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On Extending Hansel's Theorem to Hypergraphs

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by

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Dedication

For my grandmother

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ABSTRACT

For integers $n \geq k \geq 2$, let V be an n-element set, and let $\binom{V}{k}$ denote the family of all k-element subsets of V. For disjoint subsets $A, B \subseteq V$, we say that $\{A, B\}$ *covers* an element $K \in \binom{V}{k}$ if $K \subseteq A \dot{\cup} B$ and $K \cap A \neq \emptyset \neq K \cap B$. We say that a collection $\mathcal{C}$ of such pairs *covers* $\binom{V}{k}$ if every $K \in \binom{V}{k}$ is covered by at least one $\{A, B\} \in \mathcal{C}$. When $k = 2$, covers $\mathcal{C}$ of $\binom{V}{2}$ were introduced in 1961 by Rényi [24], where they were called *separating systems* of V (since every pair $u \neq v \in V$ is separated by some $\{A, B\} \in \mathcal{C}$, in the sense that $u \in A$ and $v \in B$, or vice-versa). Separating systems have since been studied by many authors.

For a cover $\mathcal{C}$ of $\binom{V}{k}$, define the *weight* $\omega(\mathcal{C})$ of $\mathcal{C}$ by $\omega(\mathcal{C}) = \sum_{\{A,B\} \in \mathcal{C}}(|A|+|B|)$. We define $h(n, k)$ to denote the minimum weight $\omega(\mathcal{C})$ among all covers $\mathcal{C}$ of $\binom{V}{k}$. In 1964, Hansel [10] determined the bounds $\lceil n \log_2 n \rceil \leq h(n, 2) \leq n \lceil \log_2 n \rceil$, which are sharp precisely when $n = 2^p$ is an integer power of two. In 2007, Bollobás and Scott [1] extended Hansel's bound to the exact formula $h(n, 2) = np + 2R$, where $n = 2^p + R$ for $p = \lfloor \log_2 n \rfloor$.

The primary result of this dissertation extends the results of Hansel and of Bollobás and Scott to the following exact formula for $h(n, k)$, for all integers $n \geq k \geq 2$. Let $n = (k-1)q + r$ be given by division with remainder, and let $q = 2^p + R$ satisfy $p = \lfloor \log_2 q \rfloor$. Then

$$h(n, k) = np + 2R(k-1) + \left\lceil \frac{r}{k-1} \right\rceil (r + k - 1).$$

A corresponding result of this dissertation proves that all optimal covers $\mathcal{C}$ of $\binom{V}{k}$, i.e., those for which $\omega(\mathcal{C}) = h(n, k)$, share a unique *degree-sequence*, as follows. For a

vertex $v \in V$, define the $\mathcal{C}$-*degree* $\deg_{\mathcal{C}}(v)$ of v to be the number of elements $\{A, B\} \in \mathcal{C}$ for which $v \in A \dot{\cup} B$. We order these degrees in non-increasing order to form $\mathbf{d}(\mathcal{C})$, and prove that when $\mathcal{C}$ is optimal, $\mathbf{d}(\mathcal{C})$ is necessarily binary with digits p and $p+1$, where uniquely the larger digits occur precisely on the first $2R(k-1) + \lceil r/(k-1) \rceil (r+k-1)$ many coordinates. That $\mathbf{d}(\mathcal{C})$ satisfies the above for optimal $\mathcal{C}$ clearly implies the claimed formula for $h(n,k)$, but in the course of this dissertation, we show these two results are, in fact, equivalent.

In this dissertation, we also consider a d-partite version of covers $\mathcal{C}$, written here as d-*covers* $\mathcal{D}$. Here, the elements $\{A, B\} \in \mathcal{C}$ are replaced by d-element families $\{A_1, \dots, A_d\} \in \mathcal{D}$ of pairwise disjoint sets $A_i \subset V$, $1 \leq i \leq d$. We require that every element $K \in \binom{V}{k}$ is covered by some $\{A_1, \dots, A_d\} \in \mathcal{D}$, in the sense that $K \subseteq A_1 \dot{\cup} \cdots \dot{\cup} A_d$ where $K \cap A_i \neq \emptyset$ holds for each $1 \leq i \leq d$. We analogously define $h_d(n,k)$ as the minimum weight $\omega(\mathcal{D}) = \sum_{D \in \mathcal{D}} \sum_{A \in D} |A|$ among all d-covers $\mathcal{D}$ of $\binom{V}{k}$. In this dissertation, we prove that for all $2 \leq d \leq k \leq n$, the bound $h_d(n,k) \geq n \log_{d/(d-1)}(n/(k-1))$ always holds, and that this bound is asymptotically sharp whenever $d = d(k) = O(k/\log^2 k)$ and $k = k(n) = O(\sqrt{\log \log n})$.

1 Introduction

Extremal combinatorics is a rich discipline within combinatorics, where both extremal and general combinatorics have experienced significant growth in the past century. Extremal problems, whether they be in combinatorics or in any other branch of mathematics, are centered on the following universal mathematical considerations. Let X be a set, and let $f\colon X \to \mathbb{R}$ be a real-valued function on X. Whatever the context, one is often tasked with determining whether or not the function f achieves a maximum value $M = \max_{x\in X} f(x)$ or a minimum value $m = \min_{x\in X} f(x)$. Such values, when they exist, are known as *extreme values*, and it is from this word that *extremal* combinatorics derives its name. It is well-known that, in many contexts, extreme values are not guaranteed to exist, and deciding whether or not they do can be a very difficult problem. When extreme values exist, it is usually of interest to compute or estimate them, and to characterize the elements $x \in X$ for which $f(x)$ is an extreme value. Problems of this character are pervasive throughout mathematics, and they are often very challenging. For example, in the case that $X \subseteq \mathbb{R}$ is a real interval and f is a reasonably well-behaved function, such problems helped motivate and shape the development of Calculus.

In extremal combinatorics, the set X above will be a (usually finite) class of combinatorially defined objects, and the function $f\colon X \to \mathbb{R}$ will be a combinatorially defined parameter. The questions asked in extremal combinatorics will, however, be precisely the same as those above. To better understand the rich character of extremal combinatorics, one should turn to some of its best-known results and problems. We hope, in fact, that these examples will draw some parallels to the results of this dissertation. We begin with what may be the first extremal combinatorial problem ever studied.

1.1 History: Origins of Extremal Graph Theory

In 1907, Mantel [17] considered the following problem: *for a fixed integer $n \geq 3$,*

> *what is the maximum number of edges $|E|$ among all n-vertex triangle-free graphs $G = (V, E)$?*

Here, a graph $G = (V, E)$ is *triangle-free* if no three of its vertices $x, y, z \in V$ admit all three pairs $\{x, y\}, \{y, z\}, \{x, z\}$ as edges in E. In equivalent language, G is triangle-free if it has no copy of the *complete graph* K_3 on three vertices (also called a *triangle*) as a subgraph. Thus, in the setting of the first paragraph, $X = X_n$ is the class of all n-vertex triangle-free graphs $G = (V, E)$, and for each $G \in X$, the parameter $f(G) = |E|$ counts the number of edges in $G = (V, E)$. Clearly, $\max_{G \in X} f(G)$ must exist, because every n-vertex graph G (whether triangle-free or not) contains $0 \leq f(G) = |E| \leq \binom{n}{2}$ many edges. (Clearly, $\min_{G \in X} f(G) = 0$ isn't an interesting problem.) For his problem, Mantel first noted that the complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ is triangle-free, and this graph achieves precisely $\lceil n/2 \rceil \lfloor n/2 \rfloor$ many edges. Mantel then showed that no n-vertex triangle-free graph G achieves more.

Theorem 1.1.1 (Mantel [17] (1907))**.** *The largest number of edges $|E|$ possible among all n-vertex triangle-free graphs $G = (V, E)$ is precisely $\lceil n/2 \rceil \lfloor n/2 \rfloor$.*

From Theorem 1.1.1, many interesting questions readily arise, and we shall outline a few of these. For example, suppose (in the context of Theorem 1.1.1) that we replace the triangle K_3 with an arbitrary fixed graph F. One may then ask

> *what is the maximum number of edges $|E|$ among all n-vertex F-free graphs $G = (V, E)$?*

(A graph $G = (V, E)$ is *F-free* if it contains no copy of F as a subgraph.) This well-studied problem has been equipped with the following customary notation and terminology. Let $\mathrm{ex}(n, F)$ denote the maximum number of edges $|E|$ in an n-vertex F-free graph $G = (V, E)$. Thus, in this notation, Mantel proved that $\mathrm{ex}(n, K_3) = \lceil n/2 \rceil \lfloor n/2 \rfloor$, and he noted that $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ is an example of a triangle-free graph achieving the extreme value $\mathrm{ex}(n, K_3)$. We shall say, more generally, that an n-vertex graph $G = (V, E)$ is an *extremal graph* (for avoiding F) if $|E| = \mathrm{ex}(n, F)$ and G contains no copies of F as a subgraph. Thus, one may also ask

> *what, if any, characterization can be given for the (n-vertex) extremal graphs which forbid F as a subgraph?*

The questions above proved to be deep, difficult, and highly influential in combinatorics. One of the great theorems in all of graph theory is due to P. Turán (1941), which precisely answers the questions above in the case that $F = K_t$ is the *clique* on $t \geq 3$ vertices. (The clique, or *complete graph*, K_t on t fixed vertices is the graph consisting of all $\binom{t}{2}$ edges on these t fixed vertices.) To state Turán's theorem, we need the following considerations. First, we fix an n-element vertex set V, which for simplicity we take to be $V = [n] = \{1, \ldots, n\}$, Second, we consider any *equitable* $(t-1)$-partition $[n] = V_1 \dot\cup \cdots \dot\cup V_{t-1}$, meaning that $|V_1| \geq \cdots \geq |V_{t-1}| \geq |V_1| - 1$ are as equal as possible. Up to relabeling the vertices, such partitions are unique, and given by division with remainder: if $n = q(t-1) + r$ has $q = \lfloor n/(t-1) \rfloor$, then the partition above satisfies $|V_1| = \cdots = |V_r| = q + 1$ and $|V_{r+1}| = \cdots = |V_{t-1}| = q$. As is customary, we denote by $T([n], t-1)$ the *Turán graph*, which is the complete $(t-1)$-partite graph with vertex $(t-1)$-partition $[n] = V_1 \dot\cup \cdots \dot\cup V_{t-1}$. (Here, the edge-set of $T([n], t-1)$ consists of all pairs $\{v_i, v_j\} \in \binom{[n]}{2}$ where $v_i \in V_i$ and $v_j \in V_j$ for some $1 \leq i < j \leq t-1$.)

Now, any $(t-1)$-partite graph is K_t-free, and basic convexity ensures that the Turán graph $T([n], t-1)$ maximizes the number of edges among all n-vertex $(t-1)$-partite graphs. Moreover, with $q = \lfloor n/(t-1) \rfloor$ and $0 \leq r < t-1$ defined above, an easy calculation shows that $T([n], t-1)$ has precisely

$$|E(T([n], t-1))| = \binom{n}{2} - r\binom{q+1}{2} - (t-1-r)\binom{q}{2}$$

many edges. Thus, $\mathrm{ex}(n, K_t)$ is at least the value above. In one of the most important theorems in all of graph theory, Turán (1941) showed that this number is precisely $\mathrm{ex}(n, K_t)$, and that the graph $T([n], t-1)$ is, up to isomorphism, the only K_t-free graph achieving $\mathrm{ex}(n, K_t)$.

Theorem 1.1.2 (Turán [27], 1941)**.** *For integers $n \geq t \geq 3$, we have $\mathrm{ex}(n, K_t) = |E(T([n], t-1))|$. In particular, if $n = q(t-1) + r$, where $q = \lfloor n/(t-1) \rfloor$, then*

$$\mathrm{ex}(n, K_t) = \binom{n}{2} - r\binom{q+1}{2} - (t-1-r)\binom{q}{2}.$$

Moreover, all n-vertex K_t-free extremal graphs $G = (V, E)$ (those having $|E| = \mathrm{ex}(n, K_t)$ many edges) are isomophic to the Turán graph $T([n], t-1)$.

Extending Theorem 1.1.2 to arbitrary fixed subgraphs F became a well-studied problem, receiving decades of attention from leading combinatorists. For the purpose of illustrating some of the difficulties in modern extremal combinatorics, we briefly sketch a few of the highlights in this area.

Two striking features of Theorem 1.1.2 are the precision it has on evaluating $\mathrm{ex}(n, K_t)$ and the uniqueness it imposes on the extremal graphs achieving $\mathrm{ex}(n, K_t)$. As extremal combinatorics continued to develop over the years, precise formulas proved to be rare, and the uniqueness of extremal examples was often not known.

For the parameter $\mathrm{ex}(n, F)$ (for a fixed but arbitrary subgraph F), it turns out that the problem fundamentally divides into two cases, depending on whether or not F is *bipartite* (or equivalently, 2-*colorable*). When F has chromatic number $\chi(F) \geq 3$, the parameter $\mathrm{ex}(n, F)$ is fairly well-understood by the following work of Erdős and Stone [7] (1946) and Erdős and Simonovits [6] (1966), which says that $\mathrm{ex}(n, F)$ behaves very closely to $\mathrm{ex}(n, K_r)$, where $r = \chi(F)$.

Theorem 1.1.3 (Erdős-Stone-Simonovits [6, 7])**.** *Let F be a fixed graph with chromatic number $r = \chi(F) \geq 3$. Then*

$$\mathrm{ex}(n, F) = \mathrm{ex}(n, K_r) + o(n^2) = \left(1 - \frac{1}{r-1} + o(1)\right)\binom{n}{2}.$$

When F is an arbitrary (non-bipartite) graph, a precise formula for $\mathrm{ex}(n, F)$ is not known. Similarly, a precise characterization of the n-vertex F-free extremal graphs $G = (V, E)$ is not known. However, a 'coarse' characterization called *stability* is known, and follows from the work in [6, 7]. In what follows, let F be a fixed graph with $\chi(F) = r \geq 3$, and let $\epsilon > 0$ be given. We denote by $\delta = \delta(F, \epsilon) > 0$ a constant depending on F and ϵ which is determined in the work in [6, 7], but which we do not explicitly specify here. Now, assume that $G = (V, E)$ is an n-vertex F-free graph, where $n \geq n_0(F, \epsilon, \delta)$ is a large integer, and where $|E| > (1 - \delta)\mathrm{ex}(n, F)$. (In other words, G is very 'close' (in its edge-count) to being an extremal graph for avoiding F.) It is then known that G is very 'close' (structurally) to being the Turán graph $T([n], r - 1)$, in the following sense: there is a graph $G' = (V, E')$ on the same vertex set V, where $|E \triangle E'| < \epsilon n^2$, and where G' is isomorphic to $T([n], r - 1)$.

When F is an arbitrary bipartite graph, estimating $\mathrm{ex}(n, F)$ is a well-known open and difficult problem. At the time of this writing, very few cases are understood, even asymptotically. However, it is known, for example, that $\mathrm{ex}(n, C_4) \sim (1/2)n^{3/2}$ and $\mathrm{ex}(n, K_{3,3}) \sim (1/2)n^{5/3}$, which follow from the works of Reiman [23] and Brown [2] and of Brown [2] and Füredi [9] (where these works span the years 1958–1996). But while these asymptotics are known, note that C_4 and $K_{3,3}$ are small bipartite graphs. We omit a further discussion of this well-studied but challenging graph-theoretic area in favor of considering some extremal *hypergraph* problems, which is our ultimate direction in this dissertation.

1.2 History: Extremal Hypergraph Problems

One of the earliest extremal problems for *hypergraphs* sought 'hypergraph' generalizations of Theorem 1.1.2, which Turán himself initiated. To describe these and other problems, we require some notation and terminology. A *hypergraph* $\mathcal{H}$ is an ordered pair $\mathcal{H} = (V, E)$, where V is a (usually finite) vertex set, and $E \subseteq 2^V$ is a family of vertex subsets. Note that the edge-set E of a graph $G = (V, E)$ is a family of pairs $E \subseteq \binom{V}{2}$. More generally, when the edge-set E of a hypergraph $\mathcal{H} = (V, E)$ is a family

$E \subseteq \binom{V}{k}$ of k-tuples, we say that $\mathcal{H}$ is *k-uniform*. (Thus, a graph $G = (V, E)$ is a 2-uniform hypergraph.) We sometimes use the notation $\mathcal{H}^{(k)}$ to denote that $\mathcal{H}$ is a k-uniform hypergraph. For a fixed k-uniform hypergraph $\mathcal{F}$, we denote by $\mathrm{ex}(n, \mathcal{F})$ the maximum number of k-tuples in an n-vertex *$\mathcal{F}$-free* hypergraph $\mathcal{H} = (V, E)$, where $\mathcal{H}$ is $\mathcal{F}$-free if it contains no subhypergraph isomorphic to $\mathcal{F}$.

In almost every non-trivial case of a hypergraph $\mathcal{F}$ (whose uniformity is three or higher), very little is known on the parameter $\mathrm{ex}(n, \mathcal{F})$. Most famously, Turán considered the case when $\mathcal{F} = K_t^{(k)}$ is the *complete k-uniform hypergraph* $K_t^{(k)}$ on $t > k$ fixed vertices, which consists of all $\binom{t}{k}$ many k-tuples on t fixed vertices. While the case $t > k = 2$ is perfectly understood by Theorem 1.1.2, the hypergraph analog is wide-open for all values of $t > k \geq 3$. In fact, Turán's following conjecture from 1941 on the case $k = 3$ and $t = 4$ remains one of the greatest open problems in all of combinatorics.

Conjecture 1.2.1 (Turán [27] (1941)).

$$\mathrm{ex}(n, K_4^{(3)}) = \left(\frac{5}{9} + o(1)\right)\binom{n}{3}.$$

Note that the density 5/9 above can be achieved by the following construction. Let $[n] = A \dot\cup B \dot\cup C$ be an equitable partition, i.e., $|A| \leq |B| \leq |C| \leq |A| + 1$. Let $\mathcal{H}^{(3)}$ consist of all triples of the form $\{a, a', b\}$, or $\{b, b', c\}$, or $\{c, c', a\}$, or $\{a, b, c\}$, where $a \neq a' \in A$, $b \neq b' \in B$, and $c \neq c' \in C$. Then $\mathrm{ex}(n, K_4^{(3)}) \geq |E(\mathcal{H}^{(3)})|$, where $|E(\mathcal{H}^{(3)})|$ can be computed precisely, but is given asymptotically by $(5/9 + o(1))\binom{n}{3}$. Turán's conjecture is that 5/9 is best possible in this context, which is widely believed but has never been confirmed. If true, Kostochka [14] (1982) and Fon-der Flaass [8] (1988) proved that there are considerably many non-isomorphic contructions which achieve the density 5/9 for that context.

Similarly to Conjecture 1.2.1, many extremal hypergraph problems have proven to be elusive. Nonetheless, extremal hypergraph research has long been active, and some difficult problems have admitted elegant resolutions. Among the most classical of examples is the following *Erdős-Ko-Rado* theorem, due to P. Erdős, C. Ko, and

R. Rado [5] (proven in 1938, but not published until 1961). In what follows, we shall say that a hypergraph $\mathcal{H} = (V, E)$ is *intersecting* if every pair of its edges overlaps. We consider the maximum size $|E|$ among all n-vertex k-uniform intersecting hypergraphs $\mathcal{H} = (V, E)$. For this problem, we impose the additional hypothesis that $n \geq 2k$, since otherwise the complete hypergraph would trivially solve the problem.

