1.12 *Let* H_2 *be the 5-dimensional Heisenberg Lie algebra with generators* P_1, P_2, Q_1, Q_2 *and* Z *subject to the following bracket relations (unspecified bracket relations are obtained by skew-symmetry or are zeros):*

$$[P_1, Q_1] = [P_2, Q_2] = Z, \text{ and } [P_1, Q_2] = [P_2, Q_1] = -Z.$$

Since Z *is the only bracket, then any linear map* $\tau : H_2 \longrightarrow \mathbb{K}$ *such that* $\tau(Z) = 0$*, is a trace function on* H_2*. Then* $(H_2, [\cdot, \cdot, \cdot]_\tau)$ *is a ternary Lie algebra with the following ternary brackets:*

$$[P_1, P_2, Q_1]_\tau = \tau(P_1 - P_2)Z$$
$$[P_1, P_2, Q_2]_\tau = \tau(P_1 + P_2)Z$$
$$[Q_1, Q_2, P_1]_\tau = \tau(Q_1 + Q_2)Z$$
$$[Q_1, Q_2, P_2]_\tau = \tau(Q_1 - Q_2)Z.$$

A converse construction is also possible in the following sense: If $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ is a ternary Lie algebra, we can induce a Lie algebra structure on $\mathfrak{g}$. Fix an element $\omega \in \mathfrak{g}$ and define the bracket $[x,y]_\omega = [x,y,\omega]$, then $(\mathfrak{g}, [\cdot,\cdot]_\omega)$ is a Lie algebra. In fact $[\cdot,\cdot]_\omega$ is clearly bilinear and skew-symmetric and by a direct computation one can see that it satisfies the Jacobi identity.

2.2 Derivations of ternary Lie algebras and ternary Lie algebras induced by Lie algebras

Now let us define derivations of a ternary Lie Algebra and some other generalizations.

2.2.1 Derivations, Central Derivations and Centroids

Definition 2.2.1 Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a ternary Lie algebra and D a linear map of $\mathfrak{g}$. D is said to be a *derivation* of $\mathfrak{g}$ if

$$D([x_1,x_2,x_3]) = [D(x_1),x_2,x_3] + [x_1,D(x_2),x_3] + [x_1,x_2,D(x_3)].$$

For all $x_1,x_2,x_3 \in \mathfrak{g}$. We denote by $\mathrm{Der}(\mathfrak{g})$ the space of derivations of $\mathfrak{g}$.

Example 2.2.2 *A straightforward computation gives the following fact: If D is a derivation of the ternary Lie algebra in Example 2.1.5 with its matrix $M = (a_{ij})_{1\le i,j\le 4}$ with respect to the basis $\{e_1,e_2,e_3,e_4\}$ then*

$$M = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Remark 2.2.3 *Der($\mathfrak{g}$) is a Lie algebra with the bracket*

$$[D_1,D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

Der($\mathfrak{g}$) can also be equipped with a ternary Lie algebra structure induced from this Lie bracket. We have the following:

Proposition 2.2.4 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Consider the map $\tau : Der(\mathfrak{g}) \longrightarrow \mathbb{K}$ defined by $\tau(D) = tr(D)$ consisting of the trace of the matrix of D. Then τ is a trace function on $Der(\mathfrak{g})$ and it follows that $(Der(\mathfrak{g}), [\cdot,\cdot,\cdot]_\tau)$ is a ternary Lie algebra.*

Proposition 2.2.5 *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a ternary Lie algebra with trivial center and which can be decomposed as the direct sum of two ideals: $\mathfrak{g} = \mathcal{I} \oplus \mathcal{J}$, then we have*

$$Der(\mathfrak{g}) = Der(\mathcal{I}) \oplus Der(\mathcal{J}).$$

To prove this proposition, we are going to use the following lemma:

Lemma 2.2.6 *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a ternary Lie algebra such that $\mathfrak{g} = \mathcal{I} \oplus \mathcal{J}$, where $\mathcal{I}$ and $\mathcal{J}$ are two ideals of $\mathfrak{g}$. Suppose that $Z(\mathfrak{g}) = \{0\}$, then for every $D \in Der(\mathfrak{g})$, we have $D(\mathcal{I}) \subseteq \mathcal{I}$ and $D(\mathcal{J}) \subseteq \mathcal{J}$.*

Proof. Let $u \in \mathcal{I}$ such that $D(u) = v_1 + v_2$, where $v_1 \in \mathcal{I}$ and $v_2 \in \mathcal{J}$. Let $x, y \in \mathfrak{g}$. We have

$$\begin{aligned}
[v_2, x, y] &= [D(u) - v_1, x, y] \\
&= [D(u), x, y] - [v_1, x, y] \\
&= D([u, x, y]) - [u, D(x), y] - [u, x, D(y)] - [v_1, x, y].
\end{aligned}$$

Since $\mathcal{I}$ is an ideal of $\mathfrak{g}$, all of $[u, D(x), y], [u, x, D(y)]$ and $[v_1, x, y]$ are in $\mathcal{I}$.
Now write $x = x_1 + x_2$ and $y = y_1 + y_2$ such that $x_1, y_1 \in \mathcal{I}$ and $x_2, y_2 \in \mathcal{J}$. Then

$$[u, x, y] = [u, x_1, y_1] + [u, x_1, y_2] + [u, x_2, y_1] + [u, x_2, y_2].$$

Each of $[u, x_1, y_2], [u, x_2, y_1], [u, x_2, y_2]$ are in $\mathcal{I} \cap \mathcal{J}$, so they are all zero, thus

$$
\begin{aligned}
D([u,x,y]) &= D([u,x_1,y_1]) \\
&= [D(u),x_1,y_1] + [u,D(x_1),y_1] + [u,x_1,D(y_1)].
\end{aligned}
$$

Hence $D([u,x,y]) \in \mathcal{I}$. It follows that $[v_2,x,y] \in \mathcal{I} \cap \mathcal{J}$, so $[v_2,x,y] = 0$. Hence $v_2 = 0$ since $Z(\mathfrak{g}) = 0$.

■

We can now prove Proposition 2.2.5.

Proof. By the previous lemma, we can see that a restriction of any derivation of $\mathfrak{g}$ to $\mathcal{I}$ (respectively to $\mathcal{J}$) is a derivation of $\mathcal{I}$ (respectively $\mathcal{J}$).

■

A natural question concerning derivations of ternary Lie algebras induced by a Lie bracket is how they are related to derivations of the original Lie algebra.

Proposition 2.2.7 *Let $(\mathfrak{g},[\cdot,\cdot])$ be a Lie algebra and $(\mathfrak{g},[\cdot,\cdot,\cdot]_\tau)$ be an induced ternary Lie algebra. Let D be a derivation of $(\mathfrak{g},[\cdot,\cdot])$. If $D(\mathfrak{g}) \subset Ker(\tau)$, then D is a derivation of $(\mathfrak{g},[\cdot,\cdot,\cdot]_\tau)$.*

Proof. Let D be a derivation of $(\mathfrak{g},[\cdot,\cdot])$. For all $x,y,z \in \mathfrak{g}$,

$$
\begin{aligned}
[D(x),y,z]_\tau + [x,D(y),z]_\tau + [x,y,D(z)]_\tau = {} & \tau(D(x))[y,z] + \tau(z)[D(x),y] \\
& + \tau(y)[z,D(x)] + \tau(z)[x,D(y)] + \\
& \tau(D(y))[z,x] + \tau(x)[y,D(z)] \\
& + \tau(D(z))[x,y] + \tau(y)[D(z),x].
\end{aligned}
$$

Now, if $D(\mathfrak{g}) \subset Ker(\tau)$, then

$$\begin{aligned}
&[D(x), y, z]_\tau + [x, D(y), z]_\tau + [x, y, D(z)]_\tau \\
&= \tau(x)([D(y), z] + [y, D(z)]) + \tau(y)([z, D(x)] + [D(z), x]) + \\
&\quad \tau(z)([D(x), y] + [x, D(y)]) \\
&= \tau(x)D([y, z]) + \tau(y)D([z, x]) + \tau(z)D([x, y]) \\
&= D(\tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y]) = D([x, y, z]_\tau).
\end{aligned}$$

Thus D is a derivation of $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\tau)$.

■

Lemma 2.2.8 *Let $D : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra derivation, then $\tau \circ D$ is a trace function on $\mathfrak{g}$.*

Proof. for all $x, y \in \mathfrak{g}$, we have :

$$\tau\left(D\left([x, y]\right)\right) = \tau\left([D(x), y] + [x, D(y)]\right) = \tau\left([D(x), y]\right) + \tau\left([x, D(y)]\right) = 0.$$

■

A more powerful criterion is given in the next theorem,

Theorem 2.2.9 ([3]) *Let $D : \mathfrak{g} \to \mathfrak{g}$ be a derivation of the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, then D is a derivation of the induced ternary Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\tau)$ if and only if:*

$$[x, y, z]_{\tau \circ D} = 0,$$

for all $x, y, z \in \mathfrak{g}$.

Proof. Let D be a derivation of $\mathfrak{g}$ and $x, y, z \in \mathfrak{g}$:

$$\begin{aligned}
D\left([x,y,z]_\tau\right) &= \tau(x)D\left([y,z]\right) + \tau(y)D\left([z,x]\right) + \tau(z)D\left([x,y]\right) \\
&= \tau(x)\left[D(y),z\right] + \tau(y)\left[D(z),x\right] + \tau(z)\left[D(x),y\right] \\
&+ \tau(x)\left[y,D(z)\right] + \tau(y)\left[z,D(x)\right] + \tau(z)\left[x,D(y)\right] \\
&+ \tau(D(x))\left[y,z\right] + \tau(D(y))\left[z,x\right] + \tau(D(z))\left[x,y\right] \\
&- \tau(D(x))\left[y,z\right] + \tau(D(y))\left[z,x\right] + \tau(D(z))\left[x,y\right] \\
&= \left[D(x),y,z\right]_\tau + \left[x,D(y),z\right]_\tau + \left[x,y,D(z)\right]_\tau - \left[x,y,z\right]_{\tau\circ D}.
\end{aligned}$$

■

Proposition 2.2.10 *Let D be a derivation of $(\mathfrak{g}, [\cdot,\cdot,\cdot])$. For $w \in \mathfrak{g}$, D is a derivation of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_w)$ if and only if $D(w) \in Z(\mathfrak{g}, [\cdot,\cdot,\cdot])$.*

Proof. Let $x, y \in \mathfrak{g}$,

$$\begin{aligned}
D([x,y]_w) = D([x,y,w]) &= [D(x),y,w] + [x,D(y),w] + [x,y,D(w)] \\
&= [D(x),y]_w + [x,D(y)]_w + [x,y,D(w)].
\end{aligned}$$

Hence if $D(w)$ is in the center of the ternary Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot])$, then D is clearly a derivation of the induced Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_w)$.

■

For any $x = (x_1, x_2) \in \mathfrak{g} \times \mathfrak{g}$, the map defined by

$$\begin{aligned}
ad_x : &\mathfrak{g} \longrightarrow \mathfrak{g} \\
&u \mapsto [x_1, x_2, u]
\end{aligned}$$

is a derivation which we call an *inner derivation.*

In fact, For $u, v, w \in \mathfrak{g}$,

$$\begin{aligned} ad_x([u,v,w]) &= [x_1, x_2, [u,v,w]] \\ &= [[x_1,x_2,u],v,w] + [u,[x_1,x_2,v],w] + [u,v,[x_1,x_2,w]] \\ &= [ad_x(u),v,w] + [u,ad_x(v),w] + [u,v,ad_x(w)]. \end{aligned}$$

Remark 2.2.11

$$\begin{aligned} ad : &\mathfrak{g} \times \mathfrak{g} \longrightarrow gl(\mathfrak{g}) \\ &(x_1, x_2) \mapsto ad_{(x_1,x_2)} \end{aligned}$$

is the adjoint representation of $\mathfrak{g}$.

It turns out that all the derivations on a semi-simple Lie algebra are inner derivations. This is also true for Lie triple systems [34] and many other algebraic structures. In particular, all the derivations of the ternary Lie algebra defined in Example 2.1.5 are inner.

Proposition 2.2.12 *The space Der($\mathfrak{g}$) is an invariant of the ternary Lie algebra* $\mathfrak{g}$.

