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Resonant Solutions to (3+1)-dimensional Bilinear Differential Equations

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by

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Abstract

In this thesis, we attempt to obtain a class of generalized bilinear differential equations in (3+1)-dimensions by D_p-operators with $p = 5$, which have resonant solutions. We construct resonant solutions by using the linear superposition principle and parameterizations of wave numbers and frequencies. We test different values of p in Maple computations, and generate three classes of generalized bilinear differential equations and their resonant solutions when $p = 5$.

Chapter 1
Introduction

1.1 The discovery and evolution of solitons

In the history of the development of natural sciences, the field of cross subject can always produce unexpected surprises. Soliton theory is one of the important research directions in nonlinear sciences, which perfectly combines applied mathematics with mathematical physics. The soliton phenomenon was first observed in nature, being one of the nonlinear phenomena that can be generated in the laboratory. Solitons are also called solitary waves, which are generated by a large class of nonlinear partial differential equations with special wave properties.

British scientist, J. Scott Russell, first discovered the soliton phenomenon. In 1844, Russell described a strange wave phenomenon he observed in 1834 in an article entitled "Report on Waves" [1]. He was observing the motion of a ship, which was quickly pulled up by two horses along the narrow river. When the boat stopped suddenly, the water pushing around in the river did not stop, but instead, it gathered alongside the front of the boat and changed dynamically. Soon, a large, circular and well-defined solitary wave began to form, and quickly left the bow to move forward. The water peaked along the river to continue its voyage, and Russell continued riding to catch up and track it. The waves travelled at a speed of approximately 8 to 9 miles per hour, while maintaining its original shape of about 30 feet long, and 1 to 1.5 feet high. While traveling, the shape and speed of the wave had no obvious changes, but the height gradually decreased. After tracking one to two miles, finally, the solitary wave disappeared in the meandering river. Russell became aware that the nature of the waves was critical to the newly discovered phenomenon. Following, he conducted a detailed investigation on waves only to conclude similar results to the prior phenomenon. He believed that this type of wave was a stable solution of fluid motion, and called it a solitary wave. Unfortunately, although Russell was able to carry out this experiment, he was never able to explain the existence of solitary waves in theory.

In the next few years, Airy, Boussinesq and Rayleigh conducted further studies to try to understand this kind of phenomenon. Airy concluded that the solitary wave mentioned by Russel does not exist [2]. However, Boussinesq and Rayleigh proved the existence of the solitary wave from the angle of mathematics [3]. In order to approximate description of solitary waves, Boussinesq proposed a one-dimensional nonlinear evolution equation, which was later named the Boussinesq equation.

It was not until sixty years later in 1895, did Holland mathematicians Korteweg and de Vries, study the wave motion of shallow waters. Under the assumption of approximation for small amplitude wave, the shallow water wave equation was established:

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2})}{\partial x},$$

where α is a small but arbitrary constant, and $\sigma = \frac{1}{3}l^3 - \frac{Tl}{\rho g}$ depends on the depth l of the liquid, the capillary tension T at its surface and its density ρ. This is the famous Korteweg-de Vries equation, or the KdV equation [4]. They made a relatively complete analysis on the solitary wave phenomenon and obtained the solitary wave solution from the equation, which is the same as that described by Russell. With this, the existence of solitary wave solutions was proved in theory. The KdV equation can be regarded as a typical example of using the solution of nonlinear evolution equations to explain the objective phenomena. However, due to nonlinear interactions, questions such as whether the wave is stable or not, and if the two wave collision involves any interference, remain unanswered. These problems led to the stop of studying the KdV equation and solitary waves.

In the 1950s, physicists Fermi, Pasta and Ulam studied the problem of heat conduction on the simple chain model [5]. They found a phenomenon that was difficult to explain by classical physics. With the passage of time, the energy does not uniformly distribute as expected, but eventually returned to the state of the initial distribution. It was also found that the use of solitary wave processing problem was necessary, which caused people to focus on the solitary wave again.

Till to 1965, the famous American physicists Kruskal and Zabusky, using computer by numerical simulation, observed and analyzed the interaction of the nonlinear solitary wave collision in plasma [6]. It was proven that after the collision of the two KdV solitary waves, each one maintains the original waveform and the wave velocity continues to propagate forward. Since the solitary waves have the same properties as the particle collisions, they are collectively named soliton. The result of this research is generally accepted by people because it not only reveals the nature of the solitary waves, but also is an important milestone in the history of the development of the soliton theory.

1.2 Methods for solving soliton equations

The soliton phenomenon is not only an important discovery in the field of physics, but also plays an important role in promoting the development of some mathematical methods. In soliton theory, it is a basic and important topic to find out exact solutions to nonlinear equations and to study properties of the solutions. In addition, the research contents and methods are very abundant. Finding exact solutions to soliton equations will not only help ones understand the essential properties of soliton equations, particularly the algebraic structures, but can also explain the natural phenomena in applications. However, it is not easy to get exact solutions of nonlinear equations directly. Ones have utilized various methods to transform and dissect the equations so that they may become easy to be solved. As researchers continue to pay more attention and research in the field of solitons, methods for solving soliton equations have become diversified, which include the Inverse scattering transformation (IST), Darboux transformation, Bäcklund transformation, Hirota's bilinear method, the homogeneous balance method, the Wronskian and Pfaffian techniques, the method of the symmetrical analysis, and the hyperbolic function method. With the emergence of various methods of solving, the soliton equations that were once difficult to solve in the past are now solvable with much more ease, and a kind of new solutions constantly being discovered and applied has an important physical significance. Those methods promoted an endless stream of momentum. The following is a brief overview of the main methods for solving nonlinear soliton equations.