Consider the following class of intersecting k-uniform hypergraphs $\mathcal{H} = (V, E)$ on a fixed vertex set V: fix an arbitrary vertex $x \in V$, and take $E = E_x$ to consist of all edges $K \in \binom{V}{k}$ which contain the vertex $x \in K$. Then $\mathcal{H} = \mathcal{H}_x$ is intersecting (because every pair of its edges overlaps in at least the vertex x), and $\mathcal{H}$ achieves precisely $|E| = \binom{n-1}{k-1}$ many edges. Such hypergraphs $\mathcal{H}$ are said to be *principle intersecting* hypergraphs. The Erdős-Ko-Rado theorem says that, in this context, $|E| = \binom{n-1}{k-1}$ cannot be improved, nor can it be achieved by any intersecting k-uniform hypergraph which isn't principle.

Theorem 1.2.2 (Erdos-Ko-Rado [5] (1961))**.** *Let $\mathcal{H} = (V, E)$ be an n-vertex, k-uniform, intersecting hypergraph. Then,*

$$|E| \leq \binom{n-1}{k-1}.$$

Moreover, equality holds if, and only if, $\mathcal{H}$ is principle (w.r.t. one of its vertices $x \in V$).

We next turn our attention to the research of this dissertation, which will have some parallels to the outcomes above.

1.3 Results: Hypergraph Covering Problems

In this dissertation, we consider several *hypergraph covering problems*, defined in a moment, which arise from classical graph problems. Our hypergraph problems are extremal in nature, and our results on these problems follow the same vein as some of those earlier in the Introduction. In particular, for some of our problems, we will determine exact formulas for the parameters we study (see Theorem 1.3.3 below).

When successful with a formula, we then determine a characterization of all extremal examples achieving our formula (see Theorem 1.3.4 below). When we are are unable to determine exact formulas, we provide some bounds (see Theorem 1.3.5 below), and in some cases, even asymptotics (see Theorem 1.3.6 below). We are then left with quite a few interesting open problems, discussed in our Concluding Remarks, Chapter 9.

Definitions and Some History

We begin with the following notation and terminology regarding the principle results of this dissertation. Fix integers $n \geq k \geq 2$, and let V be an arbitrary n-element vertex set. As before, denote by $\binom{V}{k}$ the family of all k-element subsets of V. For disjoint subsets $A, B \subseteq V$, we say that $\{A, B\}$ *covers* an element $K \in \binom{V}{k}$ if $K \subseteq A \dot\cup B$ where $K \cap A \neq \emptyset \neq K \cap B$. We say that a collection $\mathcal{C}$ of such pairs *covers* (is a *cover* of) $\binom{V}{k}$ if every $K \in \binom{V}{k}$ is covered by at least one $\{A, B\} \in \mathcal{C}$.

The concept of a cover was initiated by Rényi [24] in 1961 in the case that $k = 2$. There, he called $\mathcal{C}$ a *separating system of* V (as opposed to a cover of $\binom{V}{2}$), because every pair $u \neq v \in V$ is *separated* by some $\{A, B\} \in \mathcal{C}$, in the sense that $u \in A$ and $v \in B$, or vice-versa. Separating systems $\mathcal{C}$ of V have since been studied from various points of view by many authors (see, e.g., [1, 3, 4, 10–13, 15, 16, 19–22, 24–26, 28]). To motivate our work, we shall consider just a couple results among these, where we use the language of covers $\mathcal{C}$ of $\binom{V}{2}$ (rather than separating systems $\mathcal{C}$ of V) to be consistent with future considerations.

An early extremal problem in the area above sought the minimum size $|\mathcal{C}|$ of a cover $\mathcal{C}$ of $\binom{V}{2}$. Clearly, this minimum is at most $\binom{n}{2}$, where $|V| = n$, because $\binom{V}{2}$ is itself a cover. However, this bound is extremely poor, for it is not too hard to observe that $|\mathcal{C}| = \lceil \log_2 n \rceil$ is the exact minimum among all covers $\mathcal{C}$ of $\binom{V}{2}$. For that, observe first that every cover $\mathcal{C}$ of $\binom{V}{2}$ satisfies $|\mathcal{C}| \geq \lceil \log_2 n \rceil$, where $n = |V|$. Indeed, by definition we have $K_V^{(2)} = \bigcup_{\{A,B\}\in\mathcal{C}} K[A,B]$, and so

$$n = \chi(K_V^{(2)}) = \chi\left(\bigcup_{\{A,B\}\in\mathcal{C}} K[A,B]\right) \le \prod_{\{A,B\}\in\mathcal{C}} \chi(K[A,B]) = 2^{|\mathcal{C}|},$$

from which $|\mathcal{C}| \ge \log_2 n$ and thus $|\mathcal{C}| \ge \lceil \log_2 n \rceil$ follows. To see that this bound is sharp, consider the following cover $\mathcal{C}_0$ of $\binom{V}{2}$: let $m = \lceil \log_2 n \rceil$, and let $v \mapsto \mathbf{v}$ be any injection from V to $\{0,1\}^m$. For each $1 \le i \le m$, let $A_i = \{v \in V : \mathbf{v}(i) = 0\}$, where $\mathbf{v}(i)$ denotes the i^{th} coordinate of $\mathbf{v}$, and let $B_i = V \setminus A_i$. Now, $\mathcal{C}_0 = \{\{A_i, B_i\}\}_{i=1}^m$ is a cover of $\binom{V}{2}$ of size $m = \lceil \log_2 n \rceil$. Indeed, for each $u \ne v \in V$, we have $\mathbf{u} \ne \mathbf{v}$, in which case these vectors disagree on some coordinate $1 \le i \le m$. As such, and without loss of generality, $\mathbf{u}(i) = 0$ and $\mathbf{v}(i) = 1$, in which case $u \in A_i$ and $v \in B_i$.

In 1964, Hansel [10] considered a weighted version of the problem above, which proved to be much more challenging. For a cover $\mathcal{C}$ of $\binom{V}{2}$, define the *weight* $\omega(\mathcal{C})$ of $\mathcal{C}$ by $\omega(\mathcal{C}) = \sum_{\{A,B\}\in\mathcal{C}}(|A| + |B|)$. We set $h(n,2)$ to denote the minimum weight $\omega(\mathcal{C})$ among all covers $\mathcal{C}$ of $\binom{V}{2}$. Hansel then proved the following seminal result.

Theorem 1.3.1 (Hansel (1964), [10])**.** *For all integers $n \ge 2$, we have $\lceil n \log_2 n \rceil \le h(n,2) \le n\lceil \log_2 n \rceil$.*

Note that Theorem 1.3.1 is sharp precisely when $n = 2^p$ is an integer power of two. We mention that, independently and only slightly later, Krichevskii [15] proved a similar result to Theorem 1.3.1 in a different context. We also mention that, in 1967, Katona and Szemerédi [13] rediscovered Theorem 1.3.1 in the context of a diameter problem in graph theory. But because Hansel was the first to prove Theorem 1.3.1, we dub $h(n,2)$ as a *Hansel number*.

It is perhaps surprising that more than 40 years passed before an exact formula for $h(n,2)$ was found. However, in 2007, Bollobás and Scott [1] indeed improved Theorem 1.3.1 to the following precise formula for $h(n,2)$.

Theorem 1.3.2 (Bollobás and Scott (2007), [1])**.** *For an integer $n \ge 2$, let $n = 2^p + R$, where $p = \lfloor \log_2 n \rfloor$. Then, $h(n,2) = np + 2R$.*

For us, Theorem 1.3.2 was the principle inspiration for all of our work below. We now proceed to our first round of results.

Some Results on Optimal Covers

Let $n \geq k \geq 2$ be arbitrary integers, and let $\mathcal{C}$ cover $\binom{V}{k}$, where $|V| = n$. Identically to before, we define the *weight* $\omega(\mathcal{C})$ of $\mathcal{C}$ by $\omega(\mathcal{C}) = \sum_{\{A,B\}\in\mathcal{C}}(|A| + |B|)$. Similarly to before, we set $h(n, k)$ to denote the minimum weight $\omega(\mathcal{C})$ among all covers $\mathcal{C}$ of $\binom{V}{k}$, and we say a cover $\mathcal{C}$ of $\binom{V}{k}$ is *optimal* when $\omega(\mathcal{C}) = h(n, k)$. The principle result of this dissertation is the following exact formula for $h(n, k)$.

Theorem 1.3.3 (Main Result)**.** *For integers $n \geq k \geq 2$, let $n = q(k-1) + r$, where $q = \lfloor n/(k-1) \rfloor$, and let $q = 2^p + R$, where $p = \lfloor \log_2 q \rfloor$. Then,*

$$h(n, k) = np + 2R(k-1) + \left\lceil \frac{r}{k-1} \right\rceil (r + k - 1). \tag{1.3.1}$$

Similar to Theorems 1.1.2 and 1.2.2, we seek a characterization of all optimal covers $\mathcal{C}$ of $\binom{V}{k}$. In the case of Theorems 1.1.2 and 1.2.2, the optimal examples were unique (up to isomorphism). In the case of Theorem 1.3.3, optimal covers $\mathcal{C}$ will not be unique (regardless of vertex labels). Indeed, while the non-uniqueness of optimal covers $\mathcal{C}$ will be formally observed (see upcoming Remark 2.0.10) in the proof of upcoming Proposition 1.3.7, we say for the moment that we don't even expect an optimal $\mathcal{C}$ to be unique. Indeed, $\mathcal{C}$ has two parameters one could optimize, $\omega(\mathcal{C})$ and $|\mathcal{C}|$, and it will be the case that we can minimize $\omega(\mathcal{C})$ while simultaneously allowing $|\mathcal{C}|$ to vary. Nonetheless, we will prove that all optimal covers $\mathcal{C}$ share a unique behavior with respect to their *degree sequences*, which we now discuss.

For a vertex $v \in V$, we define the *degree of* v, denoted by $\deg_{\mathcal{C}}(v)$, to be the number of elements $\{A, B\} \in \mathcal{C}$ to which v is *incident*, meaning that $v \in A \dot{\cup} B$. Arranging these degrees in non-increasing order, we define $\mathbf{d}(\mathcal{C}) = (\deg_{\mathcal{C}}(v))_{v\in V}$ to be the *degree sequence* of $\mathcal{C}$. We will show that all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ share the common degree sequence $\mathbf{d}_0 \in \{p, p+1\}^V$, whose j^{th} coordinate $\mathbf{d}_0(j)$, $1 \leq j \leq n$, is defined by

$$\mathbf{d}_0(j) = p+1 \qquad \Longleftrightarrow \qquad 1 \le j \le 2R(k-1) + \left\lceil \frac{r}{k-1} \right\rceil (r+k-1). \qquad (1.3.2)$$

Theorem 1.3.4 (Degree–Sequence)**.** *Let integers k, n, p, q, r, and R be given as in the hypothesis of Theorem 1.3.3, where V is an n-element set. If $\mathcal{C}$ optimally covers $\binom{V}{k}$, then $\boldsymbol{d}(\mathcal{C}) = \boldsymbol{d}_0$.*

The proofs of Theorems 1.3.3 and 1.3.4 constitute the majority of the effort in this dissertation. We shall spend some time later in this introduction to outline our approach for proving them. Before doing so, however, we consider a few further results also proven in this dissertation.

Some Further Generalizations

We shall also consider the following further generalizations of Theorem 1.3.2. For that, we fix integers $n \ge k \ge d \ge 2$, and as before, we fix an n-element vertex set V. For pairwise disjoint subsets $A_1, \dots, A_d \subseteq V$, we say that the d-tuple $D = \{A_1, \dots, A_d\}$ *covers* an element $K \in \binom{V}{k}$ if $K \subseteq A_1 \dot\cup \cdots \dot\cup A_d$, where $K \cap A_i \ne \emptyset$ for every $1 \le i \le d$. We say that a collection $\mathcal{D}$ of such d-tuples $D = \{A_1, \dots, A_d\}$ is a *d-cover* of $\binom{V}{k}$ if every $K \in \binom{V}{k}$ is covered by at least one $D \in \mathcal{D}$. (Thus, covers $\mathcal{C}$ of $\binom{V}{k}$ are, in this language, 2-covers.) Similarly to before, we define the *weight* $\omega(\mathcal{D})$ of $\mathcal{D}$ by $\omega(\mathcal{D}) = \sum_{D \in \mathcal{D}} |V(D)|$, where for each $D = \{A_1, \dots, A_d\} \in \mathcal{D}$, we use $V(D) = A_1 \dot\cup \cdots \dot\cup A_d$. Similarly to before, we denote by $h_d(n,k)$ the minimum weight $\omega(\mathcal{D})$ among all d-covers $\mathcal{D}$ of $\binom{V}{k}$.

Unlike Theorem 1.3.3 when $d = 2$, we are unable to give a formula for $h_d(n,k)$ for arbitrary $n \ge k \ge d \ge 2$. We are able, however, to prove the following bound.

Theorem 1.3.5. *For all integers $2 \le d \le k \le n$,*

$$h_d(n,k) \ge n \log_{d/(d-1)} \left(\frac{n}{k-1} \right).$$

We believe it would be of interest to know, to what extent, the lower bound in Theorem 1.3.5 is close to the actual value of $h_d(n,k)$. Our next result provides multivariate asymptotics on $h_d(n,k)$ in certain ranges of $2 \leq d \leq k \leq n$. In those ranges, Theorem 1.3.5 is asymptotically sharp.

Theorem 1.3.6. *Let $d = d(k) = O(k/\log^2 k)$ be an integer function of k, which itself is a slowly diverging integer function $k = k(n) = O(\sqrt{\log\log n})$ of n. Then*

$$h_d(n,k) = (1 + o(1))n \log_{d/(d-1)} \left(\frac{n}{k-1}\right),$$

where $o(1) \to 0$ as $k, n \to \infty$.

In character, Theorems 1.3.5 and 1.3.6 are weaker results than Theorems 1.3.3 and 1.3.4, and their proofs are much easier. These proofs follow, in fact, from standard probabilistic considerations, which we give in Chapter 8.

For the remainder of the Introduction, we outline our proofs of Theorems 1.3.3 and 1.3.4. Note, in particular, that Theorem 1.3.4 immediately implies Theorem 1.3.3. Our goal is to show that these results are, in fact, equivalent, and we will prove Theorem 1.3.4 from Theorem 1.3.3 in context.

On the Proof of the Main Result

We prove Theorem 1.3.3 in steps, not all of which are difficult. First, and in Chapter 2, we use a standard construction to establish the formula in (1.3.1) as an upper bound on $h(n,k)$.

Proposition 1.3.7 (the upper bound)**.** *Let integers k, n, p, q, r, and R be given as in the hypothesis of Theorem 1.3.3, and let V be an n-element set. There exists a cover $\mathcal{C}_0$ of $\binom{V}{k}$ with weight*

$$\omega(\mathcal{C}_0) = np + 2R(k-1) + \left\lceil \frac{r}{k-1} \right\rceil (r + k - 1).$$

Second, we essentially split the formula in (1.3.1) into two cases, depending on whether or not $r = 0$ (i.e., whether or not $k-1$ divides n). In Chapter 3, we follow the approach of Bollobás and Scott [1] for Theorem 1.3.2 to prove the following *weak lower bound* on $h(n,k)$ (sharp only when $r = 0$).

Theorem 1.3.8 (Weak Lower Bound)**.** *Let integers k, n, p, q, r, and R be given as in the hypothesis of Theorem 1.3.3. Then, $h(n,k) \geq np + 2R(k-1) + 2r$. Moreover, Theorem 1.3.4 holds when $r = 0$.*

Third, and in Chapter 7, we sharpen the bound of Theorem 1.3.8 for $r \geq 1$, which claims the majority of our efforts.

Theorem 1.3.9 (the lower bound when $r \neq 0$)**.** *Let integers k, n, p, q, r, and R be given as in the hypothesis of Theorem 1.3.3, where $r \geq 1$. Then, $h(n,k) \geq np + 2R(k-1) + r + k - 1$. Moreover, Theorem 1.3.4 holds when $r \geq 1$.*

To prove Theorem 1.3.9, we enhance the proof of Theorem 1.3.8, where in particular

$$\text{we induct on the parameter } 2^p - R \geq 1. \tag{1.3.3}$$

To conduct (1.3.3), we need a handful of auxiliary results on structural properties of covers $\mathcal{C}$ of $\binom{V}{k}$. These results will be given by the *Survival Lemma* of Chapter 4, the *Extremal Lemma* of Chapter 5, and *Shifting Lemmas II* and *III* of Chapter 6. These are the pillars of Theorem 1.3.9.

In addition to the Survival Lemma, Chapter 4 introduces several structural instruments that serve as a bedrock for Chapter 4 onward. One of the most powerful tools presented here is *shifting*, which is a simple alteration process of a cover $\mathcal{C}$ not unlike that of Motzkin and Strauss [18] (which they use to prove Theorem 1.1.2). We use shifting to prove the Survival Lemma, and we make use of shifting in many of the auxiliary results we need.

In Chapter 5, we present *Shifing Lemma I*, which as the reader may suspect, has a close connection to Shifting Lemmas II and III. However, we use Shifting Lemma

I to prove the Extremal Lemma, and since Shifting Lemma I and the Extremal Lemma make use of a consideration that is unique to them, we present these two together in the same chapter. However, in Chapter 6 we use Shifting Lemma I to infer Shifting Lemma II and use Shifting Lemma II to infer Shifting Lemma III.

Finally, in Chapter 7, we prove Theorem 1.3.9.

2 The Upper Bound of the Main Result

In this Chapter we prove Proposition 1.3.7, which verifies that the formula in 1.3.1 is an upper bound on $h(n,k)$.

Proof of Proposition 1.3.7. Fix integers $n \geq k \geq 2$, and let $n = q(k-1) + r$, where $q = \lfloor n/(k-1) \rfloor$. Let $q = 2^p + R$, where $p \geq 0$ and $0 \leq R < 2^p$ are integers. We therefore have (the frequently referenced)

$$\frac{n-r}{k-1} = q = 2^p + R, \tag{2.0.1}$$

where $0 \leq r < k-1$ and $0 \leq R < 2^p$.

Let V be an n-element set. To construct the cover $\mathcal{C}_0$, we make a few auxiliary considerations. Fix an arbitrary subset $U \subseteq V$ of size r, where U may be empty. Fix a partition $V = X_1 \dot\cup \cdots \dot\cup X_{q-R}$ into $q - R = 2^p$ (cf. (2.0.1)) classes which satisfies:

(*a*) $|X_1| = \cdots = |X_{q-2R-1}| = k-1$

and $|X_{q-2R}| = r + k - 1$, where $U \subseteq X_{q-2R}$;

(*b*) if $R \neq 0$, then $|X_{q-2R+1}| = \cdots = |X_{q-R}| = 2(k-1)$,

where $X_i = Y_i \dot\cup Z_i$ is subpartitioned with $|Y_i| = |Z_i| = k-1$.