Remark 2.2.13 *Here the space of derivations is considered as ternary Lie algebra induced from the Lie algebra structure as shown in Proposition 2.2.4.*

Proof. Let $\sigma : (\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}}) \longrightarrow (\eta, [\cdot,\cdot,\cdot]_{\eta})$ be a ternary Lie algebra isomorphism and let D be a derivation of $\mathfrak{g}$. Then for any $x, y, z \in \eta$ we have:

$$\begin{aligned} \sigma D \sigma^{-1}([x,y,z]_{\eta}) &= \sigma D([\sigma^{-1}(x), \sigma^{-1}(y), \sigma^{-1}(z)]_{\mathfrak{g}}) \\ &= \sigma([D\sigma^{-1}(x), \sigma^{-1}(y), \sigma^{-1}(z)]_{\mathfrak{g}}) + \sigma([\sigma^{-1}(x), D\sigma^{-1}(y), \sigma^{-1}(z)]_{\mathfrak{g}}) \\ &+ \sigma([\sigma^{-1}(x), \sigma^{-1}(y), D\sigma^{-1}(z)]_{\mathfrak{g}}) \\ &= [\sigma D\sigma^{-1}(x), y, z]_{\eta} + [x, \sigma D\sigma^{-1}(y), z]_{\eta} + [x, y, \sigma D\sigma^{-1}(z)]_{\eta}. \end{aligned}$$

Thus $\sigma D \sigma^{-1}$ is a derivation of η, hence the mapping

$$\begin{aligned} \phi : Der(\mathfrak{g}) &\longrightarrow Der(\eta) \\ D &\longmapsto \sigma D \sigma^{-1} \end{aligned}$$

is an isomorphism of ternary Lie algebras.

In fact, it is easy to see that ϕ is linear. Moreover, let D_1, D_2, D_3 be derivations of $\mathfrak{g}$:

$$\begin{aligned}\phi([D_1, D_2, D_3]_{tr}) &= \phi(tr(D_1)[D_2, D_3]) + \phi(tr(D_3)[D_1, D_2]) + \phi(tr(D_2)[D_3, D_1]) \\ &= tr(D_1)\phi([D_2, D_3]) + tr(D_3)\phi([D_1, D_2]) + tr(D_2)\phi([D_3, D_1]) \\ &= tr(\phi(D_1))[\phi(D_2), \phi(D_3)] + tr(\phi(D_3))[\phi(D_1), \phi(D_2)] \\ &\quad + tr(\phi(D_2))[\phi(D_3), \phi(D_1)]\end{aligned}$$

since ϕ is a morphism of the Lie algebras $Der(\mathfrak{g})$ and $Der(\eta)$, and $tr(D) = tr(\sigma D \sigma^{-1})$. Then $\phi([D_1, D_2, D_3]_{tr}) = [\phi(D_1), \phi(D_2), \phi(D_3)]_{tr}$.

■

Definition 2.2.14 The *centroid* of a ternary Lie algebra $\mathfrak{g}$ is the set of all linear maps D that satisfy:

$$D([x, y, z]) = [D(x), y, z] = [x, D(y), z] = [x, y, D(z)],$$

for all $x, y, z \in \mathfrak{g}$. We denote by $C(\mathfrak{g})$ the centroid of $\mathfrak{g}$.

Remark 2.2.15 *We can define the centroid only by the equality $D([x, y, z]) = [D(x), y, z]$, and the two other equalities follow by the skew symmetry of the bracket.*

Proposition 2.2.16 *The centroid of a ternary Lie algebra $\mathfrak{g}$ is a ternary Lie subalgebra of $(Der(\mathfrak{g}), [\cdot, \cdot, \cdot]_{tr})$.*

Proof. Let $D_1, D_2, D_3 \in C(\mathfrak{g})$ and $\psi = [D_1, D_2, D_3]_{tr}$. For simplicity we let $\lambda_i = tr(D_i)$ for $i = 1, 2, 3$. So $\psi = \lambda_1(D_2D_3 - D_3D_2) + \lambda_3(D_1D_2 - D_2D_1) + \lambda_2(D_3D_1 - D_1D_3)$. Then for

$x, y, z \in \mathfrak{g}$, we have

$$
\begin{aligned}
[\psi(x), y, z] &= [\lambda_1(D_2D_3 - D_3D_2)(x) + \lambda_3(D_1D_2 - D_2D_1)(x) \\
&+ \lambda_2(D_3D_1 - D_1D_3)(x), y, z] \\
&= \lambda_1([D_2D_3(x), y, z] - [D_3D_2(x), y, z]) + \lambda_3([D_1D_2(x), y, z] \\
&- [D_2D_1(x), y, z]) + \lambda_2([D_3D_1(x), y, z] - [D_1D_3(x), y, z]) \\
&= \lambda_1(D_2([D_3(x), y, z]) - D_3([D_2(x), y, z])) + \lambda_3(D_1([D_2(x), y, z]) \\
&- D_2([D_1(x), y, z])) + \lambda_2(D_3([D_1(x), y, z]) - D_1([D_3(x), y, z])) \\
&= \lambda_1(D_2D_3 - D_3D_2)([x, y, z]) + \lambda_3(D_1D_2 - D_2D_1)([x, y, z]) + \\
&\lambda_2(D_3D_1 - D_1D_3)([x, y, z]). \\
&= \psi([x, y, z]).
\end{aligned}
$$

■

Proposition 2.2.17 *Let $D \in C(\mathfrak{g}, [\cdot, \cdot])$. If for every $u, v \in \mathfrak{g}$ we have*

$$\tau(u)D(v) = \tau(D(u))v,$$

then $D \in C(\mathfrak{g}, [\cdot, \cdot, \cdot]_\tau)$.

Proof.

$$
\begin{aligned}
D([x, y, z]_\tau) &= \tau(x)D([y, z]) + \tau(z)D([x, y])\tau(y)D([z, x]) \\
&= \tau(D(x))[y, z] + \tau(z)[D(x), y] + \tau(y)[D(z), x] \\
&= \tau(D(x))[y, z] + \tau(z)[D(x), y] + \tau(y)[z, D(x)] \\
&= [D(x), y, z]_\tau.
\end{aligned}
$$

■

Moreover, any centroid element of a ternary Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ is a centroid element of an induced Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_w)$.

The following proposition reduces the centroid of any simple ternary Lie algebra to the space of its homothety.

Proposition 2.2.18 *Let* $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ *be a simple ternary Lie algebra over an algebraically closed field* $\mathbb{K}$. *Then*

$$C(\mathfrak{g}) = \mathbb{K}\ Id,$$

where Id *is the identity map on* $\mathfrak{g}$.

Proof. First, one can see that the adjoint representation of $\mathfrak{g}$ is simple because otherwise, if a subset $\mathcal{A}$ of $\mathfrak{g}$ is stable under the action of $ad_{(x_1,x_2)}$ for any $x_1, x_2 \in \mathfrak{g}$, then $\mathcal{A}$ is an ideal. In addition, for any centroid element D we have

$$D([x, y, z]) = [x, y, D(z)].$$

Therefore

$$D \circ ad_{(x,y)} = ad_{(x,y)} \circ D.$$

Thus using the *Schur's Lemma*, we conclude that $D = \lambda Id$ for some scalar λ.

■

Definition 2.2.19 A linear map D is a *central derivation* of $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ if $D(\mathfrak{g}) \subset Z(\mathfrak{g})$ and $D(\mathfrak{g}^1) = \{0\}$.

We denote by $ZDer(\mathfrak{g})$ the set of all central derivations of $(\mathfrak{g}, [\cdot, \cdot, \cdot])$.

Example 2.2.20 *A simple ternary Lie algebra does not have a non zero central derivation since* $\mathfrak{g}^1 = \mathfrak{g}$. *Recall that* $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]$.

Proposition 2.2.21 *For a ternary Lie algebra* $(\mathfrak{g}, [\cdot, \cdot, \cdot])$, *we have*

$$ZDer(\mathfrak{g}) = Der(\mathfrak{g}) \cap C(\mathfrak{g}).$$

Proof. A central derivation is obviously a derivation of the ternary Lie algebra. Moreover, it is a centroid element since

$$D([x,y,z]) = [D(x),y,z] = [x,D(y),z] = [x,y,D(z)] = 0$$

for all $x,y,z \in \mathfrak{g}$. Conversely, if $D \in Der(\mathfrak{g}) \cap C(\mathfrak{g})$, then

$$D([x,y,z]) = 3D([x,y,z]).$$

Therefore $D([x,y,z]) = 0$. Also $[D(x),y,z] = 3[D(x),y,z]$, thus $[D(x),y,z] = 0$ and $D(x) \in Z(\mathfrak{g})$.

■

Here again we study central derivations of a ternary Lie algebra induced by a Lie algebra and vice versa.

Proposition 2.2.22 *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ be a ternary Lie algebra. For any $w \in \mathfrak{g}$, let $(\mathfrak{g}, [\cdot,\cdot]_w)$ be the induced Lie algebra. Every central derivation of $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ is also a central derivation of the induced $(\mathfrak{g}, [\cdot,\cdot]_w)$.*

Proof. If D is a central derivation of $(\mathfrak{g}, [\cdot,\cdot,\cdot])$, then for any $u,v \in \mathfrak{g}$

$$[D(u),v]_w = [D(u),v,w] = 0,$$

thus $D(u) \in Z(\mathfrak{g}, [\cdot,\cdot]_w)$.

■

Proposition 2.2.23 *Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\tau)$ be a ternary Lie algebra induced by a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ and the trace map τ. Let $D \in ZDer(\mathfrak{g}, [\cdot,\cdot])$. Then D is a central derivation of $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\tau)$ if and only if $D(\mathfrak{g}) \subset Ker(\tau)$.*

Proof. Let $D \in ZDer(\mathfrak{g}, [\cdot,\cdot])$ and $x,y,z \in \mathfrak{g}$,

$$D([x,y,z]_\tau) = \tau(x)D([y,z]) + \tau(z)D([x,y]) + \tau(y)D([z,x]) = 0.$$

In addition,

$$[D(x), y, z]_\tau = \tau(D(x))[y, z] + \tau(z)[D(x), y] + \tau(y)[z, D(x)] = \tau(D(x))[y, z].$$

■

2.2.2 Quasi-derivations and Quasi-centroids

Definition 2.2.24 A linear map D of $\mathfrak{g}$ is a *quasi-derivation* if there exists D' such that

$$[D(x), y, z] + [x, D(y), z] + [x, y, D(z)] = D'([x, y, z]),$$

for all $x, y, z \in \mathfrak{g}$. We denote by $QDer(\mathfrak{g})$ the set of all quasi derivations of $\mathfrak{g}$.

Example 2.2.25 *Let $(V, [\cdot,\cdot,\cdot])$ be the ternary Lie algebra defined in Example 2.1.5. For any linear map D of V with $M = (a_{ij})$ its matrix in the base (e_1, e_2, e_3, e_4), there exists D' such that $D \in QDer(V)$ and the matrix $M' = (b_{ij})$ of D' is given by:*

$$b_{ij} = -a_{ji} \qquad for\ 1 \leq i \neq j \leq 4;$$

and

$$b_{ii} = \sum_{\substack{j=1 \\ j\neq i}}^{4} a_{jj}.$$

As in Theorem 2.2.9, the next proposition establish the link between the quasi-derivations of a Lie algebra and the induced ternary Lie algebra.

Proposition 2.2.26 *Let D be a quasi-derivation of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ and let τ be a trace function on $\mathfrak{g}$. Then D is a quasi-derivation of $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\tau)$ if and only if $[x, y, z]_{\tau\circ D} = 0$ for any x, y, z.*

Remark 2.2.27 *Unlike Lemma 2.2.8, in this case the map $\tau \circ D$ is not necessarily a trace on $\mathfrak{g}$.*

Every derivation is obviously a quasi-derivation, so we have $Der(\mathfrak{g}) \subset QDer(\mathfrak{g})$. We will see now that a sum of any derivation and a centroid element is also a quasi-derivation:

Proposition 2.2.28 *If* $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ *is a ternary Lie algebra with trivial center, then we have*

$$Der(\mathfrak{g}) \oplus C(\mathfrak{g}) \subset QDer(\mathfrak{g}).$$

Proof. Let $D \in C(\mathfrak{g})$, for any $x, y, z \in \mathfrak{g}$, we have

$$[D(x), y, z] + [x, D(y), z] + [x, y, D(z)] = 3D([x, y, z]).$$

So D is a quasi derivation, thus $Der(\mathfrak{g}) + C(\mathfrak{g}) \subset QDer(\mathfrak{g})$. Now if $D \in Der(\mathfrak{g}) \cap C(\mathfrak{g})$, then

$$[D(x), y, z] + [D(x), y, z] + [D(x), y, z] = [D(x), y, z].$$

Thus $[D(x), y, z] = 0$, therefore $D(x) \in Z(\mathfrak{g})$, for every x, which means that $D = 0$. Hence $Der(\mathfrak{g}) \cap C(\mathfrak{g}) = \{0\}$.

■

Proposition 2.2.29 *Let* $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ *be a ternary Lie algebra with trivial center and suppose that* $\mathfrak{g} = \mathcal{I} \oplus \mathcal{J}$*, then*

(1) $QDer(\mathfrak{g}) = QDer(\mathcal{I}) \oplus QDer(\mathcal{J})$

(2) $C(\mathfrak{g}) = C(\mathcal{I}) \oplus C(\mathcal{J})$.