1.2.1 Inverse scattering transformation (IST)

In 1967, since Gardner, Greene, Kruskal and Miura (further denoted as 'GGKM') found that the initial value problem of the KdV equation can be solved by using the inverse scattering theory of Schrödinger. The method of solving nonlinear partial differential equations has been developed rapidly in recent years [7]. Now it is successfully used to solve many nonlinear equations, which are very important in applications. It is undoubtedly a major discovery of mathematical physics.

Since the Fourier transformation and inverse transformation are used in the inverse scattering transformation (IST) to solve nonlinear partial differential equations, the IST is also called the Nonlinear Fourier transformation. Using the relation between the eigenvalue problem of the Schrödinger equation in quantum mechanics and its inverse problem GGKM showed that the solution of the initial value problem of the KdV equation can be simplified to solving three linear equations, and then sucessfully obtained its N-soliton solutions. The essence of this method is to transform nonlinear partial differential equations into several linear problems. This method is based on the function transformation and has its strict physical background and mathematical rigor. It can be used to find multi soliton solutions of the whole spectrum evolution equations, which are associated with the same spectral problem. The successful application of the IST has aroused ones' interest in the study of integrable systems that have been forgotten for many years, and since ones have obtained many important results.

1.2.2 Bäcklund transformation and Darboux transformation

In 1883, Sweden geometer Bäcklund, in the study of complex constant surfaces, used the Sine-Gordon equation, $u_{\xi\eta} = \sin u$, to derive two solutions of the following relationship between u_1 and u_2:

$$u'_{2\xi} = u_{1\xi} - 2\beta \sin(\frac{u_1 + u_2}{2}),$$

$$u'_{1\eta} = -u_{2\eta} + \frac{2}{\beta} \sin(\frac{u_1 - u_2}{2}).$$

These two formulas are called the Bäcklund transformation of the Sine-Gordon equation [8]. A Bäcklund transformation is an explicit direct method of solutions. It establishes the relationship between the solution of a nonlinear partial differential equation and the other known linear partial differential equation. On the other hand, it could also establish a link between two solutions to a nonlinear partial differential equation, which can be used to derive a new solution from a known solution. The Bäcklund transformation can also introduce a simple nonlinear superposition formula. By using this formula, we can construct multi-soliton solutions form a single soliton solution. We can also construct multi soliton solutions by a single soliton solution through a purely algebraic method. Therefore, the Bäcklund transformation has become a powerful tool for the study of nonlinear partial differential equations.

The Darboux transformation is also an effective method for solving nonlinear partial differential equations, which is similar to the Bäcklund transformation. In 1882, Darboux studied the eigenvalue problem of a one-dimensional Schrdinger equation:

$$\phi_{xx} + u(x,t)\phi = \lambda\phi, \tag{1.1}$$

Darboux found: if u and ϕ are two functions which satisfy the equation (1.1), for an arbitrary constant λ_0, and let $f(x) = \phi(x, \lambda_0)$, which means f is a solution of (1.1) when $\lambda = \lambda_0$, then the functions $\bar{u}$ and $\bar{\phi}$

$$\bar{u} = u + 2(\ln f)_{xx}, \bar{\phi}(x,\lambda) = \phi_x(x,\lambda) - (\partial_x \ln f)\phi(x,\lambda), f \neq 0, \tag{1.2}$$

must satisfy (1.1). (1.2) is called the original Darboux transformation [9]. The basic idea of Darboux transformation is that from using one solution of a nonlinear equation and its Lax pair solution, one uses an algebraic algorithm and differential operation to obtain a new solution of the nonlinear equation and the corresponding new Lax pair solution. In 1975, Wadati extended the Darboux transformation for the mKdV and Sine-Gordon equations [10]. In 1986, Chinese Academy of Sciences academician, Chaohao Gu, expressed the Darboux transformation in matrix form, illustrating that the Darboux transformation is actually a gauge transformation of spectral parameters and gave the associated Bäcklund transformation [11]. The Darboux transformation is applied to the surface theory and harmonic mapping in differential geometry. Now, the Darboux transformation has become an important research topic in soliton theory.

1.2.3 The homogeneous balance method

In 1995, Mingliang Wang proposed the homogeneous balance method and used to solve many equations [12]. In 1998, Engui Fan gave a full development of this method, which was used to obtain Bäcklund transformations, similarity reductions and more forms of exact solutions [13]. The homogeneous balance method is a very important method for solving nonlinear partial differential equations, which transforms the problem of solving nonlinear evolution equations into pure algebraic operations. Using this method not only can the Bäcklund transformation equations be obtained, but also new solutions of nonlinear partial differential equations. The general steps of the method are as follows:

For a given nonlinear partial differential equation

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0 \tag{1.3}$$

where P is a polynomial in the indicated variables, which contains nonlinear terms and the highest order linear partial derivatives.