Since (2.0.1) gives $(k-1)(q-2R-1) + r + k - 1 + 2(k-1)R = n$, such a partition exists. Let $X_i \mapsto \mathbf{x}_i \in \{0,1\}^p$ be an arbitrary bijection, which exists on account of $q - R = 2^p$ (cf. (2.0.1)). This bijection induces a mapping $v \mapsto \mathbf{v} \in \{0,1\}^p$ where, for each $v \in V$, we have $\mathbf{v} = \mathbf{x}_i$ if, and only if, $v \in X_i$. We write $\mathbf{v}(j)$ to denote the j^{th} coordinate of $\mathbf{v}$.

We now define the promised cover $\mathcal{C}_0$. For each $1 \le j \le p$, set $A_j = \{v \in V : \mathbf{v}(j) = 0\}$ and $B_j = V \setminus A_j$. Then $|A_j| + |B_j| = n$. If $r \neq 0$, set $A_{p+1} = U$ and $B_{p+1} = X_{q-2R} \setminus U$, in which case $|A_{p+1}| = r$ and $|B_{p+1}| = k-1$. If $r = 0$, set $A_{p+1} = B_{p+1} = \emptyset$. Either way, $|A_{p+1}| + |B_{p+1}| = \lceil \frac{r}{k-1} \rceil (r + k - 1)$. Set $A_{p+2} = Y_{q-2R+1} \dot\cup \cdots \dot\cup Y_{q-R}$ and $B_{p+2} = Z_{q-2R+1} \dot\cup \cdots \dot\cup Z_{q-R}$, where (necessarily) $A_{p+2} = B_{p+2} = \emptyset$ when $R = 0$. Then $|A_{p+2}| = |B_{p+2}| = R(k-1)$. Now, define $\mathcal{C}_0 = \{\{A_1, B_1\}, \ldots, \{A_{p+2}, B_{p+2}\}\}$, where

$$\omega(\mathcal{C}_0) = \sum_{j=1}^{p+2} (|A_j| + |B_j|)$$

$$= np + \left\lceil \frac{r}{k-1} \right\rceil (r + k - 1) + 2R(k-1),$$

as desired. (One may similarly show that $\mathcal{C}_0$ has degree sequence $\mathbf{d}(\mathcal{C}_0) = \mathbf{d}_0$.) It remains to see that $\mathcal{C}_0$ covers $\binom{V}{k}$. For that, we fix an element $K \in \binom{V}{k}$, and consider whether or not $K \subseteq X_i$ for some $1 \le i \le q - R$.

Assume $K \subseteq X_i$ for some $1 \le i \le q - R$. Then $i \ge q - 2R$ since otherwise X_i is too small. If $i = q - 2R$, then $r \neq 0$ since otherwise X_{q-2R} is too small. Since each of A_{p+1} and B_{p+1} is too small to entirely contain K, it must be that K meets them both, and so $\{A_{p+1}, B_{p+1}\}$ covers K. Similarly, if $i > q - 2R$, then necessarily $R \neq 0$ and $X_i = Y_i \dot\cup Z_i$. Since each of Y_i and Z_i is too small to entirely contain K, it must be that K meets them both. Then, $\{Y_i, Z_i\}$ covers K, and so $\{A_{p+2}, B_{p+2}\}$ does too.

Assume K meets some two of $X_1, \ldots, X_{q-R}$, say $X_i \neq X_{i'}$. Fix $u \in K \cap X_i$ and $v \in K \cap X_{i'}$. Then, $\mathbf{u} = \mathbf{x}_i \neq \mathbf{x}_{i'} = \mathbf{v}$, in which case some coordinate $1 \le j \le p$ satisfies $\mathbf{u}(j) \neq \mathbf{v}(j)$. As such, $\{A_j, B_j\}$ separates u and v, in which case K meets each of A_j and B_j. Since $V = A_j \dot\cup B_j$ by construction, $\{A_j, B_j\}$ covers K, as desired.

■

Remark 2.0.10. One may use $\mathcal{C}_0$ to define equally-weighted covers $\mathcal{C}_0'$ which are, from the point of view of $|\mathcal{C}_0'| \neq |\mathcal{C}_0|$, distinct from $\mathcal{C}_0$. Most easily, when $R \geq 2$, the proof above shows that we can take $\mathcal{C}_0'$ to include each of $\{A_1, B_1\}, \ldots, \{A_{p+1}, B_{p+1}\} \in \mathcal{C}_0$, where we replace $\{A_{p+2}, B_{p+2}\} \in \mathcal{C}_0$ with the R elements $\{Y_{q-2R+1}, Z_{q-2R+1}\}, \ldots, \{Y_{q-R}, Z_{q-R}\}$. Alternatively, one could include $\{A_1, B_1\} \in \mathcal{C}_0$, and then inductively define covers of $\binom{A_1}{k}$ and $\binom{B_1}{k}$. $\square$

3 The Weak Lower Bound & Related Results

This chapter has primary and secondary goals. Primarily, we prove Theorem 1.3.8 (Weak Lower Bound) (cf. Section 3.1). Some details of this proof, when specialized, will be important later in this dissertation. Secondarily, we specialize precisely these details (cf. Section 3.2). Finally, we prove the latter conclusion of Theorem 1.3.8, which is that Theorem 1.3.4 (Degree–Sequence) holds when $r = 0$ (cf. Section 3.3).

3.1 Proof of the Weak Lower Bound

Proof of Theorem 1.3.8. Fix integers $n \geq k \geq 2$, fix an n-element vertex set V, and let integers p, q, r, R be given by (2.0.1). Fix an arbitrary cover $\mathcal{C}$ of $\binom{V}{k}$. To prove that $\omega(\mathcal{C}) \geq np + 2R(k-1) + 2r$, we make some auxiliary considerations which are important throughout the dissertation. Recall that for each vertex $v \in V$, we define $d_v = \deg(v) = \deg_{\mathcal{C}}(v)$ to be the number of elements $\{A, B\} \in \mathcal{C}$ to which v is *incident*, meaning that $v \in A \dot{\cup} B$. Standard double counting gives

$$\sum_{v \in V} d_v = \sum_{\{A,B\} \in \mathcal{C}} (|A| + |B|) = \omega(\mathcal{C}),$$

where

$$\alpha = \alpha(\mathcal{C}) = \frac{1}{n} \sum_{v \in V} d_v = \frac{\omega(\mathcal{C})}{n} \tag{3.1.1}$$

denotes the average degree in $\mathcal{C}$. For sake of argument, we assume

$$\alpha < p + 1, \tag{3.1.2}$$

since otherwise we would have

$$\begin{aligned}
\omega(\mathcal{C}) &\overset{(3.1.1)}{=} \alpha n \\
&\overset{\neg(3.1.2)}{\geq} (p+1)n = np + n \\
&\overset{(2.0.1)}{=} np + (2^p + R)(k-1) + r \\
&\overset{(2.0.1)}{\geq} np + (2R+1)(k-1) + r \\
&= np + 2R(k-1) + r + k - 1,
\end{aligned} \tag{3.1.3}$$

which already exceeds $np + 2R(k-1) + 2r$ on account of $r < k - 1$ in (2.0.1).

The following idea has roots in several sources [1, 10, 13, 19]: Independently for each $\{A, B\} \in \mathcal{C}$, set

$$Z_{\{A,B\}} = \begin{cases} V \setminus A & \text{with probability } 1/2, \\ V \setminus B & \text{with probability } 1/2. \end{cases} \tag{3.1.4}$$

Set $Z = \bigcap_{\{A,B\} \in \mathcal{C}} Z_{\{A,B\}}$, which is a random subset of V whose expectation $\mathbb{E}[|Z|]$ we now estimate. On the one hand, $\mathcal{C}$ covers $\binom{V}{k}$, so no k-tuple $K \in \binom{V}{k}$ can survive (3.1.4). Consequently, $|Z| \leq k - 1$ and thus $\mathbb{E}[|Z|] \leq k - 1$. On the other hand, linearity of expectation gives $\mathbb{E}[|Z|] = \sum_{v \in V} \mathbb{P}[v \in Z]$, where the event $v \in Z$ holds if, and only if, the independent events $v \in Z_{\{A,B\}}$ (cf. (3.1.4)) hold for each of the d_v

many elements $\{A, B\} \in \mathcal{C}$ to which v is incident. Thus,

$$\mathbb{E}[|Z|] = \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} \leq k - 1, \tag{3.1.5}$$

where we pause to make the following important remark.

Remark 3.1.1. Applying the Arithmetic-Geometric Mean Inequality to (3.1.5), we have

$$\frac{k-1}{n} \geq \frac{1}{n} \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v}$$

$$\geq \left(2^{-\sum_{v \in V} d_v}\right)^{1/n} \overset{(3.1.1)}{=} 2^{-\alpha}$$

$$\implies \alpha \geq \log_2 \left(\frac{n}{k-1}\right) \overset{(2.0.1)}{\geq} p, \tag{3.1.6}$$

which we reference throughout this dissertation. □

The proof of Theorem 1.3.8 hinges on the following key idea of Bollobás and Scott [1]: In (3.1.5), replace $\mathbf{d} = (d_v)_{v \in V}$ with a positive integer sequence $\mathbf{e} = (e_v)_{v \in V}$ satisfying the following properties:

(*a*) $\sum_{v \in V} e_v = \sum_{v \in V} d_v$;

(*b*) $\sum_{v \in V} \left(\frac{1}{2}\right)^{e_v} \leq \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v}$;

(*c*) $|e_w - e_x| \leq 1$ for all $w, x \in V$.

To construct $\mathbf{e} = (e_v)_{v \in V}$, fix $w, x \in V$. An easy calculation reveals that

$$d_x \geq d_w + 1 \qquad \iff \qquad \left(\frac{1}{2}\right)^{d_x} + \left(\frac{1}{2}\right)^{d_w} \geq \left(\frac{1}{2}\right)^{d_x - 1} + \left(\frac{1}{2}\right)^{d_w + 1}. \tag{3.1.7}$$

In particular, if $d_x \geq d_w + 2$, we replace d_x in $\mathbf{d}$ with $d'_x = d_x - 1$, and we replace d_w in $\mathbf{d}$ with $d'_w = d_w + 1$. Clearly, the resulting sequence $\mathbf{d}'$ satisfies Property (a), and by (3.1.7) it also satisfies Property (b). Iterating this idea on $\mathbf{d}'$, we eventually arrive at a sequence $\mathbf{e}$ which also satisfies Property (c).

We claim that $\mathbf{e}$ assumes only the values p and $p+1$. Indeed, Property (c) guarantees that $\mathbf{e}$ assumes at most two values, which we call e and $e+1$ (if $\mathbf{e}$ assumes one value we say this common value is e). Since $\mathbf{e}$ necessarily satisfies $e = \lfloor (1/n) \sum_{v \in V} e_v \rfloor$, property (a) gives

$$\frac{1}{n} \sum_{v \in V} e_v = \frac{1}{n} \sum_{v \in V} d_v \overset{(3.1.1)}{=} \alpha. \tag{3.1.8}$$

Then by (3.1.2) and (3.1.6), we have

$$\begin{aligned} p \leq e = \lfloor \alpha \rfloor < p+1 \\ \implies \quad e = p, \end{aligned} \tag{3.1.9}$$

as claimed.

We now conclude the proof of Theorem 1.3.8. Set $V^- = \{v \in V : e_v = p\}$ and $V^+ = \{v \in V : e_v = p+1\}$. Using Property (b) and (3.1.5), we have

$$|V^-| \left(\frac{1}{2}\right)^p + |V^+| \left(\frac{1}{2}\right)^{p+1} \leq k-1$$

$$\implies 2|V^-| + |V^+| \leq 2^{p+1}(k-1),$$

$$\implies 2(n - |V^+|) + |V^+| \leq 2^{p+1}(k-1)$$

$$\implies |V^+| \geq 2n - 2^{p+1}(k-1)$$

$$= 2\left(n - 2^p(k-1)\right)$$

$$\overset{(2.0.1)}{=} 2\left(R(k-1)+r\right)$$

$$= 2R(k-1)+2r. \tag{3.1.10}$$

Thus, by (3.1.1) and Property (a), we conclude with

$$\omega(\mathcal{C}) = \sum_{v\in V} d_v = \sum_{v\in V} e_v$$

$$= p|V^-| + (p+1)|V^+|$$

$$= np + |V^+|$$

$$\overset{(3.1.10)}{\geq} np + 2R(k-1) + 2r. \tag{3.1.11}$$

■

3.2 Some Related Notes

In the proof of Theorem 1.3.8, recall that $\mathcal{C}$ was an arbitrary cover of $\binom{V}{k}$. Below, we revisit (3.1.2), (3.1.3), and (3.1.5) when $\mathcal{C}$ is assumed to be optimal, i.e., $\omega(\mathcal{C}) = h(n,k)$.

Fact 3.2.1. *Let $\mathcal{C}$ optimally cover $\binom{V}{k}$, where $|V| = n \geq k \geq 2$, and where p, q, r, R are given by (2.0.1). Then the average degree α of $\mathcal{C}$ (cf. (3.1.1)) satisfies $p \leq \alpha \leq p+1$. Moreover, we have $\alpha = p$ if, and only if, $r = R = 0$. We have $\alpha = p+1$ if, and only if, $R = 2^p - 1$ and $h(n,k) = np + 2R(k-1) + r + k - 1$.*

Proof. We showed $\alpha \geq p$ in (3.1.6). If $\alpha > p+1$, then (3.1.3) would have $h(n,k) = \omega(\mathcal{C}) > np + 2R(k-1) + r + k - 1$, which contradicts Proposition 1.3.7. Since

$h(n,k) = \alpha n$, Proposition 1.3.7 and Theorem 1.3.8 show that $\alpha = p$ if, and only if, $r = R = 0$. If $\alpha = p+1$, then Proposition 1.3.7 implies that (3.1.3) must have equality throughout, which gives $R = 2^p - 1$ and $h(n,k) = \omega(\mathcal{C}) = np + 2R(k-1) + r + k - 1$. Finally, if $R = 2^p - 1$ and $h(n,k) = \omega(\mathcal{C}) = np + 2R(k-1) + r + k - 1$, then

$$\alpha n = h(n,k) = np + 2R(k-1) + r + k - 1$$

$$\implies \frac{(\alpha - p)n - r}{k-1} = 2R + 1$$

$$= 2^p + R \overset{(2.0.1)}{=} \frac{n-r}{k-1},$$

from which $\alpha = p + 1$ follows.

■

Fact 3.2.2. *Let $\mathcal{C}$ optimally cover $\binom{V}{k}$, where $|V| = n \geq k \geq 2$, and where p, q, r, R are given by (2.0.1). Then,*

$$r\left(\frac{1}{2}\right)^{p+1} + (k-1)\left(1 - \left(\frac{1}{2}\right)^{p+1}\right) \leq \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v}.$$

Proof. Let $\mathcal{C}$ optimally cover $\binom{V}{k}$, where $|V| = n \geq k \geq 2$, and where p, q, r, R are given by (2.0.1). From Fact 3.2.1, we have $p \leq \alpha \leq p+1$. If $\alpha = p+1$, then Fact 3.2.1 also gives $R = 2^p - 1$, and so (cf. (2.0.1)) $n = q(k-1) + r$ where $q = 2^{p+1} - 1$. In this case, in the Arithmetic-Geometric Mean Inequality (cf. (3.1.6)), we have

$$\sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} \geq n\left(\frac{1}{2}\right)^{p+1}$$

$$= (k-1)q\left(\frac{1}{2}\right)^{p+1} + r\left(\frac{1}{2}\right)^{p+1}$$

$$= (k-1)\left(1-\left(\frac{1}{2}\right)^{p+1}\right)+r\left(\frac{1}{2}\right)^{p+1},$$

as desired. Henceforth, we assume $p \le \alpha < p+1$, but we suppose

$$\sum_{v\in V}\left(\frac{1}{2}\right)^{d_v} < r\left(\frac{1}{2}\right)^{p+1}+(k-1)\left(1-\left(\frac{1}{2}\right)^{p+1}\right). \tag{3.2.12}$$

Construct $\mathbf{e} = (e_v)_{v\in V}$ precisely as in (3.1.7) so that $\mathbf{e}$ assumes at most two values, which are still p and $p+1$ (by Fact 3.2.1, (3.1.8), and (3.1.9)). By Property (b) and (3.2.12), we infer

$$|V^-|\left(\frac{1}{2}\right)^{p}+|V^+|\left(\frac{1}{2}\right)^{p+1} \le \sum_{v\in V}\left(\frac{1}{2}\right)^{d_v}$$

$$< r\left(\frac{1}{2}\right)^{p+1}+(k-1)\left(1-\left(\frac{1}{2}\right)^{p+1}\right),$$

or equivalently (cf. (3.1.10)), $|V^+| > 2R(k-1)+k-1+r$. By Property (a) (cf. (3.1.11)),

$$h(n,k) = \omega(\mathcal{C})$$

$$= p|V^-|+(p+1)|V^+| = np+|V^+|$$

$$> np+2R(k-1)+k-1+r,$$

which contradicts Proposition 1.3.7.

■

3.3 Proof of the Degree–Sequence Theorem When $r = 0$

As promised, we now prove the latter conclusion of Theorem 1.3.8, which is that Theorem 1.3.4 holds when $r = 0$.

Proof of Theorem 1.3.4 when $r = 0$. The proof is similar to that of Fact 3.2.2. Assume now that $r = 0$, but that k, n, p, q, and R are otherwise fixed by (2.0.1). We use that, in this case, $h(n, k) = np + 2R(k-1)$, which follows from $r = 0$, Proposition 1.3.7, and Theorem 1.3.8. Now, let $\mathcal{C}$ optimally cover $\binom{V}{k}$ with degree-sequence $\mathbf{d} = \mathbf{d}(\mathcal{C}) = (d_v)_{v \in V}$, but assume for contradiction that $\mathbf{d} \neq \mathbf{d}_0$ (cf. (1.3.2)). Using Fact 3.2.1, $\mathcal{C}$ has average degree $p \leq \alpha < p+1$, where $\alpha = p+1$ is forbidden by $h(n, k) = np + 2R(k-1)$. We again construct $\mathbf{e} = (e_v)_{v \in V}$ precisely as in (3.1.7), and observe that $\mathbf{e} = \mathbf{d}_0$. Indeed, revisiting (3.1.11),

$$np + 2R(k-1) = h(n, k) = \omega(\mathcal{C}) = np + |V^+|,$$

so that $\mathbf{e} \in \{p, p+1\}^V$ has precisely $2R(k-1)$ many $(p+1)$-digits. Since $\mathbf{d} \neq \mathbf{d}_0 = \mathbf{e}$, there must exist $x, w \in V$ with $d_x \geq d_w + 2$. As such, strict inequality holds throughout (3.1.7), and so strict inequality holds throughout (3.1.10) and (3.1.11). Now, $\omega(\mathcal{C}) > np + 2R(k-1) = h(n, k)$, contradicting the optimality of $\mathcal{C}$.

■

4 Structural Instruments

Soon, the rest of this dissertation will consist of almost entirely structural considerations. We devote this chapter to structural instruments that will be used throughout the rest of this dissertation. We begin by presenting key structural objects (cf. Section 4.1) which will be a bedrock for the rest of this dissertation; but we first use them to prove the *Pre-Extremal Lemma* (cf. Section 4.2), which greatly aids in proving the all-important *Extremal Lemma* of Chapter 5. But we make an interesting note: Although the Pre-Extremal Lemma requires these new structural objects, and although it is an aid to the stronger Extremal Lemma, the proof of the Pre-Extremal Lemma is almost entirely probabilistic, making use of considerations from Theorem 1.3.8. Thus, the Pre-Extremal Lemma is a prime example of the fascinating interplay between structural and probabilistic considerations that is presented in this dissertation.