Proof. Since Lemma 2.2.6 can be applied to any quasi-derivation and any centroid element, so the decomposition in Proposition 2.2.5 holds naturally.

■

Definition 2.2.30 The *quasi-centroid* of $\mathfrak{g}$ is the set of all linear maps D such that

$$[D(x), y, z] = [x, D(y), z] = [x, y, D(z)]$$

for all $x, y, z \in \mathfrak{g}$. We denote by $QC(\mathfrak{g})$ the quasi-centroid of $\mathfrak{g}$.

Proposition 2.2.31 *Let $D \in QC(\mathfrak{g}, [\cdot,\cdot])$. Suppose that $D(\mathfrak{g}) \subset Ker(\tau)$ and for every $u, v \in \mathfrak{g}$ we have $\tau(u).v = \tau(v).u$, then $D \in QC(\mathfrak{g}, [\cdot,\cdot,\cdot]_\tau)$.*

The proof is quite similar to the proof of 2.2.17.

Lemma 2.2.32 *The derived algebra $\mathfrak{g}^1$ is preserved under $Der(\mathfrak{g})$ and $C(\mathfrak{g})$ but not $QC(\mathfrak{g})$.*

Proof. If D is a derivation of $\mathfrak{g}$, then by definition, fo every $x, y, z \in \mathfrak{g}$ we have

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)] \in \mathfrak{g}^1.$$

Similarly, if $D \in C(\mathfrak{g})$, so $D([x, y, z]) = [D(x), y, z] \in \mathfrak{g}^1$.
We show that the derived algebra is not necessarily preserved by the quasi-centroid, by the following counter example. Let $(L, [\cdot,\cdot,\cdot])$ be the ternary Lie algebra with the basis $\{e_1, e_2, e_3\}$ such that $[e_1, e_2, e_3] = e_1$. Let φ be a linear map defined by $\varphi(e_1) = e_2$ and $\varphi(e_e) = \varphi(e_3) = 0$. Then $\varphi \in QC(L)$ since $[\varphi(e_1), e_2, e_3] + [e_1, \varphi(e_2), e_3] + [e_1, e_2, \varphi(e_3)] = 0$. However φ does not preserve $L^1 =< e_1 >$.

∎

2.2.3 $(\alpha, \beta, \gamma, \theta)$-Derivations and $(\alpha, \beta, \gamma, \theta)$-Quasiderivations

We will now define another generalization for a derivation of $(\mathfrak{g}, [\cdot,\cdot,\cdot])$.

Definition 2.2.33 Let $\mathfrak{g}$ be a ternary Lie algebra, $D \in End(\mathfrak{g})$, and $\alpha, \beta, \gamma, \theta \in \mathbb{K}$. We say that D is an $(\alpha, \beta, \gamma, \theta)$-*derivation* of $\mathfrak{g}$ if

$$\alpha D([x_1, x_2, x_3]) = \beta[D(x_1), x_2, x_3] + \gamma[x_1, D(x_2), x_3] + \theta[x_1, x_2, D(x_3)],$$

For every $x_1, x_2, x_3 \in \mathfrak{g}$.

We denote by $D(\alpha, \beta, \gamma, \theta)$ the set of $(\alpha, \beta, \gamma, \theta)$-derivations.

Remark 2.2.34 *It is clear that $D(0, 0, 0, 0) = End(\mathfrak{g})$, therefore we can assume that the parameters $\alpha, \beta, \gamma, \theta$ are not all zero. One can also see that $D(1, 1, 1, 1) = Der(\mathfrak{g})$.*

Theorem 2.2.35 *Suppose that $\alpha \neq 0$. The space $D(\alpha, \beta, \gamma, \theta)$ is one of the following spaces:*

- $D(1, \lambda, 0, 0)$
- $D(1, \lambda, \delta, \delta)$
- $D(1, \lambda, 0, 0) \cap QC(\mathfrak{g})$

for some $\lambda, \delta \in \mathbb{K}$.

Proof. Let $D \in D(\alpha, \beta, \gamma, \theta)$. Using the skew-symmetry of the bracket we have the following equalities, for every $x, y, z \in \mathfrak{g}$

$$\alpha D([x,y,z]) = \beta[D(x), y, z] + \gamma[x, D(y), z] + \theta[x, y, D(z)] \tag{2.2.3}$$

$$\alpha D([x,y,z]) = \beta[D(z), x, y] + \gamma[z, D(x), y] + \theta[z, x, D(y)] \tag{2.2.4}$$

$$-\alpha D([x,y,z]) = \beta[D(x), z, y] + \gamma[x, D(z), y] + \theta[x, z, D(y)] \tag{2.2.5}$$

$$-\alpha D([x,y,z]) = \beta[D(y), x, z] + \gamma[y, D(x), z] + \theta[y, x, D(z)] \tag{2.2.6}$$

Now by adding eq (2.2.3) to each equation, we get

$$0 = (\gamma - \theta)[x, D(y), z] + (\theta - \gamma)[x, y, D(z)] \qquad (2.2.3) + (2.2.5)$$

$$0 = (\beta - \gamma)[D(x), y, z] + (\gamma - \beta)[x, D(y), z] \qquad (2.2.3) + (2.2.6)$$

If $\gamma = \theta = 0$, then $D \in D(1, \frac{\beta}{\alpha}, 0, 0)$.
If $\gamma = \theta \neq 0$, $D \in D(1, \frac{\beta}{\alpha}, \frac{\gamma}{\alpha}, \frac{\gamma}{\alpha})$.

Similarly, if $\beta=\gamma$, then $D(\alpha,\beta,\gamma,\theta)=D(1,\frac{\theta}{\alpha},0,0)$ or $D(\alpha,\beta,\gamma,\theta)=D(1,\frac{\gamma}{\alpha},\frac{\gamma}{\alpha},\frac{\theta}{\alpha})$.
If $\theta\neq\gamma$ and $\beta\neq\gamma$, it follows from the equations above that $D\in QC(\mathfrak{g})$ hence D satisfies $D([x,y,z])=\frac{\beta+\gamma+\theta}{\alpha}[D(x),y,z]$.

■

Let us now discuss the case of a ternary Lie algebra induced by a Lie algebra. Here we denote the space of the $(\alpha,\beta,\gamma,\theta)$-derivations of the induced ternary Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot]_\tau)$ by $D_\tau(\alpha,\beta,\gamma,\theta)$.

Proposition 2.2.36 *Suppose that for every $u,v\in\mathfrak{g}$ we have $\tau(u).v=\tau(v).u$, then any (α,β,γ)-derivation D of $(\mathfrak{g},[\cdot,\cdot])$ that satisfies $D(\mathfrak{g})\subset Ker(\tau)$ is an $(\alpha',\beta',\gamma',\theta')$-derivation of $(\mathfrak{g},[\cdot,\cdot,\cdot]_\tau)$ for some $\alpha',\beta',\gamma',\theta'$. Precisely, D is in one of the following spaces:*

(i) $End(\mathfrak{g})$,

(ii) $\{f\in End(\mathfrak{g}); f(\mathfrak{g}^1)=\{0\}\}$,

(iii) $QC(\mathfrak{g},[\cdot,\cdot,\cdot]_\tau)$,

(iv) $QC(\mathfrak{g},[\cdot,\cdot,\cdot]_\tau)\cap\{f\in End(\mathfrak{g}); f(\mathfrak{g}^1)=\{0\}\}$,

(v) $D_\tau(\delta,1,1,1)$, *for some* $\delta\in\mathbb{K}$,

(vi) $QC(\mathfrak{g},[\cdot,\cdot,\cdot]_\tau)\cap D_\tau(\delta,1,1,1)$, *for some* $\delta\in\mathbb{K}$.

To prove this, we recall the following proposition stated in [39].

Proposition 2.2.37 *Let $\mathfrak{g}$ be a Lie algebra and $\alpha,\beta,\gamma\in\mathfrak{C}$. $D(\alpha,\beta,\gamma)$ is one of the following spaces:*

(i) $D(0,0,0)=End(\mathfrak{g})$,

(ii) $D(1,0,0)=\{f\in End(\mathfrak{g}); f(\mathfrak{g}^1)=\{0\}\}$,

(iii) $D(0,1,-1)=QC(\mathfrak{g},[\cdot,\cdot,\cdot]_\tau)$,

(iv) $D(1,1,-1) = QC(\mathfrak{g}, [\cdot,\cdot]) \cap \{f \in End(\mathfrak{g}); f(\mathfrak{g}^1) = \{0\}\}$,

(v) $D(\delta, 1, 1)$,

(vi) $D(\delta, 1, 0) = QC(\mathfrak{g}, [\cdot,\cdot]) \cap D(2\delta, 1, 1)$.

Proof. [Proof of Proposition 2.2.36] $(i), (ii)$ are obvious.
$(iii), (iv)$ by proposition 2.2.31.
(v) Let $D \in D(\delta, 1, 1)$. Then

$$\begin{aligned}
\delta D([x,y,z]_\tau) &= \tau(x)\delta D([y,z]) + \tau(z)\delta D([x,y]) + \tau(y)\delta D([z,x]) \\
&= \tau(x)([D(y),z] + [y,D(z)]) + \tau(z)([D(x),y] + [x,D(y)]) \\
&+ \tau(y)([D(z),x] + [z,D(x)]) \\
&= \tau(z)[D(x),y] + \tau(y)[z,D(x)] + \tau(x)[D(y),z] + \tau(z)[x,D(y)] \\
&+ \tau(y)[D(z),x] + \tau(x)[y,D(z)] \\
&= [D(x),y,z]_\tau + [x,D(y),z]_\tau + [x,y,D(z)]_\tau.
\end{aligned}$$

(vi) We apply (v) to the space $D(2\delta, 1, 1)$.

■

2.3 Derivations and Central Derivations of ternary Lie Algebras of dimension less or equal than 4

In this section, we will use the classification theorem of ternary Lie algebras of dimension less or equal 4 given in [13] to determine the spaces $Der(\mathfrak{g})$ and $C(\mathfrak{g})$. Then we determine the space of central derivations $ZDer(\mathfrak{g})$ using Proposition 2.2.21.

Theorem 2.3.1 *[13] Let $\mathfrak{g}$ be a ternary Lie algebra of dimension less or equal than 4 and let $(e_i)_{1\leq i\leq dim(\mathfrak{g})}$ be a basis of $\mathfrak{g}$. Then $\mathfrak{g}$ is isomorphic to one of the following*

1. *If dim $\mathfrak{g} < 3$, then $\mathfrak{g}$ is abelian*

2. *If dim $\mathfrak{g}$ = 3, then*

a. $\mathfrak{g}$ *is abelian.*

b. $[e_1, e_2, e_3] = e_1$.

3. *If dim* $\mathfrak{g} = 4$*, then*

a. $\mathfrak{g}$ *is abelian*

b. $[e_2, e_3, e_4] = e_1$.

c. $[e_1, e_2, e_3] = e_1$.

d. $[e_1, e_2, e_4] = \alpha e_3 + \beta e_4; [e_1, e_2, e_3] = \gamma e_3 + \delta e_4$*, where* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *is an invertible matrix.*

e. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = \alpha e_2; [e_1, e_2, e_4] = \beta e_3$ *with* $\alpha, \beta \neq 0$.

f. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = \alpha e_2; [e_1, e_2, e_4] = \beta e_3; [e_1, e_2, e_3] = \gamma e_4$ *with* $\alpha, \beta, \gamma \neq 0$.

The omitted brackets are either zeros or can be obtained by skew-symmetry.
Let D be a linear map of $\mathfrak{g}$ and let $M = (a_{i,j})$ its matrix in the basis $(e_i)_{1 \leq i \leq dim(\mathfrak{g})}$. We will compute the spaces of derivations. Each of the following items corresponds to its respective case in the previous theorem.

1. If $\dim(\mathfrak{g}) < 3$, then we have

$$Der(\mathfrak{g}) = C(\mathfrak{g}) = ZDer(\mathfrak{g}) = End(\mathfrak{g}).$$

2. $\dim(\mathfrak{g})$=3, $\{e_1, e_2, e_3\}$ a basis of $\mathfrak{g}$

a. $\mathfrak{g}$ is abelian: same as the case (1).

b. $[e_1, e_2, e_3] = e_1$.