The function $w = w(x,t)$ is called a quasi-solution of the equation (1.3), if there exists a univariate function $f = f(w)$ such that a solution u of (1.3) can be presented as follows:

$$u(x,t) = \frac{\partial^{m+n} f(w)}{\partial x^m \partial t^n} + v(x,t) \text{ (where } v(x,t) \text{ is lower order partial derivatives of } f(w)). \quad (1.4)$$

The following steps to compute m, n and $f(w)$ will uniquely determine the needed transformation:

First, let the highest power of the partial derivative of the $w = w(x,t)$ contained in the higher order partial derivative term be equal to the highest power of the partial derivative of $w = w(x,t)$ in the nonlinear term to determine the existence of the non-negative integers m, n.

Secondly, gather all the terms of the highest power of the partial derivative of $w = w(x,t)$, so that the coefficient is zero, and the ordinary differential equation that $f(w)$ satisfies is achieved. Solve it to get $f = f(w)$, which generally a logarithmic function.

Thirdly, replace the nonlinear terms of the derivatives of $f(w)$ by the higher order derivative of $f(w)$. Then gather the various derivatives of $f(w)$ and let its coefficient be zero, to obtain the partial differential equation (PDE) group of each homogeneous type of $w = w(x,t)$ and choose the appropriate coefficients of the linear combination in (1.4). Therefore, the PDE group has a solution.

Finally, if the first three steps of the answer is yes, take plus these results into (1.4), and calculate to get an exact solution of (1.3).

It can be seen from (1.4) that if $v(x,t)$ is a solution of the equation (1.3), the Bäcklund transformation of the equation can be obtained by the above steps.

1.3 Hirota's bilinear method

The bilinear method is an effective tool to construct exact solutions of soliton equations. In 1971, Hirota creatively proposed a direct method to obtain soliton solutions nowadays called Hirota's bilinear method [14]. He introduced a new kind of binary differential operators–D-operators. Nonlinear equations are transformed into bilinear differential equations by means of the transformation on the potential u. Although they are still nonlinear equations, due to the very superior bilinear properties, the perturbation expansion can be substituted into bilinear differential equations. Under certain conditions, the expansion can be truncated to finite terms. In the form of linear exponential functions, a single soliton solution, two soliton solutions and three soliton solutions are obtained, and the general expression of the N-soliton solutions is derived. For these general expressions, the mathematical induction method is used to verify its establishment. The bilinear method is a kind of algebraic non-analytical method by using bilinear differential operators. It is only concerned with solving nonlinear equations, and does not depend on the spectral problem of the equations or the Lax pairs. Furthermore, it is suitable for solving a wide range of equations including almost all the equations that can be solved by the inverse scattering transformation. In recent years, this method has been extended to the soliton equation hierarchies and the discrete and semi discrete soliton systems.

In the 1970s, Hirota combined the bilinear method with the Bäcklund transformation and proposed a bilinear Bäcklund transformation [15]. Compared with the traditional Bäcklund transformation, the bilinear Bäcklund transformation considered the linear problem of nonlinear evolution equations [16]. The linear problem is transformed into bilinear equations by an appropriate transformation for the potential function and the eigenfunction, and then multiple soliton solutions of the original equation are obtained.

Another direct method is the Wronskian technique, which is a widely used and efficient method due to the good properties of the Wronskian determinant itself [17]. Based on the Darboux transformation, we have already obtained Wronskian solutions of soliton equations [18]. Nimmo and Freemon combined the Wronskian representation with the Hirota bilinear forms of soliton equations, established the Wronskian technique and used this skill to obtain a series of development equations and their Bäcklund transformations of the Wronskian solutions [19]. This technique is based on the Hirota bilinear form or bilinear Bäcklund transformation of soliton equations, first choosing an appropriate set of functions ϕ_j to constitute a Wronskian determinant $W(\phi_1, \phi_2, \cdots, \phi_N)$, and then substituting the Wronskian determinant into bilinear equations, verified by the properties of determinants and the Laplace theorem of determinants. The direct verification of the solution and the simplicity of the proof procedure can be described as an advantage of the Wronskian technique.

The Pfaffian technique, as a new kind of algebraic tool, has been fully applied in coupling systems of soliton equations. Some soliton equations have the determinant representation of multi soliton solutions, such as the Kadomtsev-Petviashvili (KP) group equation. However, some other equations, such as the continuous and discrete BKP hierarchies, whose multi soliton solutions cannot be expressed by normal determinants, can be solved by Pfaffians. The Pfaffian technique is closely related to the determinant, and has wider properties than the determinant. The determinant of the Jacobi identity can be regarded as a special case of the Pfaffian identity. In 1989, Hirota first introduced the Pfaffian to represent the soiton solution of the BKP equation. As a result, ones found that the solution of a series of soliton equations can be expressed as Pfaffians [20]. In 1991, Hirota proposed using Pfaffians to replace the Wronskian solution of the KP equation; and at this time, the Wronskian solution, satisfying the Plücker identity, was changed into the Pfaffian identity [21]. The Pfaffian identity consists of four components, one more than the Plücker identity, so we introduce two new variables to satisfy the corresponding Pfaffian identity. This is derived in the case of the KP equation, to present Pfaffian solutions of coupled KP equations, the whole process being known as the Pfaffian technique.

The simple and practical characteristics made Hirota's bilinear method develope rapidly, allowing it to be widely applied and popularized by many scholars in the world. Based on bilinear forms, we can obtain the Wronskian determinant solution, the Pfaffian solution and many other solutions. Hirota's bilinear method is only applicable to a particular class of nonlinear equations, so it is very important and meaningful to continue to find some effective methods for solving nonlinear equations. Recently, Wen-Xiu Ma proposed generalized bilinear operators – D_p-operators, which can be used to establish more bilinear differential equations [22]. In addition, different symbols have been applied to the study of generalized Bell polynomials and even trilinear equations [23]-[24].