In addition to these, we introduce a structural alteration called *shifting* (cf. Section 4.3). We use shifting throughout this dissertation, including in the base case and inductive step of Theorem 1.3.9. But first, and at the end of this chapter (cf. Section 4.4), we use shifting to guarantee the existence of some useful covers whose structure is an intertwinement of the structural objects we now present.

4.1 Surviving Sets & Bones

Our first structural object essentially details the sample space of the random experiment in (3.1.4).

Definition 4.1.1. (surviving sets) Let V be a finite set, and let $\mathcal{C}$ cover $\binom{V}{k}$. Fix symbols a and b, and let $\{a,b\}^{\mathcal{C}}$ denote the set of all functions $\psi : \mathcal{C} \to \{a,b\}$. For $\psi \in \{a,b\}^{\mathcal{C}}$ and $\{A,B\} \in \mathcal{C}$, define

$$Z^{\psi}_{\{A,B\}} = \begin{cases} V \setminus A & \text{if } \psi(\{A,B\}) = a, \\ V \setminus B & \text{if } \psi(\{A,B\}) = b, \end{cases}$$

and define

$$Z_{\psi} = \bigcap_{\{A,B\} \in \mathcal{C}} Z^{\psi}_{\{A,B\}}. \tag{4.1.1}$$

(Soon we will note that (3.1.4) arises when $\psi \in \{a,b\}^{\mathcal{C}}$ is chosen uniformly at random.)

For $\psi \in \{a,b\}^{\mathcal{C}}$, we call the set Z_{ψ} in (4.1.1) a *surviving set*. We call $\mathcal{Z} = \left\{Z_{\psi} : \psi \in \{a,b\}^{\mathcal{C}}\right\}$ the *surviving family* of $\mathcal{C}$.

Our next definition portrays some natural and basic pieces that classify the vertices of a covering. Just as the bones of our bodies are the structural foundation for our bodies, these basic pieces are foundational to the structure of a covering.

Definition 4.1.2. (bones, skeleton) Let $\mathcal{C}$ cover $\binom{V}{k}$. Note that $\mathcal{C}$ defines the following equivalence relation $\sim_{\mathcal{C}}$ on V: For $u, v \in V$, set $u \sim_{\mathcal{C}} v$ if, and only if, for each $\{A,B\} \in \mathcal{C}$, either

$$u, v \in A, \qquad \text{or} \qquad u, v \in B, \qquad \text{or} \qquad \{u,v\} \cap (A \dot\cup B) = \emptyset. \tag{4.1.2}$$

Let $\mathcal{S} = \mathcal{S}(\mathcal{C})$ be the family of equivalence classes of V induced by $\sim_{\mathcal{C}}$. We call $\mathcal{S}$ the *skeleton* of $\mathcal{C}$, and we call elements $S \in \mathcal{S}$ the *bones* of $\mathcal{C}$.

And now we intertwine Definitions 4.1.1 and 4.1.2.

Definition 4.1.3. (surviving skeleton, surviving/strong cover) Let $\mathcal{C}$ cover $\binom{V}{k}$ with skeleton $\mathcal{S}$ and surviving family $\mathcal{Z}$. Since $\emptyset \in \mathcal{Z}$ is possible, we write $\mathcal{Z}^* = \mathcal{Z} \setminus \{\emptyset\}$ for the non-empty surviving sets of $\mathcal{Z}$. If $\mathcal{Z}^* = \mathcal{S}$, then we call $\mathcal{Z}^*$ the *surviving* *skeleton* of $\mathcal{C}$, and we call $\mathcal{C}$ a *surviving cover* of $\binom{V}{k}$. If, additionally, $\mathcal{C}$ is optimal, i.e., $\omega(\mathcal{C}) = h(|V|, k)$, then we say that $\mathcal{C}$ is a *strong cover*.

4.2 Pre-Extremal Lemma

Surviving covers are essential for the Pre-Extremal Lemma, and they will play an important role later in the base case of Theorem 1.3.9 in Chapter 7. Therefore, we will soon present and prove the *Survival Lemma*, which guarantees the existence of surviving covers. But first, we prove the Pre-Extremal Lemma, wherein we need Lemma 4.2.1 below. And know that Lemma 4.2.1 will also be used again later in the base case of Theorem 1.3.9.

Lemma 4.2.1. *Let V be a finite set and let $\mathcal{C}$ be a surviving cover of $\binom{V}{k}$ with surviving skeleton $\mathcal{Z}^* = \mathcal{S}$. Select $\psi \in \{a, b\}^{\mathcal{C}}$ uniformly at random, and let $Z = Z_\psi$ denote the random surviving set.*

Then Z satisfies (3.1.5), and for every $S \in \mathcal{S} = \mathcal{Z}^*$,

$$\mathbb{P}[Z = S] = \left(\frac{1}{2}\right)^{\deg_{\mathcal{C}}(S)}.$$

Proof. Fix $S_0 \in \mathcal{S} = \mathcal{Z}^*$. Since S_0 is a surviving set, this means (cf. Definition 4.1.1) that for some function $\psi_0 \in \{a, b\}^{\mathcal{C}}$, the set $S_0 = Z_{\psi_0} \in \mathcal{Z}^*$ has the following form (cf. (4.1.1)): for each $\{A, B\} \in \mathcal{C}$, recall $Z^{\psi_0}_{\{A,B\}} = V \setminus A$ if $\psi_0(\{A, B\}) = a$, and $Z^{\psi_0}_{\{A,B\}} = V \setminus B$ otherwise, and $S_0 = Z_{\psi_0} = \bigcap_{\{A,B\}\in\mathcal{C}} Z^{\psi_0}_{\{A,B\}}$.

And since $Z = Z_\psi$ is the random surviving set which was produced by selecting a function $\psi \in \{a, b\}^{\mathcal{C}}$ uniformly at random, it follows that Z is equivalently determined by $Z = \bigcap_{\{A,B\}\in\mathcal{C}} Z_{\{A,B\}}$ from (3.1.4), where independently for each $\{A, B\} \in \mathcal{C}$, $Z_{\{A,B\}} = V \setminus A$ with probability 1/2 and $Z_{\{A,B\}} = V \setminus B$ otherwise. Thus, Z satisfies (3.1.5).

Moreover, since $\mathcal{Z}^* = \mathcal{S}$ is the surviving skeleton (i.e., every non-empty surviving set is a single bone (cf. Definition 4.1.3)), the random surviving set $Z \in \mathcal{Z}$ satisfies

$Z = S_0$ if, and only if, $Z \supseteq S_0$. As such, $\mathbb{P}[Z = S_0] = \mathbb{P}[Z \supseteq S_0]$ is determined by the event that S_0 'survives' the procedure of (3.1.4), and so

$$\mathbb{P}[Z = S_0] = \mathbb{P}[Z \supseteq S_0] = \left(\frac{1}{2}\right)^{\deg_{\mathcal{C}}(S_0)}.$$

■

Lemma 4.2.2 (Pre-Extremal Lemma). *Fix integers $n \geq k \geq 2$, and let integers p, q, r, R be given by (2.0.1), where $r \geq 1$. Let V be an n-element vertex set, and let $\mathcal{C}$ be a strong cover of $\binom{V}{k}$. Every bone $S \in \mathcal{S}$ with subaverage degree $\deg_{\mathcal{C}}(S) < \alpha = \alpha(\mathcal{C})$ satisfies $|S| > (k-1)/2$.*

Proof. Fix $S_0 \in \mathcal{S}$ with subaverage degree $\deg_{\mathcal{C}}(S_0) < \alpha$. By Fact 3.2.1, $\alpha \leq p + 1$ and so $\deg_{\mathcal{C}}(S_0) \leq p$.

Select $\psi \in \{a, b\}^{\mathcal{C}}$ uniformly at random, and let $Z = Z_\psi$ denote the random surviving set. We will pivot $|S_0|$ against the expectation $\mathbb{E}[|Z|]$ of the random variable $|Z|$.

As ψ was chosen uniformly at random, it follows

$$\mathbb{E}[|Z|] = 2^{-|\mathcal{C}|} \sum_{\psi \in \{a,b\}^{\mathcal{C}}} |Z_\psi|$$

$$= \sum_{Z_\Psi \in \mathcal{Z}} \big(|Z_\Psi| \cdot \mathbb{P}[Z = Z_\Psi]\big), \tag{4.2.3}$$

where for each $Z_\Psi \in \mathcal{Z}$, the quantity $2^{|\mathcal{C}|}\mathbb{P}[Z = Z_\Psi]$ counts the number of functions $\psi \in \{a, b\}^{\mathcal{C}}$ for which $Z_\psi = Z_\Psi$. Since $\mathcal{S} = \mathcal{Z}^* \subseteq \mathcal{Z}$ holds by hypothesis, the element $S_0 \in \mathcal{S}$ appears in $\mathcal{Z}$, and so we further rewrite (4.2.3) as

$$\mathbb{E}[|Z|] = \big(|S_0| \cdot \mathbb{P}[Z = S_0]\big) + \sum_{S_0 \neq Z_\Psi \in \mathcal{Z}} \big(|Z_\Psi| \cdot \mathbb{P}[Z = Z_\Psi]\big). \tag{4.2.4}$$

Our remaining work bounds (4.2.4) from above and from below.

On the one hand, every surviving set $Z_\Psi \in \mathcal{Z}$ has size $|Z_\Psi| \le k-1$, and from Lemma 4.2.1 we have

$$\mathbb{P}[Z = S_0] = \left(\frac{1}{2}\right)^{\deg_{\mathcal{C}}(S)} \ge \left(\frac{1}{2}\right)^p,$$

and so (4.2.4) satisfies

$$\mathbb{E}[|Z|] \le \big(|S_0| \cdot \mathbb{P}[Z = S_0]\big) + (k-1) \sum_{S_0 \neq Z_\Psi \in \mathcal{Z}} \mathbb{P}[Z = Z_\Psi]$$

$$= \big(|S_0| \cdot \mathbb{P}[Z = S_0]\big) + (k-1)\big(1 - \mathbb{P}[Z = S_0]\big)$$

$$= k - 1 - \mathbb{P}[Z = S_0]\big(k - 1 - |S_0|\big)$$

$$\le k - 1 - \frac{1}{2^p}\big(k - 1 - |S_0|\big)$$

$$= (k-1)\left(1 - \frac{1}{2^p}\right) + \frac{1}{2^p}|S_0|. \tag{4.2.5}$$

On the other hand, by Lemma 4.2.1, Z satisfies (3.1.5), and with Fact 3.2.2 we have the lower bound

$$r\left(\frac{1}{2}\right)^{p+1} + (k-1)\left(1 - \left(\frac{1}{2}\right)^{p+1}\right) \le \sum_{v \in V} \left(\frac{1}{2}\right)^{d_v} = \mathbb{E}[|Z|]. \tag{4.2.6}$$

Comparing (4.2.5) and (4.2.6) yields $2|S_0| \ge k - 1 + r > k - 1$, as desired.

■

Remark 4.2.3. The authors wish to pause here and make a, strictly speaking, unnecessary remark. The reader may have noticed that our results thus far have benefited greatly from probabilistic tools. This organization is not a coincidence, and we note that from this point forward, this dissertation will consist of almost entirely structural considerations, with the only exception being a very mild probabilistic consideration in the base case of Theorem 1.3.9.

4.3 Shifting

We now turn our attention to structural alterations of covers, in what we call *shifting*. We use shifting to prove the Survival Lemma; but what is more, almost every result from here to Chapter 7 involves shifting in some way.

Definition 4.3.1. (shifting) Let V, $\mathcal{C}$, and $\mathcal{S}$ be given as in Definition 4.1.2, and fix $S \in \mathcal{S}$ and $U \subset V \setminus S$. For $\{A, B\} \in \mathcal{C}$, define

$$A_{U,S} = \begin{cases} A \cup U & \text{if } S \subseteq A, \\ A \setminus U & \text{if } S \cap A = \emptyset, \end{cases} \qquad \text{and} \qquad B_{U,S} = \begin{cases} B \cup U & \text{if } S \subseteq B, \\ B \setminus U & \text{if } S \cap B = \emptyset. \end{cases}$$

Define $\mathcal{C}^*_{U,S} = \{\{A_{U,S}, B_{U,S}\} : \{A, B\} \in \mathcal{C}\}$ and $\mathcal{C}_{U,S} = \big\{\{U, S\}\big\} \cup \mathcal{C}^*_{U,S}$, which we call *$S$-shifts of U in $\mathcal{C}$*. We call $\{U, S\} \in \mathcal{C}_{U,S}$ the *exceptional pair* of $\mathcal{C}_{U,S}$.

We record a few notes on Definition 4.3.1.

Remark 4.3.2. By (4.1.2), each $\{A_{U,S}, B_{U,S}\} \in \mathcal{C}^*_{U,S}$ is well-defined, where we view $\mathcal{C}^*_{U,S}$ as a multiset (with possibly $\emptyset \in \{A_{U,S}, B_{U,S}\}$) so that $\{A, B\} \mapsto \{A_{U,S}, B_{U,S}\}$ is a bijection from $\mathcal{C}$ to $\mathcal{C}^*_{U,S}$. Now, if $\mathcal{C}_{U,S}$ covers $\binom{V}{k}$, then $|U| \leq k-1$, since otherwise $\mathcal{C}_{U,S}$ is unable to cover $\binom{U}{k}$. Note that $\{U, S\} \in \mathcal{C}_{U,S}$ uniquely covers $\binom{U \cup S}{k}$, and is needed only for this purpose. When $U = \{u\} \subset V$ is a singleton, we write $A_{u,S} = A_{\{u\},S}$, $B_{u,S} = B_{\{u\},S}$, $\mathcal{C}^*_{u,S} = \mathcal{C}^*_{\{u\},S}$, and $\mathcal{C}_{u,S} = \mathcal{C}_{\{u\},S}$. □

4.4 Survival Lemma

Lemma 4.4.1 (Survival Lemma)**.** *Let V be a finite set. For every cover $\mathcal{C}$ of $\binom{V}{k}$, there exists a surviving cover $\hat{\mathcal{C}}$ of $\binom{V}{k}$ such that the following holds:*

For each $v \in V$, we have $\deg_{\hat{\mathcal{C}}}(v) \leq \deg_{\mathcal{C}}(v)$, and so $\omega(\hat{\mathcal{C}}) \leq \omega(\mathcal{C})$ (cf. (3.1.1)).

For future reference, we make the following remarks.

Remark 4.4.2. Every finite set V admits a strong cover $\mathcal{C}$. Indeed, for an optimal cover $\mathcal{C}$ of $\binom{V}{k}$, the surviving cover $\hat{\mathcal{C}}$ of Lemma 4.4.1 must also be optimal, and therefore $\hat{\mathcal{C}}$ is a strong cover. Moreover, $\mathcal{C}$ and $\hat{\mathcal{C}}$ are *degree-equivalent*, in the sense that for each $v \in V$, we have $\deg_{\hat{\mathcal{C}}}(v) = \deg_{\mathcal{C}}(v)$. $\square$

Proof of Lemma 4.4.1. Let V and $\mathcal{C}$ be given as in the hypothesis of Lemma 4.4.1, and let $\mathcal{S}$ and $\mathcal{Z}$ be the skeleton and surviving family, respectively, of $\mathcal{C}$.

We begin with an elementary observation: A surviving set $Z \in \mathcal{Z}$ is always a union of bones $S \in \mathcal{S}$. Indeed, for $Z \in \mathcal{Z}$, there clearly exists bones $S_1, \ldots, S_t \in \mathcal{S}$ such that $Z \subseteq \cup_{i\in[t]} S_i$ and $Z \cap S_i \neq \emptyset$ for every $i \in [t]$. But by (4.1.2), it follows $S_i \subseteq Z$ for every $i \in [t]$, and so $Z = \cup_{i\in[t]} S_i$.

To determine whether a skeleton is a surviving skeleton, we note that

$$\textit{if } \mathcal{Z}^* \subseteq \mathcal{S}, \textit{ then } \mathcal{Z}^* = \mathcal{S}. \tag{4.4.7}$$

Indeed, for $S \in \mathcal{S}$, let $\psi \in \{a,b\}^{\mathcal{C}}$ be defined by, for each $\{A,B\} \in \mathcal{C}$, $\psi(\{A,B\}) = a$ if, and only if, $S \cap A = \emptyset$. Then $S \subseteq Z_\psi \in \mathcal{Z}^* \subseteq \mathcal{S}$, and as equivalence classes (cf. (4.1.2)), $S = Z_\psi \in \mathcal{Z}^*$.

So if $\mathcal{Z}^* \subseteq \mathcal{S}$, then by (4.4.7) we are done, so let $Z_0 = Z_{\psi_0} \in \mathcal{Z}^* \setminus \mathcal{S}$. Since Z_0 is a union of multiple bones, let $S_0 \in \mathcal{S}$ satisfy $S_0 \subsetneq Z_0$, where

$$\deg_{\mathcal{C}}(S_0) = \min_{S \in \mathcal{S}} \{\deg_{\mathcal{C}}(S) :\, S \subseteq Z_0\}, \tag{4.4.8}$$

and set $U_0 = Z_0 \setminus S_0 \neq \emptyset$.

Let $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0} = \mathcal{C}_{U_0,S_0} \setminus \{U_0, S_0\}$ be the family given by Definition 4.3.1, where we claim the following.

Claim 4.4.3. *The family $\mathcal{C}_0$ covers $\binom{V}{k}$. Moreover, the skeleton $\mathcal{S}_0$ of $\mathcal{C}_0$ satisfies $|\mathcal{S}_0| < |\mathcal{S}|$.*

Claim 4.4.3, which we verify in a moment, quickly implies Lemma 4.4.1. Indeed, Claim 4.4.3 gives that $\mathcal{C}_0$ covers $\binom{V}{k}$, where we will infer, moreover, that $\deg_{\mathcal{C}_0}(v) \leq \deg_{\mathcal{C}}(v)$ holds for each $v \in V$. Indeed, fix $u \in U_0$ and $v \in V \setminus U_0$. The construction of $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0}$ (cf. Definition 4.3.1) gives the identities $\deg_{\mathcal{C}_0}(v) = \deg_{\mathcal{C}}(v)$ and $\deg_{\mathcal{C}_0}(u) = \deg_{\mathcal{C}_0}(S_0) = \deg_{\mathcal{C}}(S_0)$. Thus, (4.4.8) adds that $\deg_{\mathcal{C}_0}(u) = \deg_{\mathcal{C}}(S_0) \leq \deg_{\mathcal{C}}(u)$. Finally, let $\mathcal{Z}_0$ be the surviving family of $\mathcal{C}_0$. If $\mathcal{Z}^*_0 \subseteq \mathcal{S}_0$, we set $\hat{\mathcal{C}} = \mathcal{C}_0$ and we are done. Otherwise, $\mathcal{Z}^*_0 \setminus \mathcal{S}_0 \neq \emptyset$, and we repeat (4.4.8). However, Claim 4.4.3 implies that we can't repeat (4.4.8) indefinitely, and so Lemma 4.4.1 follows.

Proof of Claim 4.4.3. To prove the first part of Claim 4.4.3, we fix $K \in \binom{V}{k}$, and consider three cases.