$*$ $Der(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & -a_{22} \end{pmatrix}\}.$$

$* \; C(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} & a_{23} \\ 0 & a_{32} & a_{11} \end{pmatrix} \}.$$

$* \; ZDer(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix} \}.$$

3. dim($\mathfrak{g}$)=4, $\{e_1, e_2, e_3, e_4\}$ a basis of $\mathfrak{g}$

   a. $\mathfrak{g}$ is abelian: same as the case (1).

   b. $[e_2, e_3, e_4] = e_1$

   $* \; Der(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} a_{22}+a_{33}+a_{44} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix} \}.$$

   $* \; C(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that,

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & a_{23} & a_{24} \\ 0 & a_{32} & a_{11} & a_{34} \\ 0 & a_{42} & a_{43} & a_{11} \end{pmatrix} \}.$$

$*$ $ZDer(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix}\}.$$

c. $[e_1, e_2, e_3,] = e_1$

$*$ $Der(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that,

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & -a_{22} & 0 \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}\}.$$

$*$ $C(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & a_{23} & a_{24} \\ 0 & a_{32} & a_{11} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}\}.$$

$*$ $ZDer(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}\}.$$

d. $[e_1, e_2, e_4] = \alpha e_3 + \beta e_4; [e_1, e_2, e_3] = \gamma e_3 + \delta e_4$, with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is an invertible matrix.

The matrix of a derivation D is of the form

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where A, B, C are 2×2 matrices such that $Tr(A) = 0$ except if $\beta \neq \delta$.

$*$ $ZDer(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}\}.$$

$e.$ $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = \alpha e_2; [e_1, e_2, e_4] = \beta e_3$

$*$ $Der(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \alpha a_{12} & a_{11} & a_{23} & a_{24} \\ -\beta a_{13} & \frac{\beta}{\alpha} a_{23} & a_{11} & a_{34} \\ 0 & 0 & 0 & -a_{11} \end{pmatrix}\}.$$

$*$ $C(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{11} & 0 & a_{24} \\ 0 & 0 & a_{11} & a_{34} \\ 0 & 0 & 0 & a_{11} \end{pmatrix}\}.$$

* $ZDer(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & 0 & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}\}.$$

f. $[e_2, e_3, e_4] = e_1; [e_1, e_3, e_4] = \alpha e_2; [e_1, e_2, e_4] = \beta e_3; [e_1, e_2, e_3] = \gamma e_4$

* $Der(\mathfrak{g}) = \{D \in End(\mathfrak{g})$ such that

$$M = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ \alpha a_{12} & 0 & a_{23} & a_{24} \\ -\beta a_{13} & \frac{\beta}{\alpha} a_{23} & 0 & a_{34} \\ \gamma a_{14} & -\frac{\gamma}{\alpha} a_{24} & \frac{\gamma}{\beta} a_{34} & 0 \end{pmatrix}\}.$$

* $C(\mathfrak{g}) = \{\lambda Id\}$,
* Thus $ZDer(\mathfrak{g}) = \{0\}$.

2.3.1 Classification of $(\alpha, \beta, \gamma, \theta)$-Derivations and $(\alpha, \beta, \gamma, \theta)$-Quasiderivations

Now we classify, using Theorem 2.2.35, $(\alpha, \beta, \gamma, \theta)$-Derivations in dimension three and dimension four. For this we need to determine the spaces $D(1, \lambda, 0, 0)$ and $QC(\mathfrak{g})$.

Remark 2.3.2 *If $\lambda = 1$, then $D(1, \lambda, 0, 0) \cap QC(\mathfrak{g}) = C(\mathfrak{g})$. Therefore, in the following computations, we suppose that $\lambda \neq 1$.*

Lemma 2.3.3 *Every central derivation of a ternary Lie algebra $\mathfrak{g}$ is an $(\alpha, \beta, \gamma, \theta)$-derivation.*

Proof. Let D be a central derivation of $\mathfrak{g}$, then the image of $\mathfrak{g}$ under D is a subset of its center. Therefore, for any x, y, z in $\mathfrak{g}$ we have

$$[D(x), y, z] = [x, D(y), z] = [x, y, D(z)] = 0.$$

On the other hand, since $D(\mathfrak{g}^1) = \{0\}$, so $D([x,y,z]) = 0$. Thus D is an $(\alpha,\beta,\gamma,\theta)$-derivation.

■

Let D be an $(\alpha,\beta,\gamma,\theta)$-Derivation of $\mathfrak{g}$

1. If $dim(\mathfrak{g}) = 3$,
   - $\mathfrak{g}$ is abelian, $D(\alpha,\beta,\gamma,\theta) = End(\mathfrak{g})$.
   - $[e_1,e_2,e_3] = e_1 : D(\alpha,\beta,\gamma,\theta) = ZDer(\mathfrak{g})$.
2. If $dim(\mathfrak{g}) = 4$,
   - $[e_2,e_3,e_4] = e_1$,
     - If $\lambda = 0$, then $D(\alpha,\beta,\gamma,\theta) = ZDer(\mathfrak{g})$.
     - If $\lambda \neq 0$, then the matrix M of D has the form

$$M = \begin{pmatrix} \frac{1}{\lambda}a_{22} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{22} & a_{34} \\ 0 & a_{42} & a_{43} & a_{22} \end{pmatrix}.$$

   - $[e_1,e_2,e_3] = e_1$: $D(\alpha,\beta,\gamma,\theta) = ZDer(\mathfrak{g})$.
   - $[e_2,e_3,e_4] = e_1; [e_1,e_3,e_4] = ae_2; [e_1,e_2,e_4] = be_3 : D(\alpha,\beta,\gamma,\theta) = ZDer(\mathfrak{g})$.
   - $[e_2,e_3,e_4] = e_1; [e_1,e_3,e_4] = ae_2; [e_1,e_2,e_4] = be_3; [e_1,e_2,e_3] = ce_4 : D(\alpha,\beta,\gamma,\theta) = ZDer(\mathfrak{g})$.

3 Generalized Derivations of n-BiHom-Lie algebras

This chapter is devoted to studying the generalized derivations of n-BiHom-Lie algebras. We introduce and study properties of derivations, (α^s, β^r)-derivations and generalized derivations. We also study quasiderivations of n-BiHom-Lie algebras. Generalized derivations of $(n+1)$-BiHom-Lie algebras induced by n-BiHom-Lie algebras are also considered. Section 3.1 deals with the preliminary background including the main definitions. In Section 3.2 we study properties of derivations, (α^s, β^r)-derivations and generalized derivations. Section 3.3 is dedicated to quasi-derivations of n-BiHom-Lie algebras we prove that the quasi-derivation algebra of an n-BiHom-Lie algebra can be embedded into the derivation algebra of a larger n-BiHom-Lie algebra. In Section 3.4 we study generalized derivations of $(n+1)$-BiHom-Lie algebras induced by n-BiHom-Lie algebras.

3.1 Basic review of n-BiHom-Lie algebras

Definition 3.1.1 A quadruple $(\mathfrak{g}, [\cdot,\cdot,\cdot], \alpha, \beta)$, where $\mathfrak{g}$ is a vector space, α, β are linear maps of $\mathfrak{g}$, and $[\cdot,\cdot,\cdot] : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}$ is a 3-linear map, is called a 3-*BiHom-Lie algebra* if the following conditions are satisfied.

1. $\alpha \circ \beta = \beta \circ \alpha$.

2. $[\beta(x_1), \beta(x_2), \alpha(x_3)] = Sgn(\sigma)[\beta(x_{\sigma(1)}), \beta(x_{\sigma(2)}), \alpha(x_{\sigma(3)})]$, for all $x_1, x_2, x_3 \in \mathfrak{g}$ and $\sigma \in S_3$.

3.

$$[\beta^2(x_1), \beta^2(x_2), [\beta(y_1), \beta(y_2), \alpha(y_3)]] = [\beta^2(y_2), \beta^2(y_3), [\beta(x_1), \beta(x_2), \alpha(y_1)]]$$
$$- [\beta^2(y_1), \beta^2(y_3), [\beta(x_1), \beta(x_2), \alpha(y_2)]] + [\beta^2(y_1), \beta^2(y_2), [\beta(x_1), \beta(x_2), \alpha(y_3)]],$$

for all $x_1, x_2, y_1, y_2, y_3 \in \mathfrak{g}$.

Definition 3.1.2 An *n-BiHom-Lie algebra* is a vector space $\mathfrak{g}$ equipped with an n-linear map $[\cdot, \dots, \cdot]$ and two linear maps α and β such that

1. $\alpha \circ \beta = \beta \circ \alpha$.

2. $[\beta(x_1), \dots, \beta(x_{n-1}), \alpha(x_n)] = Sgn(\sigma)[\beta(x_{\sigma(1)}), \dots, \beta(x_{\sigma(n-1)}), \alpha(x_{\sigma(n)})]$, for any $\sigma \in S_n$.

3.

$$[\beta^2(x_1), \dots, \beta^2(x_{n-1}), [\beta(y_1), \dots, \beta(y_{n-1}), \alpha(y_n)]] =$$
$$\sum_{k=1}^{n} (-1)^{n-k} [\beta^2(y_1), \dots, \widehat{\beta^2(y_k)}, \dots, \beta^2(y_n), [\beta(x_1), \dots, \beta(x_{n-1}), \alpha(y_k)]],$$

for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in \mathfrak{g}$.

We say that $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha, \beta)$ is a *multiplicative* n-BiHom-Lie algebra if α and β are algebra morphisms and regular if they are automorphisms.

n-BiHom-Lie algebras may be induced from n-Lie algebras using two algebra morphisms as stated in the following proposition given in [28].

Proposition 3.1.3 *Let $(V, [\cdot, \dots, \cdot])$ be an n-Lie algebra and α, β two morphisms of V that commute with each other. For $x_1, \dots, x_n \in V$ define*

$$[x_1, \dots, x_n]_{\alpha\beta} = [\alpha(x_1), \dots, \alpha(x_{n-1}), \beta(x_n)].$$

Then $(V, [\cdot, \dots, \cdot]_{\alpha\beta}, \alpha, \beta)$ is an n-BiHom-Lie algebra.

In this chapter we are interested in the derivations of this particular type of n-BiHom-Lie algebras, we compare them to the derivations of the original Lie algebras, and study their inherited properties.

Example 3.1.4 *Let V be a 4-dimensional vector space with the basis $\{e_1, e_2, e_3, e_4\}$. Define the following brackets:*

$$[e_1, e_2, e_3] = -e_4 \ ; \quad [e_1, e_2, e_4] = e_3 \ ; \quad [e_1, e_3, e_4] = -e_2 \ ; \quad [e_2, e_3, e_4] = e_1.$$

With this bracket, $(V, [\cdot, \cdot, \cdot])$ is a 3-Lie algebra. Let α and β be two linear maps of V defined by :

$$\alpha(e_1) = -e_2 \ ; \quad \alpha(e_2) = -e_1 \ ; \quad \alpha(e_3) = -e_4 \ ; \quad \alpha(e_4) = -e_3 \quad \textit{and} \ , \quad \beta = -\alpha.$$

Let $[x_1, x_2, x_3]_{\alpha\beta} = [\alpha(x_1), \alpha(x_2), \beta(x_3)]$, be a twisted bracket defined on V. Then $(V, [\cdot, \cdot, \cdot]_{\alpha\beta}, \alpha, \beta)$ is a 3-BiHom-Lie algebra.

Recall that a subset $\mathcal{S} \subseteq \mathfrak{g}$ is a subalgebra of $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha, \beta)$ if $\alpha(\mathcal{S}) \subseteq \mathcal{S}$, $\beta(\mathcal{S}) \subseteq \mathcal{S}$ and $[\mathcal{S}, \mathcal{S}, \dots, \mathcal{S}] \subseteq \mathcal{S}$. We say that $\mathcal{S}$ is an ideal if $\alpha(\mathcal{S}) \subseteq \mathcal{S}$, $\beta(\mathcal{S}) \subseteq \mathcal{S}$ and $[\mathcal{S}, \mathcal{S}, \dots, \mathfrak{g}] \subseteq \mathcal{S}$.

Definition 3.1.5 The *center* of $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha, \beta)$ is the set of $u \in \mathfrak{g}$ such that

$$[u, x_1, x_2, \dots, x_{n-1}] = 0$$

for all $x_1, x_2, \dots, x_{n-1} \in \mathfrak{g}$. The center is an ideal of $\mathfrak{g}$ which we will denote by $Z(\mathfrak{g})$.

A more general definition of the center is the one involving the two morphisms α and β and we will call it the (α, β)-center.

Definition 3.1.6 The (α, β)-center of $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha, \beta)$ is the set

$$Z_{(\alpha,\beta)}(\mathfrak{g}) = \{u \in \mathfrak{g}, [u, \alpha\beta(x_1), \dots, \alpha\beta(x_{n-1})] = 0, \ for \ any \ x_1, \dots, x_{n-1} \in \mathfrak{g}\}.$$

Example 3.1.7 *A direct computation gives that the* (α, β)*-center of the* 3*-BiHom-Lie algebra given in Example 3.1.4 is trivial, that is* $Z_{(\alpha,\beta)}(\mathfrak{g}) = \{0\}$.