In this thesis, we are going to compute resonant solutions to a class of (3+1)-dimensional bilinear differential equations, based on generalized bilinear differential operators – D_p-operators. We will apply the linear superposition principle to the obtained generalized (3+1)-dimensional bilinear differential equations to analyze resonant solutions of exponential function waves. Three illustrative examples will be presented in chapter three by using the weights of dependent variables in the adopted algorithm.

Chapter 2

Bilinear Differential Equations

2.1 Hirota bilinear differential operators and equations

Hirota's bilinear method is an important method in the process of seeking exact solutions to soliton equations. In general, the key point of the application of Hirota's bilinear method is to fully understand the nature of the Hirota bilinear operators and to find the correlation variable transformation, which transforms a nonlinear equation into a Hirota bilinear form. In this section, we are going to give some basic facts about Hirota's bilinear method and resonant solutions to soliton equations.

2.1.1 Hirota D-operators and their properties

Definition 2.1.1 *Assume that $f(x,t)$ and $g(x,t)$ are differentiable functions of x, t, the Hirota bilinear D-operators are defined as*

$$\begin{aligned} D_t^m D_x^n f \cdot g &= (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n f(t,x) g(t',x') \mid_{t'=t, x'=x} \\ &= \partial_{t'}^m \partial_{x'}^n f(x+x', t+t') g(x-x', t-t') \mid_{x'=t'=0}, \end{aligned} \tag{2.1}$$

where m, n are arbitrary nonnegative integers [25].

From the definition, we can work out:

$$
\begin{aligned}
D_x f \cdot g &= f_x g - f g_x, \\
D_x^2 f \cdot g &= f_{xx} g - 2 f_x g_x + f g_{xx}, \\
D_x^3 f \cdot g &= f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx}, \\
D_x^4 f \cdot g &= f_{4x} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} - 4 f_x g_{xxx} + f g_{4x}, \\
D_t D_x f \cdot g &= f_{xt} g - f_t g_x - f_x g_t + f g_{tx}, \\
D_x^6 f \cdot g &= f_{6x} g - 6 f_{5x} g_x + 15 f_{4x} g_{xx} - 20 f_{xxx} g_{xxx} \\
&\quad + 15 f_{xx} g_{4x} - 6 f_x g_{5x} + f g_{6x}, \\
D_x^3 D_t f \cdot g &= f_{xxxt} g - 3 f_{xxt} g_x + 3 f_{xt} g_{xx} - f_t g_{xxx} \\
&\quad - f_{xxx} g_t + 3 f_{xx} g_{xt} - 3 f_x g_{xxt} + f g_{xxxt}.
\end{aligned}
\tag{2.2}
$$

Bilinear operators have many properties, and they play an important role in solving nonlinear partial differential equations. A few common properties are listed as follows:

Proposition 2.1.1 *If $m + n$ is odd, then we have*

$$D_t^m D_x^n f \cdot f = 0. \tag{2.3}$$

Proposition 2.1.2

$$D_t^m D_x^n f \cdot g = (-1)^{m+n} D_t^m D_x^n g \cdot f. \tag{2.4}$$

Proposition 2.1.3

$$D_x f \cdot g = 0 \;\; \textit{if and only if } kf = lg, \;\; \textit{where } k, l \textit{ are constants with } k^2 + l^2 \neq 0. \tag{2.5}$$

Proposition 2.1.4

$$D_t^m D_x^n f \cdot 1 = \partial_t^m \partial_x^n f. \tag{2.6}$$

Proposition 2.1.5 *If $\xi_i = \omega_i t + k_i x + \xi_i^{(0)} (i = 1, 2)$, then we have*

$$D_t^m D_x^n e^{\xi_1} \cdot e^{\xi_2} = (\omega_1 - \omega_2)^m (k_1 - k_2)^n e^{\xi_1 + \xi_2}, \tag{2.7}$$

which tells

$$D_t^m D_x^n e^{\xi_1} \cdot e^{\xi_1} = 0. \tag{2.8}$$

Based on the definition and some properties of the Hirota bilinear operators, and using the appropriate dependent variable transformation, a general integrable equation can be changed into a Hirota bilinear form. The bilinear forms of important nonlinear equations of mathematical physics that have appeared in the literature are summarized as follow:

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2.9}$$

can be expressed as

$$(D_t D_x + D_x^4) f \cdot f = 2f_{xt} f - 2f_x f_t + 2f_{4x} f - 8f_{xxx} f_x + 6f_{xx}^2 = 0, \tag{2.10}$$

under the transformation $u = 2(\ln f)_{xx}$.

The Boussinesq equation

$$u_{tt} + {u^2}_{xx} + u_{4x} = 0 \tag{2.11}$$

can be expressed as

$$(D_t^2 + D_x^4) f \cdot f = 2f_{tt} f - 2f_t^2 + 2f_{4x} f - 8f_{xxx} f_x + 6f_{xx}^2 = 0, \tag{2.12}$$

under the transformation $u = 6(\ln f)_{xx}$.

The KP equation

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0 \tag{2.13}$$

can be expressed as

$$(D_t D_x + D_x^4 + D_y^2) f \cdot f = 2f_{xt} f - 2f_x f_t + 2f_{4x} f - 8f_{xxx} f_x + 6f_{xx}^2 + 2f_{yy} f - 2f_y^2 = 0, \tag{2.14}$$

under the transformation $u = 2(\ln f)_{xx}$.