Case 1 ($K \cap U_0 = \emptyset$). Let $\{A, B\} \in \mathcal{C}$ cover K. Then

$$\{A_{U_0,S_0}, B_{U_0,S_0}\} \in \mathcal{C}_0 \text{ also covers } K \tag{4.4.9}$$

because Definition 4.3.1 gives $K \cap A_{U_0,S_0} = K \cap A$ and $K \cap B_{U_0,S_0} = K \cap B$. □

Case 2 ($K \cap U_0 \neq \emptyset : K \cap S_0 \neq \emptyset$). Let $\{A, B\} \in \mathcal{C}$ cover K. Since K meets S_0, we take without loss of generality (cf. (4.1.2)) $S_0 \subseteq A$ so that (cf. Definition 4.3.1) $A_{U_0,S_0} = A \cup U_0 \supseteq A$. We will easily infer (4.4.9) once we prove that

$$B_{U_0,S_0} = B. \tag{4.4.10}$$

Indeed, $Z_0 = Z_{\psi_0} = S_0 \cup U_0 \in \mathcal{Z}^* \setminus \mathcal{S}$ is a surving set given by $\psi_0 \in \{a, b\}^{\mathcal{C}}$ (cf. (4.1.1)), where $S_0 \subseteq A$ implies $Z_0 = S_0 \cup U_0 \subseteq Z^{\psi_0}_{\{A,B\}} = V \setminus B$, i.e., $\psi_0(\{A, B\}) = b$. Thus, $U_0 \cap B = \emptyset$, implying (4.4.10). □

Case 3 ($K \cap U_0 \neq \emptyset : K \cap S_0 = \emptyset$). Let $u \in K \cap U_0$, $v \in S_0$, $K_{u,v} = (K \setminus \{u\}) \cup \{v\}$, and $\{A, B\} \in \mathcal{C}$ cover $K_{u,v}$. By Case 1 or 2, $\{A_{U_0,S_0}, B_{U_0,S_0}\} \in \mathcal{C}_0$ covers $K_{u,v}$, where (w.l.o.g.) $S_0 \subseteq A \subseteq A_{U_0,S_0}$. Since $K \triangle K_{u,v} = \{u, v\} \subseteq A_{U_0,S_0}$, we have $u \in K \cap A_{U_0,S_0} \neq \emptyset$ and $K \cap B_{U_0,S_0} = K_{u,v} \cap B_{U_0,S_0} \neq \emptyset$, and so (4.4.9) follows. $\square$

To prove the second part of Claim 4.4.3, we first show that $Z_0 \in \mathcal{S}_0$ is a bone of $\mathcal{C}_0$. To that end, Definition 4.3.1 forces all vertices of $Z_0 = S_0 \dot\cup U_0$ to be equivalent in $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0}$, and so there exists some bone $T \in \mathcal{S}_0$ so that $Z_0 \subseteq T$. Assume, for contradiction, that there exists some $v \in T \setminus Z_0$. Since $v \notin Z_0$, we infer from (4.1.1) that some $\{A, B\} \in \mathcal{C}$ has $v \notin Z^{\psi_0}_{\{A,B\}}$, where $\psi_0 \in \{a, b\}^{\mathcal{C}}$ is the function for which $Z_0 = Z_{\psi_0}$. As such, and without loss of generality, $v \in A$ while $Z_0 \subseteq Z^{\psi_0}_{\{A,B\}} = Z \setminus A$, in which case $S_0 \cap A = \emptyset$. Definition 4.3.1 then ensures that $A_{U_0,S_0} = A \setminus U_0$, in which case $S_0 \cap A_{U_0,S_0} = \emptyset$ while necessarily $v \in A_{U_0,S_0}$ (because $v \notin Z_0 \supset U_0$). As such, v is not $\mathcal{C}_0$-equivalent to any vertex of S_0, contradicting that $\{v\}, S_0 \subseteq T \in \mathcal{S}_0$ were part of a bone T of $\mathcal{C}_0$.

To conclude the second part of Claim 4.4.3, write $U_0 = S_1 \dot\cup \cdots \dot\cup S_t$ as a union of $t \geq 1$ bones of $\mathcal{C}$, which is possible because $Z_0 = S_0 \dot\cup U_0$ is a surviving set of $\mathcal{C}$ with $U_0 \neq \emptyset$. We claim that the relation $f\colon \mathcal{S} \setminus \{S_0, S_1, \ldots, S_t\} \to \mathcal{S}_0 \setminus \{Z_0\}$ defined by $f(S) = Q$ if, and only if, $S \subseteq Q$, is a well-defined and surjective function. If true, it concludes the proof of Claim 4.4.3, since then

$$|\mathcal{S}| - (t + 1) = |\mathcal{S} \setminus \{S_0, S_1, \ldots, S_t\}| \geq |\mathcal{S}_0 \setminus \{Z_0\}| = |\mathcal{S}_0| - 1,$$

from which $|\mathcal{S}_0| \leq |\mathcal{S}| - t \leq |\mathcal{S}| - 1 < |\mathcal{S}|$ follows. Now, to see that f is well-defined, fix $S \in \mathcal{S} \setminus \{S_0, S_1, \ldots, S_t\}$. Then $S \cap Z_0 = S \cap (S_0 \dot\cup U_0) = \emptyset$, in which case S never moves upon shifting from $\mathcal{C}$ to $\mathcal{C}_0 = \mathcal{C}^*_{U_0,S_0}$, i.e., for each $\{A, B\} \in \mathcal{C}$, we have the conditions $S \subseteq A_{U_0,S_0}$ ($S \cap A_{U_0,S_0} = \emptyset$) if, and only if, $S \subseteq A$ ($S \cap A = \emptyset$). (The same statements hold for B_{U_0,S_0} and B.) Thus, the vertices of S are $\mathcal{C}_0$-equivalent, and so $S \subseteq Q$ for some unique $Q \in \mathcal{S}_0 \setminus \{Z_0\}$, where the bone Q is unique because it is an equivalence class. The proof of surjectivity is similar. Fix $Q \in \mathcal{S}_0 \setminus \{Z_0\}$, fix $v \in Q$, and let $S_v \in \mathcal{S}$ be the unique bone of $\mathcal{C}$ for which $v \in S_v$. Now, $v \notin Z_0 = S_0 \dot\cup S_1 \dot\cup \cdots \dot\cup S_t$, in which case S_v can't overlap any bone $S_i \subseteq Z_0$, $0 \leq i \leq t$. Thus, $S_v \cap (S_0 \dot\cup U_0) = \emptyset$, and S_v

doesn't move upon shifting. Thus, the vertices of S_v are $\mathcal{C}_0$-equivalent, and so $v \in Q$ implies $S_v \subseteq Q$.

■

5 Shifting Lemma I & the Extremal Lemma

In this chapter, we present and prove the *Extremal Lemma* (cf. Section 5.2), one of the most important structural results of this dissertation. To prove the Extremal Lemma, we need *Shifting Lemma I* (cf. Section 5.1), named as such because of its connection to the results of Chapter 6, *Shifting Lemmas II* and *III*.

Shifting alters a cover, and in particular, the cover that is resultant of a shift might have weight different than its predecessor. As such, we have the following fact which computes the weight of covers resultant of shifting.

Fact 5.0.4. *Let $V, \mathcal{C}, S$, and U be as in Definition 4.3.1. If $\mathcal{C}_{U,S}$ covers $\binom{V}{k}$, then*

$$\omega\left(\mathcal{C}_{U,S}\right) = \omega(\mathcal{C}) + |U| + |S| + \sum_{u \in U} \left(\deg_{\mathcal{C}}(S) - \deg_{\mathcal{C}}(u)\right).$$

If, additionally, $|U \cup S| < k$, then $\mathcal{C}^*_{U,S}$ covers $\binom{V}{k}$ with weight $\omega\left(\mathcal{C}^*_{U,S}\right) = \omega\left(\mathcal{C}_{U,S}\right) - |U| - |S|$.

The second assertion of Fact 5.0.4 is trivial, since when $\binom{U \cup S}{k} = \emptyset$ the exceptional pair $\{U, S\} \in \mathcal{C}_{U,S}$ isn't needed, and removing it (to form $\mathcal{C}^*_{U,S}$) reduces the weight by $|U| + |S|$.

Proof of Fact 5.0.4. Recall (cf. (3.1.1)) that $\omega(\mathcal{C}_{U,S})$ is the sum of the $\mathcal{C}_{U,S}$-degrees of vertices $v \in V$. We observe that

$$\deg_{\mathcal{C}_{U,S}}(v) = \begin{cases} 1 + \deg_{\mathcal{C}}(S) & \text{if } v \in U \dot{\cup} S, \\ \deg_{\mathcal{C}}(v) & \text{if } v \in V \setminus (U \dot{\cup} S). \end{cases} \tag{5.0.1}$$

Indeed, Definition 4.3.1 ensures that, for each $v \in V \setminus U$ (and because $v \notin U$), we have $v \in A_{U,S} \cup B_{U,S}$ if, and only if, $v \in A \cup B$. A vertex $v \in S$ is also incident to $\{U, S\} \in \mathcal{C}_{U,S}$, and so for $v \in S$,

$$\deg_{\mathcal{C}_{U,S}}(v) = 1 + \deg_{\mathcal{C}}(v) = 1 + \deg_{\mathcal{C}}(S), \tag{5.0.2}$$

where we used (4.1.2). The identity in (5.0.2) also holds for $v \in U$, because Definition 4.3.1 ensures $\deg_{\mathcal{C}_{U,S}}(v) = \deg_{\mathcal{C}_{U,S}}(S)$, where (5.0.2) gives $\deg_{\mathcal{C}_{U,S}}(S) = 1 + \deg_{\mathcal{C}}(S)$. Now, using (5.0.1) and (5.0.2), we see that

$$\begin{aligned} \omega\left(\mathcal{C}_{U,S}\right) &= \sum_{v \in V} \deg_{\mathcal{C}_{U,S}}(v) \\ &= \sum_{v \in U} \deg_{\mathcal{C}_{U,S}}(v) + \sum_{v \in S} \deg_{\mathcal{C}_{U,S}}(v) + \sum_{v \in V \setminus (S \dot{\cup} U)} \deg_{\mathcal{C}_{U,S}}(v) \\ &= |U|\big(1 + \deg_{\mathcal{C}}(S)\big) + \sum_{v \in S} \big(1 + \deg_{\mathcal{C}}(v)\big) + \sum_{v \in V \setminus (S \dot{\cup} U)} \deg_{\mathcal{C}}(v) \\ &= |U| + |S| + \sum_{v \in U} \deg_{\mathcal{C}}(S) + \sum_{v \in V \setminus U} \deg_{\mathcal{C}}(v) \\ &= \omega(\mathcal{C}) + |U| + |S| + \sum_{v \in U} \big(\deg_{\mathcal{C}}(S) - \deg_{\mathcal{C}}(u)\big), \end{aligned} \tag{5.0.3}$$

as desired.

■

We now introduce the central object of this chapter: *limbs*. Limbs take the spotlight for Shifting Lemma I, and are the pivotal consideration in the Extremal Lemma.

Consider $V, \mathcal{C}, S$, and U as in Fact 5.0.4, but where $U = \{u\} \subseteq V$ is a singleton. Shifting Lemma I investigates when it is possible to shift the vertex $u \notin S$ to the bone

S so that $\mathcal{C}_{u,S}$ covers $\binom{V}{k}$. If $|S| = k-1$, then it is not too difficult to show that $\mathcal{C}_{u,S}$ covers $\binom{V}{k}$. However, if $|S| < k-1$, a significant threat arises: For if S is contained in a *limb* (to be defined soon), but u is not contained in that limb, it is impossible for $\mathcal{C}_{u,S}$ to cover $\binom{V}{k}$. Thus, limbs are a thorn in the side of one who wishes to shift, and so are of crucial importance in our treatise of shifting.

Definition 5.0.5. (singular, limbs) Let $V, \mathcal{C}$, and $\mathcal{S}$ be as in Definition 4.1.2. We say that a subset $J \subset V$ is *singular* if, for each $\{A, B\} \in \mathcal{C}$, either

$$J \subseteq A, \quad \text{or} \quad J \subseteq B, \quad \text{or} \quad J \not\subseteq A \dot{\cup} B. \tag{5.0.4}$$

Let $\mathcal{J} = \mathcal{J}(\mathcal{C})$ denote the family of all singular sets $J \subset V$, where we note that each $J \in \mathcal{J}$ satisfies $|J| \leq k-1$, because $\mathcal{C}$ covers all of $\binom{V}{k}$. We call a singular set $L \in \mathcal{J}$ a *limb* of $\mathcal{C}$ if $L \notin \mathcal{S}$ is not a bone, but $|L| = k-1$ is of maximum size. We write $\mathcal{L} = \mathcal{L}(\mathcal{C}) = (\mathcal{J} \setminus \mathcal{S}) \cap \binom{V}{k-1}$ for the family of all limbs of $\mathcal{C}$.

Remark 5.0.6. All bones $S \in \mathcal{S}$, surviving sets $Z \in \mathcal{Z}$, and limbs $L \in \mathcal{L}$ are singular, but a singular set $J \in \mathcal{J}$ need be neither a bone, surviving set, nor a limb. (Indeed, fix $S \in \mathcal{S}$ and a proper subset $J \subsetneq S$ thereof.) By definition, a limb $L \in \mathcal{L}$ is not a bone, but by (4.1.2), it is a union of at least two bones. It also follows from (4.1.2) that both the intersection $L_1 \cap L_2$ and the union $L_1 \cup L_2$ of limbs $L_1, L_2 \in \mathcal{L}$ are unions of bones. □

For future reference, we observe the following fact.

Fact 5.0.7. *Let $S_0 \in \mathcal{S}$ have size $|S_0| \geq (k-1)/2$. Then, S_0 belongs to at most one limb $L \in \mathcal{L}$.*

Proof. Let $S_0 \in \mathcal{S}$ have size $|S_0| \geq (k-1)/2$, but assume for contradiction that some pair $L_1 \neq L_2 \in \mathcal{L}$ satisfies $S_0 \subseteq L_1 \cap L_2$. Using Remark 5.0.6, write $L_1 \cup L_2 = S_0 \cup S_1 \cup \cdots \cup S_t$, where $S_1, \ldots, S_t \in \mathcal{S}$ are bones of $\mathcal{C}$. Then $1 \leq t \leq k-1$ because $L_1 \cup L_2 = S_0 \in \mathcal{S}$ is impossible (cf. Remark 5.0.6) and

$$t \leq \left|(L_1 \cup L_2) \setminus S_0\right|$$

$$\leq |L_1 \setminus S_0| + |L_2 \setminus S_0|$$

$$\leq \left(k - 1 - \frac{k-1}{2}\right) + \left(k - 1 - \frac{k-1}{2}\right) = k - 1.$$

Now, let $K \subseteq L_1 \cup L_2$ be any k-tuple meeting each of $S_0, S_1, \ldots, S_t$. (Such a k-tuple exists by selecting, for each $0 \leq i \leq t$, precisely one element $v_i \in S_i$, and then selecting $|L_1 \cup L_2| - (t+1) \geq k - (t+1) \geq 0$ remaining elements arbitrarily from $(L_1 \cup L_2) \setminus \{v_0, \ldots, v_t\}$.) Let $\{A, B\} \in \mathcal{C}$ cover K. Since $K \subseteq A \dot\cup B$ meets each bone $S_0, S_1, \ldots, S_t$ of $L_1 \cup L_2$, and since $K \subseteq A \dot\cup B$, we have from (4.1.2) that all of $L_1 \cup L_2$ appears in $A \dot\cup B$. Since $L_1 \subset A \dot\cup B$ is singular, assume without loss of generality $L_1 \subseteq A$ so that $S_0 \subseteq L_1 \cap L_2 \subseteq A$. Then $S_0 \subseteq L_2 \cap A \neq \emptyset$, and since $L_2 \subset A \dot\cup B$ is singular, it must be that $L_2 \subset A$. Now, $K \subseteq L_1 \cup L_2 \subseteq A$, contradicting that $\{A, B\}$ covered K.

■

5.1 Shifting Lemma I

Lemma 5.1.1 (Shifting Lemma I)**.** *Let V, $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{L}$ be given as in Definition 5.0.5, and fix $S \in \mathcal{S}$ and $u \in V \setminus S$. If $|S| = k - 1$, then $\mathcal{C}_{u,S}$ covers $\binom{V}{k}$. More generally, for $1 \leq |S| \leq k - 1$, if*

$$\textit{every limb } L \in \mathcal{L} \textit{ which contains } S \textit{ also contains } u, \tag{5.1.5}$$

then $\mathcal{C}_{u,S}$ covers $\binom{V}{k}$.

The latter assertion of Lemma 5.1.1 implies the former, since $|S| = k-1$ implies that (5.1.5) holds vacuously. (Indeed, if $L \in \mathcal{L}$ contains $S \in \mathcal{S} \cap \binom{V}{k-1}$, then $L = S$ is itself a bone, contradicting Definition 5.0.5.) Now, the second result generalizes the former assertion of Lemma 5.1.1.

Proof of Lemma 5.1.1. Let V, $\mathcal{C}$, $\mathcal{S}$, $\mathcal{L}$, $S \in \mathcal{S}$, and $u \in V \setminus S$ be given as in the hypothesis of Lemma 5.1.1, where (5.1.5) is satisfied. We uses cases to show that $\mathcal{C}_{u,S}$ covers a fixed $K \in \binom{V}{k}$.

Case 0 ($K = \{u\}\dot{\cup}S$). Here, K is covered by the exceptional pair $\{\{u\}, S\} \in \mathcal{C}_{u,S}$. $\square$

Case 1 ($u \notin K$). Let $\{A, B\} \in \mathcal{C}$ cover K. Since $u \notin K$, Definition 4.3.1 gives that $K \cap A = K \cap A_{u,S} \neq \emptyset$ and $K \cap B = K \cap B_{u,S} \neq \emptyset$. Thus, $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}_{u,S}$ covers K. $\square$

Case 2 ($u \in K$: $S \setminus K \neq \emptyset$). Fix $v \in S \setminus K$, and let $\{A, B\} \in \mathcal{C}$ cover $K_{u,v} = (K \setminus \{u\}) \cup \{v\}$. Since $K_{u,v}$ meets S, we have by (4.1.2) that $S \subseteq A$ or $S \subseteq B$, so w.l.o.g. let $S \subseteq A$. Since $u \notin K_{u,v}$, Case 1 implies that $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}_{u,S}$ covers $K_{u,v}$. Since $K \triangle K_{u,v} = \{u, v\} \subseteq A_{u,S}$, we have $K \subseteq A_{u,S}\dot{\cup}B_{u,S}$, where $u \in K \cap A_{u,S} \neq \emptyset$ and $K \cap B_{u,S} = K_{u,v} \cap B_{u,S} \neq \emptyset$. Thus, $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}_{u,S}$ covers K. $\square$

We henceforth assume that $\{u\}\dot{\cup}S \subsetneq K$ is a proper subset. Let $S(u) \in \mathcal{S}$ be the unique bone of $\mathcal{C}$ containing u.