3.2 Derivations, (α^s, β^r)-derivations and Generalized derivations

Definition 3.2.1 Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha, \beta)$ be a 3-BiHom-Lie algebra. A linear map $D : \mathfrak{g} \to \mathfrak{g}$ is a *derivation* if for all $x, y, z \in \mathfrak{g}$

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)], \tag{3.2.1}$$

and it is called an (α^s, β^r)-derivation of $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha, \beta)$, if it satisfies

$$\begin{aligned} D \circ \alpha &= \alpha \circ D, \text{ and } D \circ \beta = \beta \circ D, & (3.2.2)\\ D([x, y, z]) &= [D(x), \alpha^s\beta^r(y), \alpha^s\beta^r(z)] + [\alpha^s\beta^r(x), D(y), \alpha^s\beta^r(z)] + \\ & \quad [\alpha^s\beta^r(x), \alpha^s\beta^r(y), D(z)]. & (3.2.3) \end{aligned}$$

Similarly, one can define (α^s, β^r)-derivations of n-BiHom-Lie algebras. Condition (3.2.3) becomes

$$\begin{aligned} &D[x_1, \ldots, x_n] = [D(x_1), \alpha^s\beta^r(x_2), \ldots, \alpha^s\beta^r(x_n)] \\ &+ \sum_{i=2}^{n} [\alpha^s\beta^r(x_1), \ldots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \ldots, \alpha^s\beta^r(x_n)]. \end{aligned}$$

Let $\mathrm{Der}_{(\alpha^s,\beta^r)}(\mathfrak{g})$ be the set of (α^s, β^r)-derivations of $\mathfrak{g}$ and set

$$\mathrm{Der}(\mathfrak{g}) := \bigoplus_{s\geq 0} \bigoplus_{r\geq 0} \mathrm{Der}_{(\alpha^s,\beta^r)}(\mathfrak{g}).$$

We show that $\mathrm{Der}(\mathfrak{g})$ is equipped with a Lie algebra structure. In fact, for $D \in \mathrm{Der}_{(\alpha^s,\beta^r)}(\mathfrak{g})$ and $D' \in \mathrm{Der}_{(\alpha^{s'},\beta^{r'})}(\mathfrak{g})$, we have $[D, D'] \in \mathrm{Der}_{(\alpha^{s+s'},\beta^{r+r'})}(\mathfrak{g})$, where $[D, D']$ is the standard commutator defined by $[D, D'] = DD' - D'D$.

Let $(V, [\cdot, \dots, \cdot])$ be an n-Lie algebra and $(V, [\cdot, \dots, \cdot]_{\alpha\beta}, \alpha, \beta)$ the induced n-BiHom-Lie algebra where $\alpha,\ \beta$ are the two morphisms used for this induction. A direct computation gives the following proposition

Proposition 3.2.2 *Any derivation of the n-Lie algebra $(V, [\cdot, \dots, \cdot])$ is a derivation of its induced n-BiHom-Lie algebra $(V, [\cdot, \dots, \cdot]_{\alpha\beta}, \alpha, \beta)$ as well.*

Definition 3.2.3 Let $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha, \beta)$ be an n-BiHom-Lie algebra and let D be an endomorphism of $\mathfrak{g}$. The linear map D is called a generalized (α^s, β^r)-derivation of $\mathfrak{g}$ if there exists $D^{(i)}$, $i \in \{1, \dots, n\}$, a family of endomorphisms of $\mathfrak{g}$, such that

$$\begin{aligned}
&D \circ \alpha = \alpha \circ D;\ D \circ \beta = \beta \circ D \\
&D^{(i)} \circ \alpha = \alpha \circ D^{(i)};\ D^{(i)} \circ \beta = \beta \circ D^{(i)}\ for\ any\ i,\ and \\
&D^{(n)}([x_1, \dots, x_n]) = [D(x_1), \alpha^s\beta^r(x_2), \dots, \alpha^s\beta^r(x_n)] \\
&+ \sum_{i=2}^{n} [\alpha^s\beta^r(x_1), \dots, \alpha^s\beta^r(x_{i-1}), D^{(i-1)}(x_i), \alpha^s\beta^r(x_{i+1}), \dots, \alpha^s\beta^r(x_n)]
\end{aligned}$$

for all $x_1, \dots, x_n \in \mathfrak{g}$.

The set of generalized (α^s, β^r)-derivations of $\mathfrak{g}$ is $\mathrm{GDer}_{(\alpha^s,\beta^r)}(\mathfrak{g})$ and as for $\mathrm{Der}(\mathfrak{g})$, we denote

$$\mathrm{GDer}(\mathfrak{g}) := \bigoplus_{s \geq 0} \bigoplus_{r \geq 0} \mathrm{GDer}_{(\alpha^s,\beta^r)}(\mathfrak{g}).$$

Definition 3.2.4 An endomorphism D of an n-BiHom-Lie algebra $\mathfrak{g}$ is called a (α^s, β^r)-*quasiderivation* if there exists an endomorphism D' of $\mathfrak{g}$ such that

$$\begin{aligned}
&D \circ \alpha = \alpha \circ D;\ D \circ \beta = \beta \circ D \\
&D' \circ \alpha = \alpha \circ D';\ D' \circ \beta = \beta \circ D',\ and \\
&D'([x_1, \dots, x_n]) = \sum_{i=1}^{n} [\alpha^s\beta^r(x_1), \dots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \dots, \alpha^s\beta^r(x_n)]
\end{aligned}$$

for any $x_1, \dots, x_n \in \mathfrak{g}$.

We then define

$$\mathrm{QDer}(\mathfrak{g}) := \bigoplus_{s\geq 0}\bigoplus_{r\geq 0} \mathrm{QDer}_{(\alpha^s,\beta^r)}(\mathfrak{g}).$$

Proposition 3.2.5 *Let $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha, \beta)$ be a regular n-BiHom-Lie algebra with trivial center. Suppose that $\mathfrak{g} = \mathcal{I} \oplus \mathcal{J}$, where $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathfrak{g}$, then*

$$GDer(\mathfrak{g}) = GDer(\mathcal{I}) \oplus GDer(\mathcal{J}).$$

Proof. To prove the proposition, first we will show that for any $D \in GDer(\mathfrak{g})$, we have $D(\mathcal{I}) \subset \mathcal{I}$ and $D(\mathcal{J}) \subset \mathcal{J}$, then it follows that the restriction of D to $\mathcal{I}$ (resp. $\mathcal{J}$) is a generalized derivation of $\mathcal{I}$ (resp. $\mathcal{J}$). Let $u \in \mathcal{I}$ and let $D(u) = a + b$, $a \in \mathcal{I}$, $b \in \mathcal{J}$ be the decomposition of $D(u)$. For any $y_1, \dots, y_{n-1} \in \mathfrak{g}$, we have $[b, y_1, \dots, y_{n-1}] \in \mathcal{J}$. On the other hand,

$$[b, y_1, \dots, y_{n-1}] = [D(u) - a, y_1, \dots, y_{n-1}] = [D(u), y_1, \dots, y_{n-1}] - [a, y_1, \dots, y_{n-1}]$$

since $\mathcal{I}$ is an ideal and $a \in \mathcal{I}$, so $[a, y_1, \dots, y_{n-1}] \in \mathcal{I}$. Moreover, for each $1 \leq i \leq n-1$, let $y_i = \alpha^s\beta^r(x_i)$, then

$$\begin{aligned}&[D(u), y_1, \dots, y_{n-1}] = [D(u), \alpha^s\beta^r(x_1), \dots, \alpha^s\beta^r(x_{n-1})]\\ &= D^{(n)}[u, x_1, \dots, x_{n-1}] - \sum_{i=1}^{n-1}[\alpha^s\beta^r(u), \alpha^s\beta^r(x_1), \dots, D^{(i)}(x_i), \alpha^s\beta^r(x_{i+1}), \dots, \alpha^s\beta^r(x_{n-1})].\end{aligned}$$

For every i, $[\alpha^s\beta^r(u), \alpha^s\beta^r(x_1), \dots, D^{(i)}(x_i), \alpha^s\beta^r(x_{i+1}), \dots, \alpha^s\beta^r(x_{n-1})] \in \mathcal{I}$, so $\sum_{i=1}^{n-1}[\alpha^s\beta^r(u), \alpha^s\beta^r(x_1), \dots, D^{(i)}(x_i), \alpha^s\beta^r(x_{i+1}), \dots, \alpha^s\beta^r(x_{n-1})] \in \mathcal{I}$.
Now let $x_i = a_i + b_i$ be the decomposition of x_i,

$$[u, x_1, \dots, x_{n-1}] = [u, a_1 + b_1, \dots, a_{n-1} + b_{n-1}] = [u, a_1 + b_1, \dots, a_{n-1}] + [u, a_1 + b_1, \dots, b_{n-1}]$$

but $[u, a_1 + b_1, \ldots, b_{n-1}] \in \mathcal{I} \cap \mathcal{J} = \{0\}$, so

$$[u, x_1, \ldots, x_{n-1}] = [u, a_1 + b_1, \ldots, a_{n-2} + b_{n-2}, a_{n-1}].$$

Similarly, $[u, a_1 + b_1, \ldots, b_{n-2}, a_{n-1}] = 0$. Thus,

$$[u, x_1, \ldots, x_{n-1}] = [u, a_1, \ldots, a_{n-2}, a_{n-1}].$$

Therefore,

$$D^{(n)}[u, x_1, \ldots, x_{n-1}] = D^{(n)}[u, a_1, \ldots, a_{n-1}] = [D(u), \alpha^s\beta^r(a_1), \ldots, \alpha^s\beta(a_{n-1})]$$
$$+ \sum_{i=1}^{n-1}[\alpha^s\beta^r(u), \alpha^s\beta^r(a_1), \ldots, D^{(i)}(a_i), \alpha^s\beta^r(a_{i+1}), \ldots, \alpha^s\beta^r(a_{n-1})] \in \mathcal{I}.$$

Then $[D(u), y_1, \ldots, y_{n-1}] \in \mathcal{I}$ and so is $[b, y_1, \ldots, y_{n-1}]$. Hence $[b, y_1, \ldots, y_{n-1}] \in \mathcal{I} \cap \mathcal{J}$. We conclude that $b \in Z(\mathfrak{g}) = \{0\}$ and so $D(\mathcal{I}) \subset \mathcal{I}$.

■

Remark: Since any derivation quasiderivation is a generalized derivation:

$$\mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{QDer}(\mathfrak{g}) \subseteq \mathrm{GDer}(\mathfrak{g}).$$

Hence Proposition 3.2.5 holds for $\mathrm{QDer}(\mathfrak{g})$ and $\mathrm{Der}(\mathfrak{g})$ as well, that is

$$\mathrm{Der}(\mathfrak{g}) = \mathrm{Der}(\mathcal{I}) \oplus \mathrm{Der}(\mathcal{J}),$$

and

$$\mathrm{QDer}(\mathfrak{g}) = \mathrm{QDer}(\mathcal{I}) \oplus \mathrm{QDer}(\mathcal{J}).$$

Definition 3.2.6 A linear map D is called an(α^s, β^r)-*central derivation* of $\mathfrak{g}$ if it satisfies

$$D([x_1, \ldots, x_n]) = [\alpha^s\beta^r(x_1), \ldots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \ldots, \alpha^s\beta^r(x_n)] = 0,$$

for all $i \in \{1, \ldots, n\}$.