And the Sawada-Kotera equation

$$u_{5x} - 15uu_{xxx} - 15u_x u_{xx} + 45u^2 u_x + u_t = 0 \tag{2.15}$$

can be expressed as

$$(D_x D_t + D_x^6) f \cdot f = 2f_{xt} f - 2f_x f_t + 2f_{6x} f - 12f_{5x} f + 30f_{4x} f_{xx} - 20f_{xxx}^2, \tag{2.16}$$

under the transformation $u = -(\ln f)_{xx}$.

2.1.2 Resonant solutions to Hirota bilinear equations

So far, many scholars have published some relevant articles on soliton interactions. The interaction of nonlinear structures generally can be divided into two categories: completely elastic collisions and inelastic collisions. When the soliton interaction is perfectly elastic, the soliton collision interaction will maintain the original nature and state. The colliding soliton mutual effect maintains the original speed and shape, and in the process of collision will not cause a phase shift. On the contrary, for inelastic interactions, it can be found that the excited states can be changed in the process of interactions:

1. partially or completely change their shape;
2. change their speed;
3. with or without phase transition;
4. there is a phenomenon of fission, e.g., a soliton fissions into several solitons with the same structure;
5. a fusion phenomenon, e.g., several structurally similar solitons aggregate into a new soliton.

Resonance is a phenomenon of interactions, which was first discovered by Miles [26]. He pointed out that the solution has some coherent structures, and described the diffraction of a soliton in the corner. Miles observed that under certain conditions, when no splits or loss of its original nature occurs, a KP soliton in the corner of the convex is unable to reflect. The structure of this kind of solution provided the solution of "Mach reflection" in water waves. This phenomenon is called resonance.

There have been many research results about resonance. Many well-known soliton equations have been found to have multi soliton solutions of the non-trivial spatial structure and interaction. The resonance phenomena of soliton equations are first presented in the (2+1)-dimensional space. Hirota was the first scientist who theoretically proved the existence of resonance phenomena of soliton solutions of soliton equations in the (1+1)-dimensional space [27]. He used the bilinear method, and taking the Sawada-Kotera equation as an example, pointed out the behavior process of the soliton resonance:

1. two solitons in the vicinity of the resonance point absorb or generate a third soliton;
2. in the process of resonance, the interaction of two solitons is aggregated as a soliton, or a soliton is divided into two solitons;

3. after the collision of two solitons, the two solitons become one soliton.

Since then there have been a lot of (1+1)-dimensional soliton equations that have also been shown to have resonance phenomena, such as the Hirota-Satsuma KdV-SK equation [28], the Boussinesq equation [29] and many others. Of course, the focus on the study of resonance is the (2+1)-dimensional KP equation hierarchy, which is shown by the following researchers. Ohkuma and Wadati clarified the essential properties of soliton resonance of the KP equation [30], and then Medina made a further study of this equation [31]. Isojima studied the parameter space of resonance condition and the "Spider-web" solution of the cKP system [32]-[33]. Pashave proposed the "four virtual" soliton resonance solution of KP-II equations [34], and then Biondini studied these equations from the Wronskian solution of the τ-function [35]. Lee showed that the resonance soliton is a complex of two dissipative solitons under the study of the MKP-II equation [36]. Honghai Hao, through the study of the two-soliton solutions of the NI-KP equation, classified the parameters under the resonance condition [37]. Wen-Xiu Ma also analyzed resonant solutions to generalized bilinear equations and trilinear equations, which are characterized by Bell polynomials [23]-[24]. Resonance phenomena can also occur in (3+1)-dimensions or even higher dimensional systems [38].

2.2 Generalized bilinear differential operators and equations

2.2.1 Generalized bilinear D_p-operators

Based on the Hirota D-operators, Wen-Xiu Ma introduced bilinear D_p-operators, which can be used to create much more generalized bilinear differential equations.

Definition 2.2.1 *Let p be a given natural number. The bilinear D_p-operators are defined as*

$$\begin{aligned}(D_{p,x}^n f\cdot g)(x) &= (\partial_x+\alpha\partial_{x'})^n f(x)g(x')\mid_{x'=x}\\ &= \sum_{i=0}^{n}\binom{n}{i}\alpha^i(\partial_x^{n-i}f)(x)(\partial_x^i g)(x), n\geq 1,\end{aligned} \tag{2.17}$$

where the powers of α are determined by

$$\alpha^i=(-1)^{r(i)},\quad \textit{where } i\equiv r(i) \mod p \tag{2.18}$$

with $0\leq r(i)<p, i\geq 0$.

It is obvious that the case of $p=1$ gives the normal derivatives and the cases of $p=2k, k\in\mathbb{N}$, reduce to the Hirota bilinear operators.