Case 3 ($\{u\}\dot{\cup}S \subsetneq K$: $|S(u) \cap K| \geq 2$). Let $\{A, B\} \in \mathcal{C}$ cover K. Since u and S appear in $A\dot{\cup}B$, we assume that $S \subseteq A$ and $u \in B$, as otherwise $\{A, B\} = \{A_{u,S}, B_{u,S}\} \in \mathcal{C}_{u,S}$ covers K. Now, fix $u \neq w \in S(u) \cap K$ so that $w \in B$ (cf. (4.1.2)). Then

$$A_{u,S}\dot{\cup}B_{u,S} = (A \cup \{u\}) \,\dot{\cup}\, (B \setminus \{u\}) = A\dot{\cup}B \supseteq K, \tag{5.1.6}$$

where $\{u\}, S \subseteq K \cap A_{u,S} \neq \emptyset$ and $w \in K \cap B_{u,S} \neq \emptyset$. Thus, $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}_{u,S}$ covers K. $\square$

We henceforth assume $\{u\}\dot{\cup}S \subsetneq K$ and $S(u) \cap K = \{u\}$. Let $S(u), S, S_1, \ldots, S_\ell \in \mathcal{S}$ be the bones of $\mathcal{C}$ meeting K, where $\ell \geq 1$ holds by our new assumption.

Case 4 ($\{u\}\dot{\cup}S \subsetneq K$: $(S_1\dot{\cup} \ldots \dot{\cup}S_\ell) \setminus K \neq \emptyset$). Fix $i \in [\ell]$ with $S_i \setminus K \neq \emptyset$, and let $x \in S_i \cap K$, $x' \in S_i \setminus K$, and $\{A, B\} \in \mathcal{C}$ cover $K_{u,x'} = (K \setminus \{u\}) \cup \{x'\}$. By Case 1,

$\{A_{u,S}, B_{u,S}\} \in \mathcal{C}_{u,S}$ covers $K_{u,x'}$, where since $S \subset K_{u,x'}$, we take $S \subseteq A \subset A_{u,S}$. Since $u \in A_{u,S}$, we have $K \subseteq K_{u,x'} \cup \{u\} \subseteq A_{u,S} \dot{\cup} B_{u,S}$, where $\{u\}, S \subseteq K \cap A_{u,S} \neq \emptyset$. Also, $x' \in A_{u,S} \dot{\cup} B_{u,S}$, and we consider the two possibilities: If $x' \in A_{u,S}$, then $\{u, x'\} \subseteq A_{u,S}$ and so $K \cap B_{u,S} = K_{u,x'} \cap B_{u,S} \neq \emptyset$. If $x' \in B_{u,S}$, then $x' \in B$ so that $x \in B$ (cf. (4.1.2)), in which case $x \in K \cap B_{u,S} \neq \emptyset$. Either way, $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}_{u,S}$ covers K. □

Case 5 ($K = \{u\} \dot{\cup} S \dot{\cup} S_1 \dot{\cup} \ldots \dot{\cup} S_\ell$). Let $\{A, B\} \in \mathcal{C}$ cover K, where (as in Case 3) we assume that $S \subseteq A$ and $u \in B$. Then (5.1.6) holds, and if some $j \in [\ell]$ has $S_j \subset B$, then $\{u\}, S \subseteq K \cap A_{u,S} \neq \emptyset$, $S_j \subseteq K \cap B_{u,S} \neq \emptyset$, and $\{A_{u,S}, B_{u,S}\} \in \mathcal{C}_{u,S}$ covers K. Suppose every $\{A, B\} \in \mathcal{C}$ covering K satisfies $K \cap A = \{u\}$ or $K \cap B = \{u\}$. Then $K \setminus \{u\}$ has size $k-1$ and is singular in $\mathcal{C}$. Since $\ell \geq 1$, it follows that $K \setminus \{u\}$ is not a bone, so Definition 5.0.5 says that $K \setminus \{u\}$ is a limb in $\mathcal{C}$. In particular, $K \setminus \{u\}$ is a limb which contains S but which does not contain u, contradicting (5.1.5).

■

5.2 Extremal Lemma

Lemma 5.2.1 (Extremal Lemma)**.** *Fix integers $n \geq k \geq 2$, and let integers p, q, r, R be given by (2.0.1), where $r \geq 1$. Let V be an n-element vertex set, and let $\mathcal{C}$ be a strong cover of $\binom{V}{k}$. Every bone $S \in \mathcal{S}$ with subaverage degree $\deg_{\mathcal{C}}(S) < \alpha = \alpha(\mathcal{C})$ has maximum size $|S| = k-1$.*

Proof. Let n, k, p, q, r, R, and V be given as in the hypothesis of Lemma 5.2.1, where $r \geq 1$. Let $\mathcal{C}$ be a strong cover of $\binom{V}{k}$ (cf. Definition 4.1.3). Assume, on the contrary, that some bone $S_0 \in \mathcal{S}$ satisfies

$$\deg_{\mathcal{C}}(S_0) < \alpha \qquad \text{and} \qquad |S_0| < k-1, \tag{5.2.7}$$

where $\alpha = \alpha(\mathcal{C})$ is the average degree in $\mathcal{C}$. By Lemma 4.2.2 (Pre-Extremal Lemma) and Fact 3.2.1, we can sharpen (5.2.7) and say

$$\deg_{\mathcal{C}}(S_0) \le p \qquad \text{and} \qquad \frac{k-1}{2} < |S_0| < k-1. \tag{5.2.8}$$

Thus, by Fact 5.0.7, we have that S_0 is contained in at most one limb $L \in \mathcal{L}$ of $\mathcal{C}$. We consider two very similar cases.

Case 1 **(**S_0 *belongs to no limbs* $L \in \mathcal{L}$**).** Since $\alpha > p$ (cf. Fact 3.2.1 and $r \ge 1$), let $u \in V$ satisfy $\deg_{\mathcal{C}}(u) \ge p+1$. By the hypothesis of Case 1, every limb $L \in \mathcal{L}$ containing S_0 also contains u (cf. (5.1.5)), and so by Lemma 5.1.1 (Shifting Lemma I), $\mathcal{C}_{u,S_0}$ covers $\binom{V}{k}$ with weight

$$\begin{aligned} \omega(\mathcal{C}_{u,S_0}) &= \omega(\mathcal{C}) + 1 + |S_0| + \deg_{\mathcal{C}}(S_0) - \deg_{\mathcal{C}}(u) \\ &\overset{(5.2.8)}{\le} \omega(\mathcal{C}) + |S_0| = h(n,k) + |S_0|, \end{aligned} \tag{5.2.9}$$

where we used that $\mathcal{C}$ is optimal (because $\mathcal{C}$ is strong). However, we assumed in (5.2.8) that $|S_0| < k-1$, and therefore $\binom{\{u\}\cup S_0}{k} = \emptyset$. As such, Fact 5.0.4 gives that $\mathcal{C}^*_{u,S_0}$ covers $\binom{V}{k}$ with weight

$$\omega(\mathcal{C}^*_{u,S_0}) = \omega(\mathcal{C}_{u,S_0}) - 1 - |S_0| \overset{(5.2.9)}{\le} h(n,k) - 1,$$

which is impossible. □

Case 2 **(**S_0 belongs to precisely one limb $L_0 \in \mathcal{L}$**).** Let $L_0 \in \mathcal{L}$ be the unique limb containing S_0. It suffices to show that there exists a vertex $u \in L_0 \setminus S_0$ with $\deg_{\mathcal{C}}(u) \ge p+1$. (Indeed, $L_0 \in \mathcal{L}$ is the only limb to contain S_0, where $u \in L_0$, and so we could apply Lemma 5.1.1 (Shifting Lemma I) to u and S_0 identically as in Case 1.) For that, recall from Remark 5.0.6 that L_0 is a union of at least two bones $S \in \mathcal{S}$, one of which is S_0. Let $S_1 \in \mathcal{S}$ be any bone satisfying $S_1 \subseteq L_0 \setminus S_0$. If $\deg_{\mathcal{C}}(S_1) < \alpha$, then Lemma 4.2.2 (Pre-Extremal Lemma) would say $|S_1| > (k-1)/2$, which is impossible on account that $|S_0 \dot{\cup} S_1| \leq |L_0| = k - 1$, where already $|S_0| > (k-1)/2$. Thus, $\deg_{\mathcal{C}}(S_1) \geq \alpha > p$, and we may choose any element $u \in S_1 \subseteq L_0 \setminus S_0$.

■

6 Shifting Lemmas II & III

In this chapter we present and prove *Shifting Lemmas II* (cf. Section 6.1) and *III* (cf. Section 6.2), which will be used in the base case and inductive step of Theorem 1.3.9, respectively. These statements can be proven directly from Definition 4.3.1, but we infer Shifting Lemma II from Lemma 5.1.1 (Shifting Lemma I), and we infer Shifting Lemma III from Shifting Lemma II.

6.1 Shifting Lemma II

Corollary 6.1.1 (Shifting Lemma II)**.** *Let V, $\mathcal{C}$, and $\mathcal{S}$ be given as in Definition 4.1.2. Fix $S \in \mathcal{S}$ of size $|S| = k-1$, and let $U \subset V \setminus S$ have size $1 \leq |U| \leq k-1$. Then, $\mathcal{C}_{U,S}$ covers $\binom{V}{k}$.*

Proof. Let V, $\mathcal{C}$, $\mathcal{S}$, $S \in \mathcal{S}$, and $U \subset V \setminus S$ be given as in the hypothesis of Corollary 6.1.1. To prove that $\mathcal{C}_{U,S}$ covers $\binom{V}{k}$, we proceed by induction on $|U|$, where the base case is immediate from Lemma 5.1.1 (Shifting Lemma I). For the inductive step, fix any $u \in U$, and set $U' = U \setminus \{u\}$. Let $\mathcal{C}' = \mathcal{C}_{U',S}$ be the S-shift of U' in $\mathcal{C}$, which by induction covers $\binom{V}{k}$. Let $\mathcal{C}'_{u,S}$ be the S-shift of u in $\mathcal{C}'$, which by Lemma 5.1.1 (Shifting Lemma I) covers $\binom{V}{k}$. We claim that

$$\mathcal{C}'_{u,S} \setminus \mathcal{C}_{U,S} = \Big\{ \big\{ \{u\}, S \big\}, \big\{ U', \{u\} \cup S \big\} \Big\}, \tag{6.1.1}$$

which would complete our induction. Indeed, each fixed $K \in \binom{V}{k}$ is covered by some element of $\mathcal{C}'_{u,S}$, which for sake of argument we assume belongs to $\mathcal{C}'_{u,S} \setminus \mathcal{C}_{U,S}$. If

$\{\{u\}, S\}$ covers K, then so does $\{U, S\} \in \mathcal{C}_{U,S}$ (because $u \in U$), and if $\{U', S \cup \{u\}\}$ covers K, then so does $\{U, S\} \in \mathcal{C}_{U,S}$ (because K meets S on account of $|U| \leq k-1$).

To see (6.1.1), consider $\{A, B\} \in \mathcal{C}$ and the corresponding elements (cf. Definition 4.3.1)

$$\{A', B'\} \stackrel{\text{def}}{=} \{A_{U',S}, B_{U',S}\} \in \mathcal{C}', \qquad \{A'_{u,S}, B'_{u,S}\} \in \mathcal{C}'_{u,S}, \qquad \text{and} \qquad \{A_{U,S}, B_{U,S}\} \in \mathcal{C}_{U,S},$$

each of which is a non-exceptional pair of its respective family. We observe that $A'_{u,S} = A_{U,S}$ (and similarly $B'_{u,S} = B_{U,S}$) since Definition 4.3.1 gives either

$$A'_{u,S} = A' \cup \{u\} = (A \cup U') \cup \{u\} = A \cup U = A_{U,S},$$

or

$$A'_{u,S} = A' \setminus \{u\} = (A \setminus U') \setminus \{u\} = A \setminus U = A_{U,S}. \qquad (6.1.2)$$

Thus, $\mathcal{C}'_{u,S} \setminus \mathcal{C}_{U,S}$ consists of elements in $\mathcal{C}'_{u,S}$ which in some way arise from exceptional pairs. The exceptional pair $\{U', S\} \in \mathcal{C}'$ bears the element $\{U', S \cup \{u\}\} \in \mathcal{C}'_{u,S}$, and this element can't appear in $\mathcal{C}_{U,S}$. The exceptional pair $\{\{u\}, S\} \in \mathcal{C}'_{u,S}$ also can't appear in $\mathcal{C}_{U,S}$. By (6.1.2), these are the only two elements of $\mathcal{C}'_{u,S} \setminus \mathcal{C}_{U,S}$, which proves (6.1.1), and hence Corollary 6.1.1.

■

6.2 Shifting Lemma III

For Shifting Lemma III, we extend Definition 4.3.1 (shifting) for when U (written here as W) is *disjoint* from V.

Definition 6.2.1. (immersion) Let V, $\mathcal{C}$, and $\mathcal{S}$ be given as in Definition 4.1.2. Fix $S \in \mathcal{S}$, and let W be a set which is disjoint from V. For $\{A, B\} \in \mathcal{C}$, define

$$A^{W,S} = \begin{cases} A \cup W & \text{if } S \subseteq A, \\ A & \text{if } S \cap A = \emptyset, \end{cases} \qquad \text{and} \qquad B^{W,S} = \begin{cases} B \cup W & \text{if } S \subseteq B, \\ B & \text{if } S \cap B = \emptyset. \end{cases}$$

Define $\mathcal{C}^{W,S} = \big\{\{W,S\}\big\} \cup \big\{\{A^{W,S}, B^{W,S}\} : \{A,B\} \in \mathcal{C}\big\}$, which we call the *$S$-immersion of W into $\mathcal{C}$*.

Corollary 6.2.2 (Shifting Lemma III)**.** *Let V, $\mathcal{C}$, and $\mathcal{S}$ be given as in Definition 4.1.2. Fix $S \in \mathcal{S}$ of size $|S| = k-1$, and let W be a set of size $|W| \leq k-1$ which is disjoint from V. Then $\mathcal{C}^{W,S}$ covers $\binom{V \cup W}{k}$ with weight $\omega(\mathcal{C}^{W,S}) = \omega(\mathcal{C}) + |W|(1 + \deg_{\mathcal{C}}(S)) + |S|$.*

Proof. Let V, $\mathcal{C}$, $\mathcal{S}$, $S \in \mathcal{S}$, and W be given as in Corollary 6.2.2, where $W \cap V = \emptyset$ and $1 \leq |W| \leq k-1 = |S|$. We construct $\mathcal{C}^{W,S}$ (the S-immersion of W) indirectly via Definition 4.3.1 (shifting). For that, set $X = W \cup V$ and $\mathcal{C}^X = \{\{W,V\}\} \cup \mathcal{C}$. Then $\mathcal{C}^X$ covers $\binom{X}{k}$ by construction, and $\mathcal{C}^X_{W,S}$ covers $\binom{X}{k}$ by Corollary 6.1.1 (Shifting Lemma II). We claim that

$$\mathcal{C}^X_{W,S} = \{\{\emptyset, X\}\} \cup \mathcal{C}^{W,S}, \tag{6.2.3}$$

which would imply that $\mathcal{C}^{W,S}$ covers $\binom{X}{k}$. Indeed, $\mathcal{C}^X_{W,S}$ covers $\binom{X}{k}$ while $\{\emptyset, X\} \in \mathcal{C}^X_{W,S}$ covers nothing.

To see (6.2.3), we have, by Definition 4.3.1,

$$\mathcal{C}^X_{W,S} = \{\{W,S\}\} \cup \{\{W_{W,S}, V_{W,S}\}\} \cup \{\{A_{W,S}, B_{W,S}\} : \{A,B\} \in \mathcal{C}\}.$$

As $S \cap W = V \cap W = \emptyset$ and $S \subseteq V$, we have, again by Definition 4.3.1, $W_{W,S} = W \setminus W = \emptyset$ and $V_{W,S} = V \cup W = X$. But for each $\{A,B\} \in \mathcal{C}$, we have $A_{W,S} = A^{W,S}$ (and similarly $B_{W,S} = B^{W,S}$) since $A \cap W = V \cap W = \emptyset$ and so Definitions 4.3.1 and 6.2.1 agree that either

$$A_{W,S} = A \cup W = A^{W,S} \qquad \text{or} \qquad A_{W,S} = A \setminus W = A = A^{W,S}.$$

Thus,

$$\mathcal{C}^X_{W,S} = \{\{W,S\}\} \cup \{\{W_{W,S}, V_{W,S}\}\} \cup \{\{A_{W,S}, B_{W,S}\} : \{A,B\} \in \mathcal{C}\}$$

$$= \{\{W, S\}\} \cup \{\{\emptyset, X\}\} \cup \{\{A^{W,S}, B^{W,S}\} : \{A, B\} \in \mathcal{C}\}$$

$$= \{\{\emptyset, X\}\} \cup \mathcal{C}^{W,S}$$

It remains to determine $\omega(\mathcal{C}^{W,S})$. For that, we apply Fact 5.0.4 to $\mathcal{C}^{X}_{W,S}$ to infer

$$\omega(\mathcal{C}^{X}_{W,S}) = \omega(\mathcal{C}^{X}) + |W| + |S| + \sum_{w \in W} \left(\deg_{\mathcal{C}^X}(S) - \deg_{\mathcal{C}^X}(w)\right). \tag{6.2.4}$$

By construction, we have

$$\omega(\mathcal{C}^{X}) = \omega(\mathcal{C}) + |W| + |V|, \qquad \deg_{\mathcal{C}^X}(S) = \deg_{\mathcal{C}}(S) + 1, \qquad \text{and} \qquad \deg_{\mathcal{C}^X}(w) = 1 \tag{6.2.5}$$

for each $w \in W$, where $|W| + |V| = |X|$. Applying (6.2.5) to (6.2.4), we infer

$$\omega(\mathcal{C}^{X}_{W,S}) = \omega(\mathcal{C}) + |X| + |S| + |W| \left(1 + \deg_{\mathcal{C}}(S)\right), \tag{6.2.6}$$

and applying (6.2.3) to (6.2.6), we infer

$$\omega(\mathcal{C}^{W,S}) + |X| = \omega(\mathcal{C}^{X}_{W,S}) = \omega(\mathcal{C}) + |X| + |S| + |W| \left(1 + \deg_{\mathcal{C}}(S)\right),$$

which implies the desired formula for $\omega(\mathcal{C}^{W,S})$, and concludes the proof of Corollary 6.2.2.

■

7 Proof of the Main Result and the Degree–Sequence Theorem When $r \neq 0$

In this chapter we prove Theorem 1.3.9 which, with the upper bound provided by Proposition 1.3.7, verifies Theorem 1.3.3 (Main Result) when $r \neq 0$. In the context of the proof of Theorem 1.3.9, we also prove the latter conclusion of Theorem 1.3.3, which is that Theorem 1.3.4 (Degree–Sequence) holds when $r \neq 0$.

Proof of Theorem 1.3.9. Let integers k, n, p, q, r, and R be given as in (2.0.1), where $r \geq 1$, and fix an n-element vertex set V. Recall from (1.3.3) that to prove Theorem 1.3.9 we will induct on the parameter $2^p - R \geq 1$.

7.1 Base Case: $2^p - R = 1$

Since all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ have weight $\omega(\mathcal{C}) = h(n,k)$, they also have common average degree $\alpha = \alpha(n,k) = (1/n)h(n,k)$ (cf. (3.1.1)), which by Fact 3.2.1 satisfies $p \leq \alpha \leq p+1$. Also by Fact 3.2.1, Theorem 1.3.9 holds when $\alpha = p+1$, so we shall prove that $\alpha = p+1$ must hold when $r \geq 1$ and $2^p - R = 1$, and in the following strong form.