The set of (α^s, β^r)-central derivations is denoted by $\mathrm{ZDer}_{(\alpha^s,\beta^r)}(\mathfrak{g})$ and we set

$$\mathrm{ZDer}(\mathfrak{g}) := \bigoplus_{s\geq 0}\bigoplus_{r\geq 0}\mathrm{ZDer}_{(\alpha^s,\beta^r)}(\mathfrak{g}).$$

Definition 3.2.7 The (α^s, β^r)-*centroid* of $(\mathfrak{g}, [\cdot, \ldots, \cdot], \alpha, \beta)$ denoted by $C_{(\alpha^s,\beta^r)}(\mathfrak{g})$ is the set of linear maps D satisfying:

$$D([x_1, \ldots, x_n]) = [\alpha^s\beta^r(x_1), \ldots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \ldots, \alpha^s\beta^r(x_n)]$$

for all $i \in \{1, \ldots, n\}$. We set

$$\mathrm{C}(\mathfrak{g}) := \bigoplus_{s\geq 0}\bigoplus_{r\geq 0}\mathrm{C}_{(\alpha^s,\beta^r)}(\mathfrak{g}).$$

Proposition 3.2.8 *For any r, s, we have*

$$\mathrm{ZDer}_{(\alpha^s,\beta^r)}(\mathfrak{g}) = \mathrm{Der}_{(\alpha^s,\beta^r)}(\mathfrak{g}) \cap \mathrm{C}_{(\alpha^s,\beta^r)}(\mathfrak{g}).$$

Proof. It is clear that $\mathrm{ZDer}_{(\alpha^s,\beta^r)}(\mathfrak{g}) \subseteq \mathrm{Der}_{(\alpha^s,\beta^r)}(\mathfrak{g})$ and $\mathrm{ZDer}_{(\alpha^s,\beta^r)}(\mathfrak{g}) \subseteq C_{(\alpha^s,\beta^r)}(\mathfrak{g})$. Conversely, let $D \in \mathrm{Der}_{(\alpha^s,\beta^r)}(\mathfrak{g}) \cap \mathrm{C}_{(\alpha^s,\beta^r)}(\mathfrak{g})$, so for each i we have

$$D([x_1, \ldots, x_n]) = [\alpha^s\beta^r(x_1), \ldots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \ldots, \alpha^s\beta^r(x_n)].$$

In addition,

$$D([x_1, \ldots, x_n]) = \sum_{i=1}^{n}[\alpha^s\beta^r(x_1), \ldots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \ldots, \alpha^s\beta^r(x_n)].$$

Then $D([x_1, \ldots, x_n]) = nD([x_1, \ldots, x_n])$. Thus $D([x_1, \ldots, x_n]) = 0$ and $D \in \mathrm{ZDer}_{(\alpha^s,\beta^r)}(\mathfrak{g})$. ∎

Definition 3.2.9 The (α^s, β^r)-*quasicentroid* $\mathrm{QC}_{(\alpha^s,\beta^r)}(\mathfrak{g})$ is the set of linear maps D such that

$$[D(x_1), \alpha^s\beta^r(x_2) \ldots, \alpha^s\beta^r(x_n)] = [\alpha^s\beta^r(x_1), \ldots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \ldots, \alpha^s\beta^r(x_n)]$$

for all $i \in \{1, \ldots, n\}$. We set

$$\mathrm{QC}(\mathfrak{g}) := \bigoplus_{s\geq 0}\bigoplus_{r\geq 0} \mathrm{QC}_{(\alpha^s,\beta^r)}(\mathfrak{g}).$$

Lemma 3.2.10 *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \alpha, \beta)$ be an n-BiHom-Lie algebra.*

(1) $[\mathrm{Der}(\mathfrak{g}), \mathrm{C}(\mathfrak{g})] \subseteq \mathrm{C}(\mathfrak{g})$;

(2) $\mathrm{C}(\mathfrak{g}) \oplus \mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{Der}(\mathfrak{g})$.

Proof. Let $D \in Der_{(\alpha^s,\beta^r)}(\mathfrak{g})$ and $D' \in C_{(\alpha^{s'},\beta^{r'})}(\mathfrak{g})$ for some s, s', r, r'. Let $x_1, \ldots, x_n \in \mathfrak{g}$.

(1) Compute

$$\begin{aligned}
&[DD'(x_1), \alpha^{s+s'}\beta^{r+r'}(x_2), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)] \\
&= D([D'(x_1), \alpha^{s'}\beta^{r'}(x_2), \ldots, \alpha^{s'}\beta^{r'}(x_n)]) - \sum_{i=2}^{n}[\alpha^s\beta^r D'(x_1), \ldots, D(x_i), \ldots, \alpha^s\beta^r(x_n)] \\
&= DD'([x_1, \ldots, x_n]) - \sum_{i=2}^{n}[\alpha^s\beta^r(x_1), \ldots, D'D(x_i), \ldots, \alpha^s\beta^r(x_n)].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&[D'D(x_1), \alpha^{s+s'}\beta^{r+r'}(x_2), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)] = D'([D(x_1), \alpha^s\beta^r(x_2), \ldots, \alpha^s\beta^r(x_n)]) \\
&= DD'([x_1, \ldots, x_n]) - D'(\sum_{i=2}^{n}[\alpha^s\beta^r(x_1), \ldots, D(x_i), \ldots, \alpha^s\beta^r(x_n)])
\end{aligned}$$

but since for each i,

$$D'([\alpha^s\beta^r(x_1), \ldots, D(x_i), \ldots, \alpha^s\beta^r(x_n)]) = [\alpha^s\beta^r(x_1), \ldots, D'D(x_i), \ldots, \alpha^s\beta^r(x_n)],$$

so

$$D'(\textstyle\sum_{i=2}^{n}[\alpha^s\beta^r(x_1), \ldots, D(x_i), \ldots, \alpha^s\beta^r(x_n)]) = \sum_{i=2}^{n}[\alpha^s\beta^r(x_1), \ldots, D'D(x_i), \ldots, \alpha^s\beta^r(x_n)].$$

Hence $[[D,D'](x_1), \alpha^{s+s'}\beta^{r+r'}(x_2), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)] = [D,D']([x_1,\ldots,x_n])$.

The same proof holds for any $i \in \{1,\ldots,n\}$. Thus $[D,D'] \in C_{(\alpha^{s+s'},\beta^{r+r'})}(\mathfrak{g})$.

(2) Now

$$\begin{aligned}
&D'D([x_1,\ldots,x_n]) = D'([D(x_1), \alpha^s\beta^r(x_2)\ldots,\alpha^s\beta^r(x_n)]) \\
&+ D'(\sum_{i=2}^{n}[\alpha^s\beta^r(x_1),\ldots,D(x_i),\ldots,\alpha^s\beta^r(x_n)]) \\
&= [D'D(x_1), \alpha^{s+s'}\beta^{r+r'}(x_2),\ldots,\alpha^{s+s'}\beta^{r+r'}(x_n)] \\
&+ \sum_{i=2}^{n}[\alpha^{s+s'}\beta^{r+r'}(x_1),\ldots,D'D(x_i),\ldots,\alpha^{s+s'}\beta^{r+r'}(x_n)].
\end{aligned}$$

Thus $D'D \in \mathrm{Der}_{(\alpha^{s+s'},\beta^{r+r'})}(\mathfrak{g})$.

■

In the following lemma, we provide some properties and relations of the subspaces of $Der(\mathfrak{g})$ involving in particular the subalgebra of quasiderivations $QDer(\mathfrak{g})$.

Lemma 3.2.11 *Let* $(\mathfrak{g}, [\cdot,\ldots,\cdot], \alpha, \beta])$ *be a multiplicative n-BiHom-Lie algebra.*

(1) $[\mathrm{QDer}(\mathfrak{g}), \mathrm{QC}(\mathfrak{g})] \subseteq \mathrm{QC}(\mathfrak{g})$;

(2) $\mathrm{C}(\mathfrak{g}) \subseteq \mathrm{QDer}(\mathfrak{g})$;

(3) $[\mathrm{QC}(\mathfrak{g}), \mathrm{QC}(\mathfrak{g})] \subseteq \mathrm{QDer}(\mathfrak{g})$;

(4) $\mathrm{QDer}(\mathfrak{g}) + \mathrm{QC}(\mathfrak{g}) \subseteq \mathrm{GDer}(\mathfrak{g})$.

Proof. (1) This inclusion is similar to (1) of Lemma 3.2.10.

(2) It is an immediate consequence of the definition of a quasiderivation. If $D \in C_{(\alpha^s,\beta^r)}(\mathfrak{g})$, then $\sum_{i=1}^{n}[\alpha^s\beta^r(x_1),\ldots,D(x_i),\ldots,\alpha^s\beta^r(x_n)] = nD([x_1,\ldots,x_n])$.

(3) Let $D \in QC_{(\alpha^s,\beta^r)}(\mathfrak{g})$ and $D' \in QC_{(\alpha^{s'},\beta^{r'})}(\mathfrak{g})$. For any $x_1, \ldots, x_n \in \mathfrak{g}$ we have

$$[DD'(x_1), \alpha^{s+s'}\beta^{r+r'}(x_2), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)] = [\alpha^s\beta^r D'(x_1), D\alpha^{s'}\beta^{r'}(x_2), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)]$$
$$= [\alpha^{s+s'}\beta^{r+r'}(x_1), D\alpha^{s'}\beta^{r'}(x_2), D'\alpha^s\beta^r(x_3), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)]$$
$$= [D\alpha^{s'}\beta^{r'}(x_1), \alpha^{s+s'}\beta^{r+r'}(x_2), D'\alpha^s\beta^r(x_3), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)]$$
$$= [D'D(x_1), \alpha^{s+s'}\beta^{r+r'}(x_2), \alpha^{s+s'}\beta^{r+r'}(x_3), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)].$$

Then $[[D, D'](x_1), \alpha^{s+s'}\beta^{r+r'}(x_2), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)] = 0$.

In the same way we have $[\alpha^{s+s'}\beta^{r+r'}(x_1), \ldots, [D, D'](x_i), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)] = 0$ for all i. Hence $\sum_{i=1}^n [\alpha^{s+s'}\beta^{r+r'}(x_1), \ldots, [D, D'](x_i), \ldots, \alpha^{s+s'}\beta^{r+r'}(x_n)] = 0$. And so $[D, D'] \in QDer_{(\alpha^{s+s'},\beta^{r+r'})}(\mathfrak{g})$.

(4) The inclusion is straightforward.

■

Proposition 3.2.12 *If $(\mathfrak{g}, [\cdot, \ldots, \cdot], \alpha, \beta)$ is an n-BiHom-Lie algebra with trivial center, then*

$$\mathrm{Der}(\mathfrak{g}) \oplus \mathrm{C}(\mathfrak{g}) \subseteq \mathrm{QDer}(\mathfrak{g}).$$

Proof. Both $\mathrm{Der}(\mathfrak{g})$ and $\mathrm{C}(\mathfrak{g})$ are subspaces of $\mathrm{QDer}(\mathfrak{g})$. Moreover, if $D \in \mathrm{Der}(\mathfrak{g}) \cap \mathrm{C}(\mathfrak{g})$ then for $u \in \mathfrak{g}$ we have $\sum_{i=1}^n [D(u), x_1, \ldots, x_{n-1}] = [D(u), x_1, \ldots, x_{n-1}] = 0$, for all $x_1, \ldots, x_{n-1} \in \mathfrak{g}$. Therefore $D(u) \in Z(\mathfrak{g})$, hence $D = 0$.

■

3.3 Quasiderivations of n-BiHom-Lie Algebras

The main goal in this section is to prove that the space of quasi-derivations of an n-BiHom Lie algebra can be embedded in the space of derivation of a larger algebra. We start by giving an n-BiHom Lie structure to the vector space that we define in the following proposition.

Proposition 3.3.1 *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot]_{\mathfrak{g}}, \alpha, \beta)$ be an n-BiHom-Lie algebra over $\mathbb{K}$ and t be an indeterminate. Define* $\breve{\mathfrak{g}} = \{\Sigma(x \otimes t + y \otimes t^n) \mid x, y \in \mathfrak{g}\}$, $\breve{\alpha}(\breve{\mathfrak{g}}) = \{\Sigma(\alpha(x) \otimes t + \alpha(y) \otimes t^n) \mid x, y \in \mathfrak{g}\}$, *and* $\breve{\beta}(\breve{\mathfrak{g}}) = \{\Sigma(\beta(x) \otimes t + \beta(y) \otimes t^n) \mid x, y \in \mathfrak{g}\}$. *Then* $(\breve{\mathfrak{g}}, [\cdot, \ldots, \cdot]_{\breve{\mathfrak{g}}}, \breve{\alpha}, \breve{\beta})$ *is a multiplicative n-BiHom-Lie algebra where the bracket is given by*

$$[x_1 \otimes t^{i_1}, x_2 \otimes t^{i_2}, \ldots, x_n \otimes t^{i_n}]_{\breve{\mathfrak{g}}} = [x_1, x_2, \ldots, x_n]_{\mathfrak{g}} \otimes t^{\sum i_j},$$

for $i_1, \ldots, i_n \in \{1, n\}$ *. If* $k > n$*, we let* $t^k = 0$.

Proof. For any $x, x_1, \ldots, x_n \in \mathfrak{g}$ and $i, i_1, \ldots, i_n \in \{1, n\}$, we have

$$\begin{aligned}\breve{\alpha} \circ \breve{\beta}(x \otimes t^i) &= \breve{\alpha}(\beta(x) \otimes t^i)\\ &= \alpha \circ \beta(x) \otimes t^i\\ &= \beta \circ \alpha(x) \otimes t^i = \breve{\beta} \circ \breve{\alpha}(x \otimes t^i).\end{aligned}$$

Then $\breve{\alpha} \circ \breve{\beta} = \breve{\beta} \circ \breve{\alpha}$. Also,

$$\begin{aligned}\breve{\alpha}([x_1 \otimes t^{i_1}, x_2 \otimes t^{i_2}, \ldots, x_n \otimes t^{i_n}]) &= \breve{\alpha}([x_1, x_2, \ldots, x_n] \otimes t^{\sum i_j})\\ &= [\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)] \otimes t^{\sum i_j}\\ &= [\breve{\alpha}(x_1 \otimes t^{i_1}), \breve{\alpha}(x_2 \otimes t^{i_2}), \ldots, \breve{\alpha}(x_n \otimes t^{i_n})].\end{aligned}$$

The same argument holds for $\breve{\beta}$.

$$\begin{aligned}&[\breve{\beta}(x_1 \otimes t^{i_1}), \ldots, \breve{\beta}(x_{n-1} \otimes t^{i_{n-1}}), \breve{\alpha}(x_n \otimes t^{i_n})] = [\beta(x_1), \ldots, \beta(x_{n-1}), \alpha(x_n)] \otimes t^{\sum i_j}\\ &= Sgn(\sigma)[\beta(x_{\sigma(1)}), \ldots, \beta(x_{\sigma(n-1)}), \alpha(x_{\sigma(n)})] \otimes t^{\sum i_j}\\ &= Sgn(\sigma)[\breve{\beta}(x_{\sigma(1)} \otimes t^{i_1}, \ldots, \breve{\beta}(x_{\sigma(n-1)} \otimes t^{i_{n-1}}), \breve{\alpha}(x_{\sigma(n)} \otimes t^{i_n})]\end{aligned}$$

for any $\sigma \in S_n$. Note that if $i_j = n$ for some j, then the bracket would be zero since in that case the sum $\sum i_j \geq n + 1$ therefore $t^{\sum i_j} = 0$. So one may assume that $i_1 = \ldots = i_n = 1$.