According to the definition of D_p-operators, when $p=3$, the powers of α^i read

$$\alpha^0=1, \alpha=-1, \alpha^2=\alpha^3=1, \alpha^4=-1, \alpha^5=\alpha^6=1, \alpha^7=-1, \alpha^8=1\cdots. \tag{2.19}$$

When $p=5$, we have

$$\alpha^0=1, \alpha=-1, \alpha^2=1, \alpha^3=-1, \alpha^4=\alpha^5=1, \alpha^6=-1, \alpha^7=1, \alpha^8=-1,\cdots. \tag{2.20}$$

When $p=7$, we have

$$\alpha^0=1, \alpha=-1, \alpha^2=1, \alpha^3=-1, \alpha^4=1, \alpha^5=-1, \alpha^6=1=\alpha^7=1, \alpha^8=-1,\cdots. \tag{2.21}$$

and thus we have the expressions:

$$\begin{aligned}
&D_{5,x}f\cdot g=f_xg-fg_x,\\
&D_{5,x}D_{5,t}f\cdot g=f_{xt}g-f_xg_t-f_tg_x+fg_{xt},\\
&D_{5,x}^2f\cdot g=f_{xx}g-2f_xg_x+fg_{xx},\\
&D_{5,x}^2D_{5,t}f\cdot g=f_{xxt}g-f_{xx}g_t-2f_{xt}g_x+2f_xg_{xt}+f_tg_{xx}-fg_{xxt},\\
&D_{5,x}^3D_{5,t}f\cdot g=f_{xxxt}g-f_{xxx}g_t-3f_{xxt}g_x+3f_{xt}g_{xx}\\
&\quad+3f_{xx}g_{xt}-3f_xg_{xxt}-f_tg_{xxx}+fg_{xxxt}.
\end{aligned}\tag{2.22}$$

$$\begin{aligned}
&D_{5,x}^4f\cdot g=f_{4x}g-4f_{xxx}g_x+6f_{xx}g_{xx}-4f_xg_{xxx}+fg_{4x},\\
&D_{5,x}^4D_{5,t}f\cdot g=f_{4x,t}g-f_{4x}g_t+4f_{xxx}g_{xt}-6f_{xx}g_{xxt}\\
&\quad+4f_xg_{xxxt}+f_tg_{4x}-4f_{xt}g_{xxx}+6f_{xxt}g_{xx}-4f_{xxxt}g_x+fg_{4x,t},\\
&D_{5,x}^6f\cdot g=f_{6x}g-6f_{5x}g_x+15f_{4x}g_{xx}-20f_{xxx}g_{xxx}\\
&\quad+15f_{xx}g_{4x}+6f_xg_{5x}-fg_{6x},\\
&D_{5,x}^5D_{5,t}f\cdot g=f_{5x,t}g-5f_{4x,t}g_x+10f_{xxxt}g_{xx}-10f_{xxt}g_{xxx}\\
&\quad+5f_{xt}g_{4x}+f_tg_{5x}+5f_xg_{4x,t}+10f_{xx}g_{xxxt}\\
&\quad-10f_{xxx}g_{xxt}+5f_{4x}g_{xt}-f_{5x}g_t-fg_{5x,t},\\
&D_{5,x}^7f\cdot g=f_{7x}g-7f_{6x}g_x+21f_{5x}g_{xx}-35f_{4x}g_{xxx}\\
&\quad+35f_{xxx}g_{4x}+21f_{xx}g_{5x}-7f_xg_{6x}+fg_{7x}.
\end{aligned}\tag{2.23}$$

$$
\begin{aligned}
&D_{5,x}^4 D_{5,t}^2 f \cdot g = f_{4x,tt} g - 4 f_{xxxtt} g_x + 6 f_{xxtt} g_{xx} - 4 f_{xtt} g_{xxx} + f_{tt} g_{4x} \\
&\quad + 2 f_t g_{4x,t} + 8 f_{xt} g_{xxxt} - 12 f_{xxt} g_{xxt} + 8 f_{xxxt} g_{xt} - 2 f_{4x,t} g_t \\
&\quad + 4 f_x g_{xxxtt} + 6 f_{xx} g_{xxtt} - 4 f_{xxx} g_{xtt} + f_{4x} g_{tt} - f g_{4x,tt}, \\
&D_{5,x}^2 D_{5,t} D_{5,y} f \cdot g = f_{xxty} g - 2 f_{xty} g_x - f_{xxt} g_y + f_{ty} g_{xx} + 2 f_{xt} g_{xy} - f_{xxy} g_t \\
&\quad + f_{xx} g_{ty} + 2 f_{xy} g_{xt} - f_y g_{xxt} - 2 f_x g_{xty} - f_t g_{xxy} + f g_{xxty}, \\
&D_{5,x}^3 f \cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx}, \\
&D_{5,x} D_{5,t} D_{5,y} f \cdot g = f_{xty} g - f_{xt} g_y - f_{ty} g_x - f_{xy} g_t + f_t g_{xy} + f_y g_{xt} + f_x g_{ty} - f g_{xty}.
\end{aligned}
\tag{2.24}
$$

All the expressions above are different from the Hirota bilinear differential expressions. However, there are some special expressions, which are the same as the Hirota bilinear differential expressions when $p = 5$ and $f(x, y, t) = g(x, y, t)$, e.g.,

$$
\begin{aligned}
&D_{5,x}^3 f \cdot f = 0, \\
&D_{5,x}^2 D_{5,t} f \cdot f = 0, \\
&D_{5,x} D_{5,y} D_{5,t} f \cdot f = 0.
\end{aligned}
\tag{2.25}
$$