Proposition 7.1.1. *Let k, n, p, q, r, R, and V be given as above, where $r \geq 1$ and $R = 2^p - 1$. Then, all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ are $(p+1)$-regular, i.e., $\deg_{\mathcal{C}}(v) = p+1$ for all $v \in V$.*

Proposition 7.1.1 implies all conclusions of Theorem 1.3.9 when $R = 2^p - 1$. Indeed, Proposition 7.1.1 ensures that $\alpha = p+1$, which by Fact 3.2.1 ensures that

$h(n,k) = np + 2R(k-1) + r + k - 1$, which is the former conclusion of Theorem 1.3.9 (when $R = 2^p - 1$). Moreover, we observe that Proposition 7.1.1 also implies that Theorem 1.3.4 holds when $R = 2^p - 1$ and $r \geq 1$, which is the latter conclusion of Theorem 1.3.9. Indeed, Proposition 7.1.1 guarantees that all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ are $(p+1)$-regular, where we recall from (1.3.2) that the number of $(p+1)$-digits of $\mathbf{d}_0 \in \{p, p+1\}^V$ when $R = 2^p - 1$ (and $r \neq 0$) is precisely

$$
\begin{aligned}
& 2R(k-1) + r + k - 1 \\
&= (2R+1)(k-1) + r \\
&= (2^p + R)(k-1) + r \\
&\overset{(2.0.1)}{=} n - r + r = n.
\end{aligned}
$$

Thus, we proceed with the proof of Proposition 7.1.1.

Proof of Proposition 7.1.1. Assume, to the contrary, that there exist optimal covers $\mathcal{C}$ of $\binom{V}{k}$ which are not $(p+1)$-regular. Observe that we may restrict our attention to strong covers $\mathcal{C}$ of $\binom{V}{k}$ (cf. Definition 4.1.3). Indeed, if $\mathcal{C}$ is an optimal cover of $\binom{V}{k}$ which is not $(p+1)$-regular, then the strong cover $\hat{\mathcal{C}}$ of $\binom{V}{k}$ guaranteed by Lemma 4.4.1 is optimal and also not $(p+1)$-regular, because $\mathcal{C}$ and $\hat{\mathcal{C}}$ are degree-equivalent (cf. Remark 4.4.2). Thus, going forward in our proof,

$$\textit{we assume that there exist strong covers } \mathcal{C} \textit{ of } \binom{V}{k} \textit{ which are not } (p+1)\textit{-regular.} \tag{7.1.1}$$

Below in (7.1.4), we choose a particular such strong cover $\mathcal{C}^{\diamond}$ with which to derive the promised contradiction, but for this we require several preparations.

First, Fact 3.2.1 ensures that an optimal cover $\mathcal{C}$ has (the common) average degree $p < \alpha = \alpha(\mathcal{C}) \leq p+1$, where $\alpha = p$ is forbidden by $r \geq 1$. (It is also forbidden by $R = 2^p - 1$.) Second, for an optimal cover $\mathcal{C}$ of $\binom{V}{k}$, define the sets

$$V_-(\mathcal{C}) = \{v : \deg_{\mathcal{C}}(v) \le p\},$$
$$V_0(\mathcal{C}) = \{v : \deg_{\mathcal{C}}(v) = p+1\},$$
$$V_+(\mathcal{C}) = \{v : \deg_{\mathcal{C}}(v) \ge p+2\}. \tag{7.1.2}$$

Third, observe that

when $\mathcal{C}$ is strong, every bone $S \in \mathcal{S}(\mathcal{C})$ of $\mathcal{C}$ with $S \subseteq V_-(\mathcal{C})$ has size $|S| = k-1$. (7.1.3)

Indeed, when $S \in \mathcal{S}(\mathcal{C})$ has $S \subseteq V_-(\mathcal{C})$, then $\deg_{\mathcal{C}}(S) \le p < \alpha = \alpha(\mathcal{C})$ holds in a strong cover $\mathcal{C}$ satisfying $r \ge 1$, and so Lemma 5.2.1 (Extremal Lemma) ensures $|S| = k-1$. Finally,

we choose $\mathcal{C} = \mathcal{C}^\diamond$ to minimize $|V_+(\mathcal{C})|$ among all strong covers $\mathcal{C}$ of $\binom{V}{k}$ which are not $(p+1)$-regular (cf. (7.1.1)). (7.1.4)

We proceed with the following claim.

Claim 7.1.2. *The strong cover $\mathcal{C}^\diamond$ chosen in (7.1.4) satisfies $|V_+(\mathcal{C}^\diamond)| \le k-1$.*

Proof of Claim 7.1.2. Assume, on the contrary, that $|V_+(\mathcal{C}^\diamond)| \ge k$. Now, fix any subset $U \subseteq V_+(\mathcal{C}^\diamond)$ of size $|U| = k-1$, and fix $v_0 \in V_+(\mathcal{C}^\diamond) \setminus U$ to be any additional vertex, where both U and v_0 are ensured by our assumption that $|V_+(\mathcal{C}^\diamond)| \ge k$. Then $V_-(\mathcal{C}^\diamond) \ne \emptyset$ is not possible on account of $\alpha = \alpha(\mathcal{C}^\diamond) \le p+1$, so let $S \in \mathcal{S}(\mathcal{C}^\diamond)$ be any bone of $\mathcal{C}^\diamond$ satisfying $S \subseteq V_-(\mathcal{C}^\diamond) \ne \emptyset$. Then (7.1.3) gives $|S| = k-1$, and so Corollary 6.1.1 (Shifting Lemma II) says that the family $\mathcal{C}^\diamond_{U,S}$ covers $\binom{V}{k}$ with weight (cf. Fact 5.0.4)

$$\omega(\mathcal{C}^\diamond_{U,S}) = \omega(\mathcal{C}^\diamond) + |U| + |S| + \sum_{u \in U} \left(\deg_{\mathcal{C}^\diamond}(S) - \deg_{\mathcal{C}^\diamond}(u)\right)$$
$$\le h(n,k) + |U| + |S| - 2|U| = h(n,k),$$

where we used $\deg_{\mathcal{C}^\diamond}(S) \leq p$ (from $S \subseteq V_-(\mathcal{C}^\diamond)$), $\deg_{\mathcal{C}^\diamond}(u) \geq p+2$ for each $u \in U \subseteq V_+(\mathcal{C}^\diamond)$, and $|U| = |S| = k-1$. Then $\mathcal{C}^\diamond_{U,S}$ is an optimal cover of $\binom{V}{k}$, where by Definition 4.3.1,

$$\deg_{\mathcal{C}^\diamond_{U,S}}(U) = \deg_{\mathcal{C}^\diamond_{U,S}}(S) = 1 + \deg_{\mathcal{C}^\diamond}(S) \overset{(7.1.2)}{\leq} p+1,$$

while

$$\deg_{\mathcal{C}^\diamond_{U,S}}(v) = \deg_{\mathcal{C}^\diamond}(v) \tag{7.1.5}$$

holds for each $v \in V \setminus (U \dot{\cup} S)$. (In particular, $\deg_{\mathcal{C}^\diamond_{U,S}}(v_0) = \deg_{\mathcal{C}^\diamond}(v_0) \geq p+2$ holds for the fixed vertex $v_0 \in V_+(\mathcal{C}^\diamond) \setminus U$, which will be important in a moment.)

Thus,

$$\begin{aligned} & V_+(\mathcal{C}^\diamond_{U,S}) = V_+(\mathcal{C}^\diamond) \setminus U \\ \Longrightarrow \ & |V_+(\mathcal{C}^\diamond_{U,S})| = |V_+(\mathcal{C}^\diamond)| - (k-1) < |V_+(\mathcal{C}^\diamond)|. \end{aligned} \tag{7.1.6}$$

To the optimal cover $\mathcal{C}^\diamond_{U,S}$ of $\binom{V}{k}$, we apply Lemma 4.4.1 (the Survival Lemma) to obtain the strong cover $\hat{\mathcal{C}}^\diamond_{U,S}$ of $\binom{V}{k}$. Remark 4.4.2 says that $\hat{\mathcal{C}}^\diamond_{U,S}$ and $\mathcal{C}^\diamond_{U,S}$ are degree-equivalent, and so

$$\begin{aligned} & V_+(\hat{\mathcal{C}}^\diamond_{U,S}) = V_+(\mathcal{C}^\diamond_{U,S}) \\ \Longrightarrow \ & |V_+(\hat{\mathcal{C}}^\diamond_{U,S})| = |V_+(\mathcal{C}^\diamond_{U,S})| \overset{(7.1.6)}{<} |V_+(\mathcal{C}^\diamond)|. \end{aligned} \tag{7.1.7}$$

Thus, $\hat{\mathcal{C}}^\diamond_{U,S}$ is a strong cover of $\binom{V}{k}$ satisfying (7.1.7), which we now observe is not $(p+1)$-regular (which would contradict our choice of $\mathcal{C}^\diamond$ in (7.1.4)). Indeed, the fixed vertex $v_0 \in V_+(\mathcal{C}^\diamond) \setminus U$ satisfies

$$\begin{aligned} \deg_{\hat{\mathcal{C}}^\diamond_{U,S}}(v_0) &= \deg_{\mathcal{C}^\diamond_{U,S}}(v_0) \\ &= \deg_{\mathcal{C}^\diamond}(v_0) \geq p+2 \end{aligned}$$

by Remark 4.4.2 and (7.1.5), while

$$\deg_{\hat{\mathcal{C}}^\diamond_{U,S}}\left(U \dot{\cup} S\right) = \deg_{\mathcal{C}^\diamond_{U,S}}\left(U \dot{\cup} S\right)$$

$$= 1 + \deg_{\mathcal{C}^\diamond}(S) \leq p + 1$$

holds by Remark 4.4.2 and (7.1.5). Thus, strict inequality in (7.1.7) contradicts our choice of $\mathcal{C}^\diamond$ in (7.1.4).

■

For the remainder of the proof, we consider no further possible alterations to the strong cover $\mathcal{C}^\diamond$ of $\binom{V}{k}$ chosen in (7.1.4), so we relax the notation and say let $\mathcal{C}$ denote a strong cover such that $|V_+(\mathcal{C})| \leq k-1$ and where $\mathcal{C}$ is not $(p+1)$-regular. Also, we relax the notation in (7.1.2) to $V_- = V_-(\mathcal{C})$, $V_0 = V_0(\mathcal{C})$, and $V_+ = V_+(\mathcal{C})$ (and we continue to write $\mathcal{S} = \mathcal{S}(\mathcal{C})$ and $\mathcal{Z} = \mathcal{Z}(\mathcal{C})$, as usual). Since each of V_-, V_0, and V_+ is defined in terms of $\mathcal{C}$-degrees, each of these sets is a union of bones $S \in \mathcal{S}$. Analogously to (7.1.2), define

$$\mathcal{S}_- = \{S \in \mathcal{S} : \ S \subseteq V_-\},$$

$$\mathcal{S}_0 = \{S \in \mathcal{S} : \ S \subseteq V_0\},$$

$$\mathcal{S}_+ = \{S \in \mathcal{S} : \ S \subseteq V_+\}, \tag{7.1.8}$$

where we claim the following inequality.

Claim 7.1.3. $|S_0| \geq 2^{p+1} - |\mathcal{S}_-| - 1$.

Proof of Claim 7.1.3. Indeed, (7.1.3) gives $|V_-| = (k-1) \cdot |\mathcal{S}_-|$, and since every bone $S \in \mathcal{S}$ has size $|S| \leq k-1$, we similarly have $|V_0| \leq (k-1) \cdot |S_0|$. As such, Claim 7.1.2 provides

$$(k-1) \cdot |S_0| \geq |V_0|$$

$$= n - |V_-| - |V_+|$$

$$\geq n - |V_-| - (k-1)$$

$$= n - (k-1)|\mathcal{S}_-| - (k-1)$$

$$> n - r - (k-1)|\mathcal{S}_-| - (k-1),$$

where the strict inequality holds from our hypothesis that $r \geq 1$. Thus, with $R = 2^p - 1$ in (2.0.1), the inequality above gives

$$|S_0| > \frac{n-r}{k-1} - |\mathcal{S}_-| - 1$$

$$\stackrel{(2.0.1)}{=} 2^p + R - |\mathcal{S}_-| - 1$$

$$= 2^{p+1} - |\mathcal{S}_-| - 2,$$

from which Claim 7.1.3 follows.

■

We now conclude the proof of Proposition 7.1.1. Since $\mathcal{C}$ is a strong cover of $\binom{V}{k}$, its surviving family $\mathcal{Z}$ consists of the skeleton $\mathcal{S}$, together with possibly the empty set (cf. Definition 4.1.3). Now, consider the random surviving set $Z = Z_\psi \in \mathcal{Z}$ obtained by selecting $\psi \in \{a,b\}^{\mathcal{C}}$ uniformly at random. Then

$$1 = \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}]$$

$$= \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_-] + \mathbb{P}[Z \in \mathcal{S}_0] + \mathbb{P}[Z \in \mathcal{S}_+]$$

$$= \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_+] + \Big(\sum_{S \in \mathcal{S}_-} \mathbb{P}[Z = S] \Big) + \sum_{S \in \mathcal{S}_0} \mathbb{P}[Z = S]. \tag{7.1.9}$$

For each bone $S \in \mathcal{S} = \mathcal{Z}^*$, we have from Lemma 4.2.1 that $\mathbb{P}[Z = S] = 1/2^{\deg_{\mathcal{C}}(S)}$, and so we infer from (7.1.2), (7.1.8), (7.1.9), and Claim 7.1.3, that

$$1 \geq \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_+] + |\mathcal{S}_-| \left(\frac{1}{2}\right)^p + |\mathcal{S}_0| \left(\frac{1}{2}\right)^{p+1}$$

$$\geq \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_+] + |\mathcal{S}_-| \left(\frac{1}{2}\right)^p + \left(2^{p+1} - |S_-| - 1\right) \left(\frac{1}{2}\right)^{p+1}$$

$$= \mathbb{P}[Z = \emptyset] + \mathbb{P}[Z \in \mathcal{S}_+] + 1 + \frac{1}{2^{p+1}} \left(|\mathcal{S}_-| - 1\right). \tag{7.1.10}$$

Consequently, $\mathbb{P}[Z = \emptyset] = \mathbb{P}[Z \in \mathcal{S}_+] = 0$ and $|\mathcal{S}_-| \leq 1$. Then $\mathcal{S}_+ = \emptyset$, and our hypothesis in (7.1.4) that $\mathcal{C}$ is not $(p+1)$-regular implies $|\mathcal{S}_-| = 1$. As such, $\mathcal{S}_-$

consists of a single $(k-1)$-tuple (cf. (7.1.3)) of vertices of (common) degree at most p, and all remaining $n-(k-1)$ vertices have degree precisely $p+1$. As such,

$$h(n,k) = \omega(\mathcal{C})$$

$$\leq (k-1)p + (n-(k-1))(p+1)$$

$$= n(p+1) - (k-1)$$

$$= np + 2R(k-1) + r,$$

because $n(p+1) = np + 2R(k-1) + r + k - 1$ when $R = 2^p - 1$ (cf. (2.0.1)). Since $r \geq 1$, the bound $h(n,k) \leq np + 2R(k-1) + r$ contradicts the bound $h(n,k) \geq np + 2R(k-1) + 2r$ of Theorem 1.3.8 (Weak Lower Bound).

■

7.2 Inductive Step: $2^p - R > 1$

The proof of Theorem 1.3.9 will follow from the recurrence

$$h(n,k) \geq h(n+k-1,k) - (k-1)(2+p), \tag{7.2.11}$$

which (in a moment) we show holds when $0 \leq R < 2^p - 1$. Indeed, (2.0.1) gives $n = q(k-1) + r$, where $1 \leq r < k-1$, $q = 2^p + R$, and $0 \leq R < 2^p - 1$. Thus, $n+k-1 = (q+1)(k-1)+r$ has the same modulus r, and $q+1 = 2^p+(R+1)$ has the same exponent p, but with remainder $1 \leq R+1 \leq 2^p - 1$. Since $1 \leq 2^p - (R+1) < 2^p - R$, induction gives

$$h(n,k) \overset{(7.2.11)}{\geq} (n+k-1)p + 2(R+1)(k-1) + r + k - 1 - (k-1)(2+p)$$

$$= np + 2R(k-1) + r + k - 1,$$

as desired. Thus, it remains to prove (7.2.11).

Proof of (7.2.11). Let $\mathcal{C}$ be a strong cover of $\binom{V}{k}$ (guaranteed by Lemma 4.4.1 (Survival Lemma)), and let $\mathcal{C}$ have average degree $\alpha = \alpha(\mathcal{C})$. Since $R < 2^p - 1$, Fact 3.2.1 gives $\alpha < p + 1$. Thus, some bone $S \in \mathcal{S}$ satisfies $\deg_{\mathcal{C}}(S) \leq p$, where Lemma 5.2.1 (Extremal Lemma) ensures that $|S| = k - 1$. Let W be a set of $|W| = k - 1$ new vertices, i.e., $W \cap V = \emptyset$, and let $\mathcal{C}^{W,S}$ be the S-immersion of W into $\mathcal{C}$. On the one hand, Corollary 6.2.2 (Shifting Lemma III) ensures that $\mathcal{C}^{W,S}$ covers $\binom{V \cup W}{k}$ with weight

$$\begin{aligned} \omega(\mathcal{C}^{W,S}) &= \omega(\mathcal{C}) + (k-1)(2 + \deg_{\mathcal{C}}(S)) \\ &\leq h(n,k) + (k-1)(2+p), \end{aligned} \tag{7.2.12}$$

On the other hand, by definition, $\omega(\mathcal{C}^{W,S}) \geq h(n+k-1,k)$, so that (7.2.11) follows. ■

We now conclude the proof of the latter conclusion of Theorem 1.3.9, which is that Theorem 1.3.4 (Degree–Sequence) holds when $r \neq 0$. We continue with the considerations above, where $\mathcal{C}$ is a strong cover of $\binom{V}{k}$, $S \in \mathcal{S} = \mathcal{S}(\mathcal{C})$ is a $(k-1)$-bone of $\mathcal{C}$ with degree $\deg_{\mathcal{C}}(S) \leq p$, and $\mathcal{C}^{W,S}$ is the $\mathcal{S}$-immersion of a set of $k-1$ new vertices W into the cover $\mathcal{C}$. In (7.2.12), we observed that

$$\begin{aligned} h(n+k-1,k) &\leq \omega(\mathcal{C}^{W,S}) \\ &= \omega(\mathcal{C}) + (k-1)(2 + \deg_{\mathcal{C}}(S)) \\ &\leq h(n,k) + (k-1)(2+p), \end{aligned} \tag{7.2.13}$$

where $n + k - 1$ has the same modular remainder r, the same exponent p, but with remainder $1 \leq R+1 \leq 2^p - 1$ (with respect to base 2 expansion). Since Theorem 1.3.3 (Main Result) has now been proven in full, we apply it both sides of (7.2.13) to obtain

$$\begin{aligned} (n+k-1)p + 2(R+1)(k-1) + r + k - 1 &\leq \omega(\mathcal{C}^{W,S}) \\ &\leq np + 2R(k-1) + r + k - 1 + (k-1)(2+p), \end{aligned} \tag{7.2.14}$$

and so equality holds throughout (7.2.13)–(7.2.14). As such, we infer that $\mathcal{C}^{W,S}$ is an optimal cover of $\binom{V\cup W}{k}$, and that it was necessarily the case that $\deg_{\mathcal{C}}(S) = p$. Since $\mathcal{C}^{W,S}$ is optimal with $2^p - (R+1) < 2^p - R$, we infer from induction that its degree sequence $\mathbf{d}(\mathcal{C}^{W,S})$ is the unique element of $\{p, p+1\}^{V\cup W}$ which has precisely

$$2(R+1)(k-1) + r + k - 1$$

$$= 2R(k-1) + r + k - 1 + 2(k-1)$$

many $(p+1)$-digits (cf. (1.3.2)). We now compare the sequences $\mathbf{d}(\mathcal{C}^{W,S})$ and $\mathbf{d}(\mathcal{C})$, which by Definition 6.2.1 differ only on the $|W \cup S| = 2(k-1)$ many coordinates corresponding to $W \cup S$. First, note that each $(W\cup S)$-coordinate of $\mathcal{C}^{W,S}$ is a $(p+1)$-digit, since we observed that $\deg_{\mathcal{C}}(S) = p$, where Definition 6.2.1 gives $\deg_{\mathcal{C}^{W,S}}(W \cup S) = 1 + \deg_{\mathcal{C}}(S)$. Second, the $|W| = (k-1)$ many W-coordinates of $\mathbf{d}(\mathcal{C}^{W,S})$ don't appear in $\mathbf{d}(\mathcal{C})$ at all. Third, the $|S| = (k-1)$ many S-coordinates of $\mathbf{d}(\mathcal{C}^{W,S})$ do appear in $\mathbf{d}(\mathcal{C})$, but as p-digits (as noted above). Thus, $\mathbf{d}(\mathcal{C})$ consists of precisely

$$2R(k-1) + r + k - 1 + 2(k-1) - 2(k-1)$$

$$= 2R(k-1) + r + k - 1$$

many $(p+1)$-digits, and all remaining coordinates are p-digits. In other words, $\mathbf{d}(\mathcal{C}) = \mathbf{d}_0$ (cf. (1.3.2)), as desired.