Finally,

$$
\begin{aligned}
&[\breve{\beta}^2(x_1 \otimes t^{i_1}), \ldots, \breve{\beta}^2(x_{n-1} \otimes t^{i_{n-1}}), [\breve{\beta}(y_1 \otimes t^{i'_1}), \ldots, \breve{\beta}(y_{n-1} \otimes t^{i'_{n-1}}), \breve{\alpha}(y_n \otimes t^{i'_n})]] \\
&= [\beta^2(x_1) \otimes t^{i_1}, \ldots, \beta^2(x_{n-1}) \otimes t^{i_{n-1}}, [\beta(y_1), \ldots, \beta(y_{n-1}), \alpha(y_n)] \otimes t^{\sum i'_j}] \\
&= [\beta^2(x_1), \ldots, \beta^2(x_{n-1}), [\beta(y_1), \ldots, \beta(y_{n-1}), \alpha(y_n)]] \otimes t^{\sum i_j + \sum i'_j} \\
&= \sum_{k=1}^{n} (-1)^{n-k} [\beta^2(y_1), \ldots, \widehat{\beta^2(y_k)}, \ldots, \beta^2(y_n), [\beta(x_1), \ldots, \beta(x_{n-1}), \alpha(y_k)]] \otimes t^{\sum i_j + \sum i'_j} \\
&= \sum_{k=1}^{n} (-1)^{n-k} [\breve{\beta}^2(y_1 \otimes t^{i'_1}), \ldots, \breve{\beta}^2\widehat{(y_k \otimes t^{i'_k})}, \ldots, \breve{\beta}^2(y_n \otimes t^{i'_n}), [\breve{\beta}(x_1 \otimes t^{i_1}), \ldots \\
&\ldots, \breve{\beta}(x_{n-1} \otimes t^{i_{n-1}}), \breve{\alpha}(y_k \otimes t^{i'_k})]].
\end{aligned}
$$

Thus, $(\breve{\mathfrak{g}}, [\cdot, \ldots, \cdot], \breve{\alpha}, \breve{\beta})$ is a multiplicative n-BiHom-Lie algebra.

■

For the sake of convenience, we will write xt (xt^n) instead of $x \otimes t$ ($x \otimes t^n$). If U is a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = U \oplus [\mathfrak{g}, \ldots, \mathfrak{g}]$, then

$$\breve{\mathfrak{g}} = \mathfrak{g}t + \mathfrak{g}t^n = \mathfrak{g}t + Ut^n + [\mathfrak{g}, \ldots, \mathfrak{g}]t^n.$$

Let a map $\varphi : \mathrm{QDer}(\mathfrak{g}) \to \mathrm{End}(\breve{\mathfrak{g}})$ be defined by

$$\varphi(D)(at + ut^n + bt^n) = D(a)t + D'(b)t^n,$$

where $D \in \mathrm{QDer}(\mathfrak{g})$, D' is a map related to D by the definition of quasiderivation, $a \in \mathfrak{g}, u \in U, b \in [\mathfrak{g}, \ldots, \mathfrak{g}]$.

Proposition 3.3.2 *Let $\mathfrak{g}, \breve{\mathfrak{g}}, \varphi$ be as above. Then*

(1) *φ is injective and $\varphi(D)$ does not depend on the choice of D';*

(2) $\varphi(\mathrm{QDer}(\mathfrak{g})) \subseteq \mathrm{Der}(\breve{\mathfrak{g}})$.

Proof. (1) If $\varphi(D_1) = \varphi(D_2)$, then for all $a \in \mathfrak{g}, b \in [\mathfrak{g}, \dots, \mathfrak{g}]$ and $u \in U$ we have

$$\varphi(D_1)(at + ut^n + bt^n) = \varphi(D_2)(at + ut^n + bt^n),$$

so

$$D_1(a)t + D_1'(b)t^n = D_2(a)t + D_2'(b)t^n,$$

therefore $D_1(a) = D_2(a)$. Thus $D_1 = D_2$.

Now suppose that there exists D'' such that

$$\varphi(D)(at + ut^n + bt^n) = D(a)t + D''(b)t^n,$$

and

$$D''([x_1, \dots, x_n]) = \sum_{i=1}^{n} [\alpha^s\beta^r(x_1), \dots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \dots, \alpha^s\beta^r(x_n)],$$

for any $x_1, \dots, x_n \in \mathfrak{g}$, then $D''([x_1, \dots, x_n]) = D'([x_1, \dots, x_n])$. Hence $D''(b) = D'(b)$ and so

$$\varphi(D)(at + ut^n + bt^n) = D(a)t + D'(b)t^n = D(a)t + D''(b)t^n.$$

(2) Let $x_1t^{i_1}, \dots, x_nt^{i_n} \in \breve{\mathfrak{g}}$. Again, here we consider only the case when $i_1 = \dots = i_n = 1$ since otherwise $[x_1t^{i_1}, \dots, x_nt^{i_n}] = 0$.

$$\begin{aligned}
\varphi(D)([x_1t, \dots, x_nt]) &= \varphi(D)([x_1, \dots, x_n]t^n) = D'([x_1, \dots, x_n])t^n \\
&= \sum_{i=1}^{n} [\alpha^s\beta^r(x_1), \dots, \alpha^s\beta^r(x_{i-1}), D(x_i), \alpha^s\beta^r(x_{i+1}), \dots, \alpha^s\beta^r(x_n)]t^n \\
&= \sum_{i=1}^{n} [\alpha^s\beta^r(x_1)t, \dots, \alpha^s\beta^r(x_{i-1})t, D(x_i)t, \alpha^s\beta^r(x_{i+1})t, \dots, \alpha^s\beta^r(x_n)t] \\
&= \sum_{i=1}^{n} [\breve{\alpha}^s\breve{\beta}^r(x_1t), \dots, \breve{\alpha}^s\breve{\beta}^r(x_{i-1}t), \varphi(D)(x_it), \breve{\alpha}^s\breve{\beta}^r(x_{i+1}t), \dots, \breve{\alpha}^s\breve{\beta}^r(x_nt)].
\end{aligned}$$

Hence $\varphi(D) \in \mathrm{Der}_{(\alpha^s,\beta^r)}(\breve{\mathfrak{g}})$.

■

Proposition 3.3.3 *Let $\mathfrak{g}$ be a multiplicative n-BiHom-Lie algebra with trivial center and let $\breve{\mathfrak{g}}, \varphi$ be as defined above. Then*

$$\mathrm{Der}(\breve{\mathfrak{g}}) = \varphi(\mathrm{QDer}(\mathfrak{g})) \oplus \mathrm{ZDer}(\breve{\mathfrak{g}}).$$

Proof. It is obvious that $\varphi(\mathrm{QDer}(\mathfrak{g})) + \mathrm{ZDer}(\breve{\mathfrak{g}}) \subseteq \mathrm{Der}(\breve{\mathfrak{g}})$ since both $\varphi(\mathrm{QDer}(\mathfrak{g}))$ and $\mathrm{ZDer}(\breve{\mathfrak{g}})$ are subsets of $\mathrm{Der}(\breve{\mathfrak{g}})$.

Moreover, since $\mathrm{Z}(\mathfrak{g}) = \{0\}$, we have $\mathrm{Z}(\breve{\mathfrak{g}}) = \mathfrak{g}t^n$. Let $g \in \mathrm{Der}(\breve{\mathfrak{g}})$, so $g(\mathrm{Z}(\breve{\mathfrak{g}})) \subseteq \mathrm{Z}(\breve{\mathfrak{g}})$, then $g(Ut^n) \subseteq g(\mathrm{Z}(\breve{\mathfrak{g}})) \subseteq \mathrm{Z}(\breve{\mathfrak{g}}) = \mathfrak{g}t^n$. Define a map $f : \mathfrak{g}t + Ut^n + [\mathfrak{g}, \dots, \mathfrak{g}]t^n \to \mathfrak{g}t^n$ by

$$f(x) = \begin{cases} g(x) \cap \mathfrak{g}t^n, & x \in \mathfrak{g}t; \\ g(x), & x \in Ut^n; \\ 0, & x \in [\mathfrak{g}, \dots, \mathfrak{g}]t^n. \end{cases}$$

f is linear and we know that

$$f([\breve{\mathfrak{g}}, \dots, \breve{\mathfrak{g}}]) = f([\mathfrak{g}, \dots, \mathfrak{g}]t^n) = 0,$$

$$[\breve{\alpha}^s\breve{\beta}^r(\breve{\mathfrak{g}}), \dots, f(\breve{\mathfrak{g}}), \dots, \breve{\alpha}^s\breve{\beta}^r(\breve{\mathfrak{g}})] \subseteq [\alpha^s\beta^r(\mathfrak{g})t + \alpha^s\beta^r(\mathfrak{g})t^n, \dots, \mathfrak{g}t^n, \dots, \alpha^s\beta^r(\mathfrak{g})t + \alpha^s\beta^r(\mathfrak{g})t^n] = 0,$$

then $f \in \mathrm{ZDer}(\breve{\mathfrak{g}})$. We claim that $g - f \in \varphi(\mathrm{QDer}(\mathfrak{g}))$. This implies that $g \in \varphi(\mathrm{QDer}(\mathfrak{g})) + \mathrm{ZDer}(\breve{\mathfrak{g}})$, hence we have equality. In fact, since

$$(g - f)(\mathfrak{g}t) = g(\mathfrak{g}t) - g(\mathfrak{g}t) \cap \mathfrak{g}t^n = g(\mathfrak{g}t) - \mathfrak{g}t^n \subseteq \mathfrak{g}t, \ (g - f)(Ut^n) = 0,$$

and

$$(g - f)([\mathfrak{g}, \dots, \mathfrak{g}]t^n) = g([\breve{\mathfrak{g}}, \dots, \breve{\mathfrak{g}}]) \subseteq [\breve{\mathfrak{g}}, \dots, \breve{\mathfrak{g}}] = [\mathfrak{g}, \dots, \mathfrak{g}]t^n,$$

there exists $D, \ D' \in \mathrm{End}(\mathfrak{g})$ such that for all $a \in \mathfrak{g}, \ b \in [\mathfrak{g}, \dots, \mathfrak{g}]$,

$$(g - f)(at) = D(a)t, \ (g - f)(bt^n) = D'(b)t^n.$$

$g - f \in \mathrm{Der}(\breve{\mathfrak{g}})$, then

$$\sum_{i=1}^{n}[\breve{\alpha}^s\breve{\beta}^r(a_1t), \ldots, \breve{\alpha}^s\breve{\beta}^r(a_{i-1}t), (g-f)(a_it), \breve{\alpha}^s\breve{\beta}^r(a_{i+1}t), \ldots, \breve{\alpha}^s\breve{\beta}^r(a_nt)]$$
$$= (g-f)([a_1t, \ldots, a_nt]),$$

for all $a_1, \ldots, a_n \in \mathfrak{g}$. Then

$$\sum_{i=1}^{n}[\alpha^s\beta^r(a_1), \ldots, D(a_i), \ldots, \alpha^s\beta^r(a_n)]t^n = D'([a_1, \ldots, a_n])t^n,$$

thus

$$\sum_{i=1}^{n}[\alpha^s\beta^r(a_1), \ldots, D(a_i), \ldots, \alpha^s\beta^r(a_n)] = D'([a_1, \ldots, a_n]),$$

which means that $D \in \mathrm{QDer}(\mathfrak{g})$. Therefore, $g - f = \varphi(D) \in \varphi(\mathrm{QDer}(\mathfrak{g}))$.