2.2.2 Generalized bilinear equations

A bilinear differential equation related to a multivariate polynomial

$$
P = P(x_1, x_2, \cdots, x_l) \tag{2.26}
$$

is defined by

$$
P(D_{p,x_1}, D_{p,x_2}, \cdots, D_{p,x_l}) f \cdot f = 0. \tag{2.27}
$$

In particular, when $p = 3$, we can get the generalized bilinear KdV equation [39]

$$
(D_{3,x} D_{3,t} + D_{3,x}^4) f \cdot f = 2 f_{xt} f - 2 f_x f_t + 6 f_{xx}^2 = 0, \tag{2.28}
$$

the generalized bilinear Boussinesq equation [40]

$$
(D_{3,t}^2 + D_{3,x}^4) f \cdot f = 2 f_{tt} f - 2 f_t^2 + 6 f_{xx}^2 = 0, \tag{2.29}
$$

the generalized bilinear KP equation [41]

$$
(D_{3,t} D_{3,x} + D_{3,x}^4 + D_{3,y}^2) f \cdot f = 2 f_{xt} f - 2 f_x f_t + 6 f_{xx}^2 + 2 f_{yy} f - 2 f_y^2 = 0, \tag{2.30}
$$

and the generalized bilinear Sawada-Kotera equation

$$(D_{3,x}D_{3,t} + D_{3,x}^6)f \cdot f = 2f_{xt}f - 2f_xf_t + 2f_{6x}f + 20f_{xxx}^2. \tag{2.31}$$

From these equations, we can see that they are different from the Hirota bilinear differential equations. Therefore, we can obtain more diverse classes of new bilinear differential equations. Since the D_p-operators enable us to obtain much more generalized bilinear differential equations, they also extends our research to nonlinear partial differential equations.

Chapter 3

Resonant Solutions to Generalized Bilinear Differential Equations

3.1 Linear superposition principle

The superposition principle is prevalent in the physical phenomena, so that the mathematical equations of the reaction of physical phenomena should be universally applicable. As we all know, the linear superposition principle is an important part of the theory of linear differential equations. Wen-Xiu Ma applied the linear superposition principle to bilinear equations to obtain their exact solutions [42]-[43].

Let us introduce a multivariate polynomial

$$P = P(x_1, x_2, \cdots, x_l) \tag{3.1}$$

and the corresponding bilinear differential equation will be

$$P(D_{p,x_1}, D_{p,x_2}, \cdots, D_{p,x_l})f \cdot f = 0. \tag{3.2}$$

For a fixed $N \in \mathbb{N}$, define a set of N wave variables

$$\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{l,i}x_l, 1 \leq i \leq N, \tag{3.3}$$

where the $k_{j,i}$'s are constants, and N exponential wave functions can be given as

$$f_i = e^{\eta_i} = e^{k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{l,i}x_l}, 1 \leq i \leq N, \tag{3.4}$$

noting a bilinear identity:

$$\begin{aligned} &P(D_{p,x_1}, D_{p,x_2}, \cdots, D_{p,x_l})e^{\eta_i} \cdot e^{\eta_j} \\ &= P(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j})e^{\eta_i + \eta_j}, \end{aligned} \tag{3.5}$$

where the powers of α follow the rule (2.18). Then we form a linear combination of N exponential waves:

$$\begin{aligned} f &= \epsilon_1 f_1 + \epsilon_2 f_2 + \cdots + \epsilon_N f_N \\ &= \sum_{i=1}^{N} \epsilon_i e^{k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{l,i}x_l}, \end{aligned} \tag{3.6}$$

where the ϵ_i's are arbitrary constants. Substituting (3.6) into (3.2), we can get

$$\begin{aligned} & P(D_{p,x_1}, D_{p,x_2}, \cdots, D_{p,x_l}) f \cdot f \\ = & \sum_{i,j=1}^{N} \epsilon_i \epsilon_j P(D_{p,x_1}, D_{p,x_2}, \cdots, D_{p,x_l}) e^{\eta_i} \cdot e^{\eta_j} \\ = & \sum_{i,j=1}^{N} \epsilon_i \epsilon_j P(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) e^{\eta_i + \eta_j} \\ = & \sum_{i=1}^{N} \epsilon_i^2 P(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) e^{2\eta_i} \\ & + \sum_{1 \le i < j \le N} \epsilon_i \epsilon_j [P(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) \\ & + P(k_{1,j} + \alpha k_{1,i}, \cdots, k_{l,j} + \alpha k_{l,i})] e^{\eta_i + \eta_j}. \end{aligned} \tag{3.7}$$

If the linear combination function f defined by (3.6) with arbitrary constants ϵ_i solves the generalized bilinear differential equation (3.2), all solutions of exponential waves in (3.6) are resonant. Then f is called a resonant solution to the generalized bilinear differential equation (3.2).

From the bilinear identity (3.7), we can obtain that the linear combination function f defined by (3.6) can be a solution of the generalized bilinear differential equation (3.2) if and only if for $1 \le i \le j \le N$, all conditions

$$P(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) + P(k_{1,j} + \alpha k_{1,i}, \cdots, k_{l,j} + \alpha k_{l,i}) = 0 \tag{3.8}$$

are satisfied. Therefore, we can have the following criterion on resonant solutions:

Theorem 3.1.1 *[21] Let $P(x_1, x_2, \cdots, x_l)$ be a multivariate polynomial in the indicated variables and the N wave variables $\eta_i, 1 \le i \le N$, be defined by $\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{l,i}x_l, 1 \le i \le N$, where the $k_{i,j}$'s are all constants. Then any linear combination of the N exponential waves $\eta_i, 1 \le i \le N$, solves the generalized bilinear differential equation $P(D_{p,x_1}, D_{p,x_2}, \cdots, D_{p,x_l}) f \cdot$*