∎

8 Some Results on $h_d(n,k)$ *

In this chapter we prove Theorems 1.3.5 and 1.3.6, which address d-covers and, in particular, $h_d(n,k)$. Since d-covers have not been discussed since the introduction, we reintroduce their terminology: Fix integers $n \geq k \geq d \geq 2$, fix an n-element vertex set V, and let $\binom{V}{k}$ denote the set of all k-element subsets of V, as usual. For disjoint subsets $A_1, \ldots, A_d \subseteq V$, we say that $\{A_1, \ldots, A_d\}$ *covers* an element $K \in \binom{V}{k}$ if $K \subseteq A_1 \dot\cup \cdots \dot\cup A_d$ where $K \cap A_i \neq \emptyset$ for every $i \in [d]$. We say that a collection $\mathcal{D}$ of such d-element families is a *d-cover* of $\binom{V}{k}$ if every $K \in \binom{V}{k}$ is covered by at least one $\{A_1, \ldots, A_d\} \in \mathcal{D}$. We write $\mathcal{D} = \{D_1, \ldots, D_t\}$ for the members of $\mathcal{D}$, and for each $D_i \in \mathcal{D}$ we write $D_i = \{V_{i,1}, \ldots, V_{i,d}\}$ and $V_i = V(D_i) = V_{i,1} \dot\cup \cdots \dot\cup V_{i,d}$. Like for covers, we denote $\omega(\mathcal{D}) = \sum_{D \in \mathcal{D}} |V(D)|$ as the *weight* of a d-cover and we denote $h_d(n,k)$ as the minimum weight $\omega(\mathcal{D})$ over all d-covers of $\binom{V}{k}$.

8.1 A Lower Bound on $h_d(n,k)$

To prove Theorem 1.3.5, we evoke most of the tools of Theorem 1.3.8 (Weak Lower Bound), in particular, the probabilistic tools.

Proof of Theorem 1.3.5. Fix $2 \leq d \leq k \leq n$, fix an n-element vertex set V, and let $\mathcal{D} = \{D_1, \ldots, D_t\}$ be a d-covering of $\binom{V}{k}$. We prove

$$\sum_{i=1}^{t} |V_i| \geq n \log_{d/(d-1)} \left(\frac{n}{k-1} \right).$$

*Sections of this chapter are taken from [3], which has been submitted to "Congressus Numerantium", 2016.

Consider the following random subset $Z \subseteq [n]$: select $\boldsymbol{j} = (j_1, \ldots, j_t) \in [d]^t$ uniformly at random; for each $i \in [t]$, set $Z_i = V \setminus V_{i,j_i}$; set $Z = \bigcap_{i=1}^t Z_i$. Observe that $|Z| \leq k-1$; indeed, if there exists a k-tuple $K \in \binom{Z}{k}$, then there exists $i \in [t]$ so that $K \subseteq V_i$ and K meets each of $V_{i,1}, \ldots, V_{i,d}$. On the other hand, $K \subseteq Z \subseteq Z_i = V \setminus V_{i,j_i}$, so that $K \cap V_{i,j_i} = \emptyset$, a contradiction. Therefore,

$$\mathbb{E}[|Z|] \leq k - 1. \tag{8.1.1}$$

We next develop an exact expression for $\mathbb{E}[|Z|]$ (see (8.1.4) below).

Fix $v \in V$, and set $X_v = 1$ if $v \in Z = \bigcap_{i=1}^t Z_i$, and $X_v = 0$ otherwise. Note that, since $\boldsymbol{j} \in [d]^t$ is selected uniformly at random, the events $v \in Z_i$, over $i \in [t]$, are independent. Then $|Z| = \sum_{v \in V} X_v$, and by linearity of expectation,

$$\begin{aligned}
\mathbb{E}[|Z|] &= \sum_{v \in V} \mathbb{E}[X_v] \\
&= \sum_{v \in V} \mathbb{P}\left[v \in \bigcap_{i=1}^t Z_i\right] \\
&= \sum_{v \in [n]} \prod_{i=1}^t \mathbb{P}\left[v \in Z_i\right] \\
&= \sum_{v \in V} \prod_{i=1}^t \mathbb{P}\left[v \notin V_{i,j_i}\right].
\end{aligned} \tag{8.1.2}$$

For fixed $(v, i) \in V \times [t]$, observe that

$$\mathbb{P}\left[v \notin V_{i,j_i}\right] = \begin{cases} 1 & \text{if } v \notin V_i, \\ (d-1)/d & \text{if } v \in V_i. \end{cases} \tag{8.1.3}$$

Indeed, to avoid triviality, let $v \in V_i$, and let $j_v \in [d]$ be the unique index for which $v \in V_{i,j_v}$. Then, $\mathbb{P}[v \notin V_{i,j_i}] = \mathbb{P}[j_i \neq j_v] = 1 - \mathbb{P}[j_i = j_v] = 1 - (1/d)$, as promised in (8.1.3).

Returning to (8.1.2), define auxiliary family $\mathcal{F} = \{V_1, \ldots, V_t\}$, so that a fixed $v \in V$ belongs to precisely $\deg_{\mathcal{F}}(v)$ many elements $V_i \in \mathcal{F}$. As such,

$$\prod_{i=1}^{t} \mathbb{P}\left[v \notin V_{i,j_i}\right] \overset{(8.1.3)}{=} \left(\frac{d-1}{d}\right)^{\deg_{\mathcal{F}}(v)}$$

$$\overset{(8.1.2)}{\Longrightarrow} \mathbb{E}[|Z|] = \sum_{v \in [n]} \left(\frac{d-1}{d}\right)^{\deg_{\mathcal{F}}(v)}. \tag{8.1.4}$$

Comparing (8.1.1) and (8.1.4), and using the Arithmetic-Geometric mean inequality, we infer

$$\frac{k-1}{n} \geq \frac{1}{n}\mathbb{E}[|Z|]$$

$$= \frac{1}{n} \sum_{v \in [n]} \left(\frac{d-1}{d}\right)^{\deg_{\mathcal{F}}(v)}$$

$$\geq \left[\prod_{v \in [n]} \left(\frac{d-1}{d}\right)^{\deg_{\mathcal{F}}(v)}\right]^{1/n}$$

$$= \left(\frac{d-1}{d}\right)^{(1/n)\sum_{v \in [n]} \deg_{\mathcal{F}}(v)}.$$

By standard double-counting, we have $\sum_{v \in [n]} \deg_{\mathcal{F}}(v) = \sum_{i=1}^{t} |V_i|$, from which it follows that

$$\sum_{i=1}^{t} |V_i| \geq n \log_{d/(d-1)} \left(\frac{n}{k-1}\right)$$

.

■

8.2 On Upper Bounds on $h_d(n,k)$

For $d = 2$, we have computed $h_2(n,k) = h(n,k)$ precisely in this dissertation. However, when $d > 2$ much is unknown about $h_d(n,k)$. In fact, even for $d = 3$ much is unknown, including a concrete example of a non-trival 3-covering like as in Proposition 1.3.7. And of course we are far from a concrete example of a d-covering for general $d \geq 3$. In other words, for all $d \geq 3$ we lack a *constructive upper bound* on $h_d(n,k)$.

However, for a certain range of $2 \leq d \leq k \leq n$, the proof of Theorem 1.3.6 shows that we can probabilistically guarantee the existence of some d-covers whose weight we can bound from above. And from these d-covers we can compute an asymptotic for $h_d(n,k)$ which holds for that certain range of $2 \leq d \leq k \leq n$. What is most noteworthy is that this asymptotic mirrors the lower bound value of Theorem 1.3.5. Thus, for a certain range of $2 \leq d \leq k \leq n$, the Hansel number $h_d(n,k)$ is virtually $n \log_{d/(d-1)} n/(k-1)$. Although this is verified only for a certain range of $2 \leq d \leq k \leq n$, it nonetheless shows that the lower bound of Theorem 1.3.5 is asymptotically sharp. So in determining a lower bound on $h_d(n,k)$ that holds for all $2 \leq d \leq k \leq n$, one cannot greatly improve the bound of Theorem 1.3.5. But the authors wonder if the bound of Theorem 1.3.5 can be improved for $d \approx k$.

Proof of Theorem 1.3.6. Let n, $k = k(n) = O(\sqrt{\log\log n})$, and $2 \leq d = d(k) = O(k/\log^2 k)$ be given as in Theorem 1.3.6. Fix an n-element vertex set V. To bound $h_d(n,k)$, we use a standard random construction to produce a d-covering $\mathcal{D}$ of $\binom{V}{k}$ for which $\sum_{D \in \mathcal{D}} |V(D)|$ is not too large. To that end, define auxiliary positive integer parameter

$$m = \left\lceil -\frac{(k-1)\log(n/k)}{\log\left(d(1-(1/d))^k\right)} \right\rceil, \tag{8.2.5}$$

where for simplicity in calculations, we will ignore the ceilings. For a function $\phi : V \to [d]^m$, we write $\mathbf{v} = \phi(v) = (\mathbf{v}(1), \ldots, \mathbf{v}(m))$. For a fixed $K = \{v_1, \ldots, v_k\} \in \binom{V}{k}$, we say that K is *ϕ-separated* if, for some $i \in [m]$, we have $\{\mathbf{v}_1(i), \ldots, \mathbf{v}_k(i)\} = [d]$. Moreover, we define $X_{\phi,K}$ to be the indicator variable for when K is not ϕ-separated, and we set $X_\phi = \sum_{K \in \binom{[n]}{k}} X_{\phi,K}$.

Select $\varphi : V \to [d]^m$ uniformly at random. We will observe that for each $K \in \binom{V}{k}$,

$$\mathbb{E}[X_{\varphi,K}] = \mathbb{P}[K \text{ is not } \varphi\text{-separated}] \leq (k/n)^{k-1},$$

in which case

$$\begin{aligned}
\mathbb{E}[X_\varphi] &= \sum_{K \in \binom{V}{k}} \mathbb{E}[X_{\varphi,K}] \\
&\leq \binom{n}{k} \left(\frac{k}{n}\right)^{k-1} \\
&\leq \left(\frac{en}{k}\right)^k \left(\frac{k}{n}\right)^{k-1} \\
&= e^k \frac{n}{k}. \qquad (8.2.6)
\end{aligned}$$

Indeed, there are at least $d^k - d(d-1)^k$ many surjections $K \overset{\text{onto}}{\to} [d]$, and so

$$\begin{aligned}
\mathbb{E}[X_{\varphi,K}] &\leq \frac{\left(d^k - (d^k - d(d-1)^k)\right)^m \times d^{m(n-k)}}{d^{mn}} \\
&= \left[d\left(1 - \frac{1}{d}\right)^k\right]^m \\
&= \exp\left\{m \log\left[d\left(1 - \frac{1}{d}\right)^k\right]\right\} \\
&\overset{(8.2.5)}{=} \exp\left\{-(k-1)\log(n/k)\right\} \\
&= (k/n)^{k-1}.
\end{aligned}$$

We now define the promised family $\mathcal{D}$. Fix any $\phi : V \to [d]^m$ for which $X_\phi \leq \mathbb{E}[X_\varphi]$. For each $(i,j) \in [m] \times [d]$, set $V_{i,j} = \{v \in V : \mathbf{v}(i) = j\}$, and set $D_i = \{V_{i,1}, \ldots, V_{i,d}\}$. For each $K = \{v_1, \ldots, v_k\} \in \binom{V}{k}$, define

$D_K = \{\{v_1\}, \ldots, \{v_{d-1}\}, \{v_d, \ldots, v_k\}\}$. Define

$$\mathcal{D} = \mathcal{D}_\phi = \{D_1, \ldots, D_m\} \cup \{D_K : K \in \binom{[n]}{k} \text{ is not } \phi\text{-separated}\}.$$

By construction, $\mathcal{D}$ is a d-covering of $\binom{V}{k}$, which satisfies

$$\sum_{D \in \mathcal{D}} |V(D)| = kX_\phi + \sum_{i=1}^{m} |V(D_i)|$$

$$\leq k\mathbb{E}[X_\varphi] + mn$$

$$\overset{(8.2.6)}{\leq} mn + e^k n = mn\left(1 + \frac{e^k}{m}\right).$$

We claim that

$$m = \left(1 + O\left(\frac{d \log d}{k}\right)\right) \log_{d/(d-1)}\left(\frac{n}{k-1}\right)$$

and

$$\frac{e^k}{m} = O\left(\frac{1}{k}\right), \tag{8.2.7}$$

which, if true, immediately implies Theorem 1.3.6. To see (8.2.7), note first that the denominator $-\log(d(1-(1/d))^k)$ in (8.2.5) equals

$$k \log\left(\frac{d}{d-1}\right)\left(1 + \frac{\log d}{k \log(1-(1/d))}\right)$$

$$= k \log\left(\frac{d}{d-1}\right)\left(1 - \Theta\left(\frac{d \log d}{k}\right)\right), \tag{8.2.8}$$

where we used that $\log(1+x) \approx x$ for $x \approx 0$. Since $d(\log d)/k = o(1)$ holds by hypothesis,

$$m \overset{(8.2.5)}{=} \frac{(k-1)\log(n/k)}{-\log(d(1-(1/d))^k)}$$

$$\leq \frac{k\log(n/(k-1))}{k\log(d/(d-1))(1-\Theta(d(\log d)/k))}$$

satisfies (8.2.7), using $(1-x)^{-1} \leq 1+2x$ (on $[0,1/2]$). Moreover, since (8.2.8) is

$$(1-o(1))k\log\left(\frac{d}{d-1}\right),$$

where $k = O(\sqrt{\log\log n})$ diverges, we have

$$m \geq (1-o(1))\log_{d/(d-1)} n$$

$$\geq (1-o(1))\log n,$$

while $ke^k = O(e^{k^2}) = O(\log n) = O(m)$. Thus, e^k/m satisfies (8.2.7).

$\blacksquare$

9 Conclusion and Future Work

Recall that Theorems 1.3.3 (Main Result) and 1.3.4 (Degree–Sequence) generalized Theorem 1.3.2 of Bollobás and Scott [1] by providing an exact formula for $h(n, k)$, together with a characterization of all optimal covers $\mathcal{C}$. Our proofs employed the elegant techniques of Bollobás and Scott [1] for proving Theorem 1.3.8 (Weak Lower Bound), which verified the lower bound of Theorem 1.3.3 when $k - 1$ divides n. However, the probabilistic tools of Theorem 1.3.8 do not seem sufficient for providing the desired conclusions when $k - 1$ does not divide n. For that, we needed several structural results, namely the Survival Lemma, the Extremal Lemma, and the Shifting Lemmas. Although these auxiliary results are structural, the proofs of some employed probabilistic considerations similar to those in Theorem 1.3.8. And in fact, even the base case of Theorem 1.3.9 employed probabilistic considerations. Thus, this dissertation provides yet another example of the familiar theme in combinatorics when structural conclusions arise from probabilistic considerations. Moreover, the details between the structural and probabilistic elements of our proofs were fairly non-trivial. Thus, we pose the following problem, which could be of general interest.

Problem 9.0.1. *Find alternative, and in fact simpler, proofs of Theorems 1.3.3 and 1.3.4. In particular, find proofs which are purely deterministic.*

Recall that Theorem 1.3.4 (Degree–Sequence) showed that all optimal covers $\mathcal{C}$ of $\binom{V}{k}$ share the unique degree sequence $\mathbf{d}(\mathcal{C}) = \mathbf{d}_0$, defined in (1.3.2). Since optimal covers need not be unique (recall Remark 2.0.10), it may be interesting to know what other structural properties all optimal covers must share.

Problem 9.0.2. *For a finite set V, determine structural properties common to all optimal covers $\mathcal{C}$ of $\binom{V}{k}$.*

In Theorems 1.3.5 and 1.3.6, we addressed bounds and partial (multivariate) asymptotics on the parameter $h_d(n, k)$. We believe it would be of interest to sharpen these bounds, and expand them for a wider range of $2 \leq d \leq k \leq n$. In particular, we would prefer our multivariate asymptotics on $h_d(n, k)$ to be single-variable when k is constant in the variable n.

Problem 9.0.3. *Let $d \leq k = \Theta(1)$ be a pair of bounded functions of the integer variable n. Determine such pairs of functions which admit asymptotic evaluations of $h_d(n, k)$ (now in the single variable n).*

Note that Theorem 1.3.3 solves Problem 9.0.3 when $d = 2$, and it does so with an exact evaluation of $h_2(n, k)$ for *any* fixed integer $k \geq 2$. Perhaps there is hope that something could be said for $d = 3$, at least if we take k large enough (but not diverging in n).

Problem 9.0.4. *For $d = 3$, determine the asymptotics of $h_3(n, k)$, for any large enough range of $k = \Theta(1)$.*

Our proof of Theorem 1.3.6 was non-constructive (even for $d = 3$), while our proof of Proposition 1.3.7 was purely constructive (for $d = 2$). Perhaps it would be interesting to know what competitive upper bounds could be deterministically established for $h_d(n, k)$, again when $k = k(d)$ is a function of d alone. In fact, perhaps this problem is already interesting when $d = 3$.

Problem 9.0.5. *Give a competitive constructive upper bound on $h_3(n, k)$, for any large enough range of $k = \Theta(1)$.*

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Appendix

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