Now if $f \in \varphi(\mathrm{QDer}(\mathfrak{g})) \cap \mathrm{ZDer}(\breve{\mathfrak{g}})$, then $f = \varphi(D)$ for some $D \in \mathrm{QDer}(\mathfrak{g})$. So

$$f(at + ut^n + bt^n) = \varphi(D)(at + ut^n + bt^n) = D(a)t + D'(b)t^n,$$

where $a \in \mathfrak{g}, b \in [\mathfrak{g}, \ldots, \mathfrak{g}]$. Also, since $f \in \mathrm{ZDer}(\breve{\mathfrak{g}})$, we have

$$f(at + bt^n + ut^n) \in \mathrm{Z}(\breve{\mathfrak{g}}) = \mathfrak{g}t^n.$$

That is, $D(a) = 0$, for all $a \in \mathfrak{g}$ and so $D = 0$. Hence $f = 0$.

We conclude that

$$\mathrm{Der}(\breve{\mathfrak{g}}) = \varphi(\mathrm{QDer}(\mathfrak{g})) \oplus \mathrm{ZDer}(\breve{\mathfrak{g}}).$$

■

3.4 Generalized Derivations of $(n+1)$-BiHom-Lie Algebras induced by n-BiHom-Lie algebras

In [5], the authors investigated a construction of $(n+1)$-Hom-Lie algebras induced by n-Hom-Lie algebras. The construction of $(n+1)$-BiHom-Lie Algebras induced by n-BiHom-Lie algebras was studied in [28]. In this section, we discuss (α^s, β^r)-derivations of n-BiHom-Lie algebras that give (α^s, β^r)-derivations on the induced $(n+1)$-BiHom-Lie algebras.

Definition 3.4.1 Let A be a vector space, $\phi : A^n \to A$ be an n-linear map and τ be a linear form. The map τ is said to be an (α, β)-twisted ϕ-trace if it satisfies the following condition:

$$\forall x_1, \dots, x_n \in A, \tau(\phi(\beta(x_1), \dots, \beta(x_{n-1}), \alpha(x_n))) = 0.$$

We set ϕ_τ be an $(n+1)$-linear map defined by

$$\forall x_1, \dots, x_{n+1} \in A, \phi_\tau(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} \tau(x_i) \phi(x_1, \dots, \widehat{x_i}, \dots, x_{n+1}).$$

We recall the construction of an $(n+1)$-BiHom-Lie algebra using an n-BiHom-Lie algebra and an (α, β)-twisted trace given in [28]:

Theorem 3.4.2 *Let $(A, [\cdot, \dots, \cdot], \alpha, \beta)$ be an n-BiHom-Lie algebra and τ an (α, β)-twisted $[\cdot, \dots, \cdot]$-trace. If the following conditions are satisfied*

$$For all x, y \in A, \tau(\alpha(x))\beta(y) = \tau(\beta(x))\alpha(y),$$

$$\tau \circ \alpha = \tau \text{ and } \tau \circ \beta = \tau,$$

then $(A, [\cdot, \dots, \cdot]_\tau, \alpha, \beta)$ is an $(n+1)$-BiHom-Lie algebra. We say that this algebra is induced by $(A, [\cdot, \dots, \cdot], \alpha, \beta)$.

We first focus on the ternary case. For a given BiHom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and a

$[\cdot,\cdot]$-trace map $\tau : \mathfrak{g} \to \mathbb{K}$, the ternary induced bracket is then given by

$$[x, y, z]_\tau := \tau(x)[y, z] - \tau(y)[x, z] + \tau(z)[x, y]. \tag{3.4.4}$$

Now we have the following theorem.

Theorem 3.4.3 *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha, \beta)$ be a BiHom-Lie algebra. Let $D : \mathfrak{g} \to \mathfrak{g}$ be an (α^s, β^r)-derivation of $(\mathfrak{g}, [\cdot,\cdot], \alpha, \beta)$. If the following identity holds, for all $x, y, z \in \mathfrak{g}$,*

$$\alpha^s\beta^r(\tau(D(x))[y, z]) - \alpha^s\beta^r(\tau(D(y))[x, z]) + \alpha^s\beta^r(\tau(D(z))[x, y]) = 0,$$

then D is an (α^s, β^r)-derivation of the induced ternary BiHom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot,\cdot]_\tau, \alpha, \beta)$.

Proof. In the sequel, for simplicity we drop the τ from the ternary bracket. We have to prove that

$$D[x, y, z] = [D(x), \alpha^s\beta^r(y), \alpha^s\beta^r(z)] + [\alpha^s\beta^r(x), D(y), \alpha^s\beta^r(z)] + [\alpha^s\beta^r(x), \alpha^s\beta^r(y), D(z)].$$

By applying D to each side of equation (3.4.4), we get

$$\begin{aligned} LHS &= D[x, y, z] = \tau(x)D[y, z] - \tau(y)D[x, z] + \tau(z)D[x, y] \\ &= \tau(x)([D(y), \alpha^s\beta^r(z)] + [\alpha^s\beta^r(y), D(z)]) - \\ &\quad \tau(y)([D(x), \alpha^s\beta^r(z)] - [\alpha^s\beta^r(x), D(z)]) + \\ &\quad \tau(z)([D(x), \alpha^s\beta^r(y)] + [\alpha^s\beta^r(x), D(y)]), \end{aligned}$$

while

$$RHS = [D(x), \alpha^s\beta^r(y), \alpha^s\beta^r(z)] + [\alpha^s\beta^r(x), D(y), \alpha^s\beta^r(z)] + [\alpha^s\beta^r(x), \alpha^s\beta^r(y), D(z)]$$
$$= \tau(D(x))[\alpha^s\beta^r(y), \alpha^s\beta^r(z)] - \tau(\alpha^s\beta^r(y))[\alpha^s\beta^r(x), D(z)] + \tau(\alpha^s\beta^r(z))[D(x), \alpha^s\beta^r(y)]$$
$$+ \tau(\alpha^s\beta^r(x))[D(y), \alpha^s\beta^r(z)] - \tau(D(y))[\alpha^s\beta^r(x), \alpha^s\beta^r(z)] + \tau(\alpha^s\beta^r(z))[\alpha^s\beta^r(x), D(y)]$$
$$+ \tau(\alpha^s\beta^r(x))[\alpha^s\beta^r(y), D(z)] - \tau(\alpha^s\beta^r(y))[D(x), \alpha^s\beta^r(z)] + \tau(D(z))[\alpha^s\beta^r(x), \alpha^s\beta^r(y)].$$

Using the fact that

$$\tau \circ \alpha = \tau \text{ and } \tau \circ \beta = \tau,$$

we can rewrite the right hand side as

$$\begin{aligned} RHS &= \tau(D(x))[\alpha^s\beta^r(y), \alpha^s\beta^r(z)] - \tau(y)[\alpha^s\beta^r(x), D(z)] + \tau(z)[D(x), \alpha^s\beta^r(y)] \\ &+ \tau(x)[D(y), \alpha^s\beta^r(z)] - \tau(D(y))[\alpha^s\beta^r(x), \alpha^s\beta^r(z)] + \tau(z)[\alpha^s\beta^r(x), D(y)] \\ &+ \tau(x)[\alpha^s\beta^r(y), D(z)] - \tau(y)[D(x), \alpha^s\beta^r(z)] + \tau(D(z))[\alpha^s\beta^r(x), \alpha^s\beta^r(y)], \end{aligned}$$

Thus the difference between the right hand side and the left hand side is given by

$$\begin{aligned} RHS - LHS &= \tau(D(x))[\alpha^s\beta^r(y), \alpha^s\beta^r(z)] - \tau(D(y))[\alpha^s\beta^r(x), \alpha^s\beta^r(z)] + \\ &\quad \tau(D(z))[\alpha^s\beta^r(x), \alpha^s\beta^r(y)], \\ &= \tau(D(x))\alpha^s\beta^r[y, z] - \tau(D(y))\alpha^s\beta^r[x, z] + \tau(D(z))\alpha^s\beta^r[x, y]. \end{aligned}$$

We then obtain the result by assuming that the following identity holds, $\forall x, y, z \in \mathfrak{g}$,

$$\alpha^s\beta^r(\tau(D(x))[y, z] - \tau(D(y))[x, z] + \tau(D(z))[x, y]) = 0.$$

This completes the proof.

■

Similar computations lead to a generalization of Theorem 3.4.3 to n-ary case.

Theorem 3.4.4 *Let $(\mathfrak{g}, [\cdot, \ldots, \cdot], \alpha, \beta)$ be an n-BiHom-Lie algebra. Let $D : \mathfrak{g} \to \mathfrak{g}$ be an*

(α^s, β^r)-derivation of $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha, \beta)$. If the following identity holds

$$\forall x_1, \dots, x_n \in \mathfrak{g}, \sum_{i=1}^{n} (-1)^{i-1} \alpha^s \beta^r (\tau(D(x_i))[x_1, \dots, \hat{x}_i \dots, x_n]) = 0,$$

then D is a derivation of the induced $(n+1)$-BiHom-Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot, \cdot]_\tau, \alpha, \beta)$.

Example 3.4.5 *We consider the 2-dimensional BiHom-Lie algebra $\mathfrak{g}$ with a basis $\{e_1, e_2\}$, where the map α is given by $\alpha(e_1) = e_1$ and $\alpha(e_2) = \frac{1}{m}e_1 + \frac{n-1}{n}e_2$ and β is the identity map (see [41]). The bracket is given by $[e_1, e_1] = 0,\ [e_1, e_2] = me_2 - ne_1,\ [e_2, e_1] = (n-1)e_1 - \frac{m(n-1)}{n}e_2$ and $[e_2, e_2] = -\frac{n}{m}e_1 + e_2$, where m and n are scalars such that $m, n \neq 0$.*
A direct computation gives that

$$\alpha^s(e_1) = e_1, \quad \alpha^s(e_2) = \frac{n^s - (n-1)^s}{n^{(s-1)}m} e_1 + \frac{(n-1)^s}{n^s} e_2.$$

We set $D(e_1) = ae_1 + be_2$ and $D(e_2) = ce_2 + de_2$. Finding the conditions on the parameters a, b, c, d such that D is an (α^s, β^r)-derivation and solving the system, we obtain that the following (α^s, β^r)-derivations:

$$\begin{aligned} &\text{For } n = 1, \quad D(e_1) = ae_1; \quad D(e_2) = be_1 + (a - mb)e_2. \\ &\text{For } n \neq 1, \quad D(e_1) = 0; \qquad D(e_2) = a(e_1 + \tfrac{m}{n}e_2). \end{aligned}$$

Now, seeking a linear form τ that are $[\cdot, \cdot]$-trace, we obtain first

$$n\tau(e_1) = m\tau(e_2).$$

The condition $\tau(\alpha(x))\beta(y) = \tau(\beta(x))\alpha(y)$ implies that $\tau(e_1) = 0 = \tau(e_2)$ and thus the form τ is identically trivial. Therefore the ternary bracket is trivial and any linear map is a derivation.

4 Conclusion and Future Work

The main subject in this dissertation is the generalized derivations and the study of their algebras. We explore the abgebras of generalized derivations and quasi-derivations of ternary Lie algebras and n-BiHom Lie algebras and study their properties. We introduce examples and a classification of the algebras of generalized derivations of low dimensional ternary Lie algebras.

Despite the research that has been done on the algebras of generalized derivations of different algebraic structures, it continues to draw the attention of many researchers. As generalized derivations of numerous generalizations of Lie algebras have been investigated, we believe that one can study these objects in a general framework. For that purpose, we plan on studying generalized derivations of the set of algebra varieties of BiHom algebras, which includes BiHom-associative algebras, BiHom-alternative algebras, BiHom-Lie algebras, BiHom-Leibniz algebras, BiHom-preLie.

Moreover, a research on some concrete examples can be fruitful. For instance, the quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$ introduced by Larsson and Silvestrov in [32] where they presented a complete set of equations can be used to compute the algebras of generalized derivations of the deformed algebra.

Also, we plan to apply some of the methods used in the deformations of Lie algebras to other structures related to Knot theory like Quandles and Racks.

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Appendices

4.0.1 Appendix A - Copyright and Permissions

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About the Author

Amine Ben Abdeljelil was born and raised in M'saken, Tunisia. He graduated from a local high school in 2007 then entered the College of Sciences at the University of Monastir, Tunisia where he earned his BSc degree majoring in Mathematics in 2010. He moved to France and entered the graduate school at the University of Paul Verlaine, Metz where he received his Master's degree in 2012. After one year as a Ph.D. student at the University of Haut-Alsace, Mulhouse, France, He joined the Ph.D program at the University of South Florida in Fall 2014.

Amine studied the algebras of generalized derivations of Lie algebras and their generalizations under the supervision of Professor Mohamed ELHAMDADI, University of South Florida, and Professor Abdenacer MAKHLOUF, University of Haut-Alsace, France. As a Graduate Teaching Associate at the department of Mathematics and Statistics at USF, he taught several undergraduate courses. His research interests include Deformations of Lie algebras, algebraic Varieties and Quandles.