$f = 0$ *if and only if the condition* $P(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) + P(k_{1,j} + \alpha k_{1,i}, \cdots, k_{l,j} + \alpha k_{l,i}) = 0$, $1 \le i \le j \le N$, *are satisfied.*

Now we are going to consider how to construct a multivariate polynomial $P(x_1, x_2, \cdots, x_l)$ such that $P(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) + P(k_{1,j} + \alpha k_{1,i}, \cdots, k_{l,j} + \alpha k_{l,i}) = 0$, $1 \le i \le j \le N$. We can use Theorem 3.1.1 to compute generalized bilinear differential equations defined by (3.2) with linear subspaces of solutions. In order to achieve generalized bilinear differential equations and their resonant solutions, we list the following steps:

Firstly, we are going to apply weights for the independent variables:

$$(w(x_1), w(x_2), \cdots, w(x_l)) = (w_1, w_2, \cdots, w_l), \tag{3.9}$$

where the weights $w(x_j) = w_j$ can be both positive and negative integers.

Secondly, construct a multivariate polynomial $P = P(x_1, x_2, \cdots, x_l)$, which is homogeneous in some weight.

Thirdly, we parameterize $k_{1,i}, k_{2,i}, \cdots, k_{l,i}$, composed of wave numbers and frequencies, by using a parameter k_i:

$$k_{j,i} = b_j k_i^{w_j}, 1 \le j \le l, \tag{3.10}$$

where the b_j's are constants to be determined.

Then we put (3.10) into (3.8), collect all the terms by the powers of the parameters k_i and let the coefficient of each power to be zero. This way, we can get algebraic equations on the constants b_j's and the coefficients of the homogeneous multivariate polynomial P. After solving the resulting algebraic equations, we can obtain the polynomial we want to construct. Thus, the resulting solution of b_j's and the coefficients of the polynomial yield the resonant solution f defined by (3.6) and the generalized bilinear differential equation defined by (3.2). In addition, we can derive the corresponding nonlinear differential equation under the transformation $u = 2(\ln f)_x$.

3.2 Resonant solutions to (3+1)-dimensional bilinear equations

Here we are going to present three illustrative examples in (3+1)-dimensions by applying the linear superposition principle we discussed above.

3.2.1 Examples with positive weights

EXAMPLE 1 Now we apply the weights of the independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 3, 5), \tag{3.11}$$

and take a general homogeneous polynomial in weight 6:

$$P = c_1x^6 + c_2x^4y + c_3x^3z + c_4z^2 + c_5xt + c_6y^3 + c_7xyz. \tag{3.12}$$

Following the steps we mentioned above, we set the wave variables:

$$\eta_i = k_ix + b_1k_i^2y + b_2k_i^3z + b_3k_i^5t, 1 \leq i \leq N, \tag{3.13}$$

where the k_i's are arbitrary constants, but the constants b_1, b_2, b_3 are to be determined. Then we obtain the corresponding generalized bilinear differential equation with $p = 5$:

$$\begin{aligned} &P(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t})f \cdot f \\ &= (c_1D_{5,x}^6 + c_2D_{5,x}^4D_{5,y} + c_3D_{5,x}^3D_{5,z} + c_4D_{5,z}^2 + c_5D_{5,x}D_{5,t} + c_6D_{5,y}^3 + c_7D_{5,x}D_{5,y}D_{5,z})f \cdot f \\ &= 30c_1f_{4x}f_{xx} - 20c_1f_{xxx}^2 + 2c_2f_{4x,y}f + 2c_3f_{xxxz}f - 6c_3f_{xxz}f_x \\ &\quad - 2c_3f_{xxx}f_z + 6c_3f_{xx}f_{xz} + 2c_4f_{zz}f - 2c_4f_z^2 + 2c_5f_{xt}f - 2c_5f_xf_t. \end{aligned} \tag{3.14}$$

In the generalized bilinear differential equation (3.14), we can see that

$$\begin{aligned} c_6D_{5,y}^3f \cdot f = 0, \\ c_7D_{5,x}D_{5,y}D_{5,z}f \cdot f = 0. \end{aligned} \tag{3.15}$$

The corresponding resonant solution of N exponential waves is defined by

$$f = \sum_{i=1}^{N} \epsilon_ie^{\eta_i} = \sum_{i=1}^{N} \epsilon_ie^{k_ix+b_1k_i^2y+b_2k_i^3z+b_3k_i^5t}, \tag{3.16}$$

where the ϵ_i's and k_i's, $1 \leq i \leq N$, are arbitrary constants, but by a Maple computation, the proportional constants b_1, b_2, b_3 have to satisfy

$$\begin{cases} b_1c_2 + 5c_1 = 0, \\ b_2c_3 + 5c_1 = 0, \\ 3b_2c_3 + b_3c_5 = 0, \\ b_2^2c_4 + b_2c_3 + 10c_1 = 0. \end{cases} \tag{3.17}$$

From (3.15), which we can use the software Maple to get, and after calculation, derive the solution to (3.17):

$$\begin{cases} b_1 = \dfrac{c_3^2}{c_4 c_2}, \\ b_2 = \dfrac{c_3}{c_4}, \\ b_3 = -\dfrac{3c_3^2}{c_4 c_5}, \end{cases} \tag{3.18}$$

when the coefficients of the polynomial P satisfy

$$c_1 = -\frac{c_3^2}{5c_4}, \tag{3.19}$$

where c_2, c_3, c_4, c_5 are arbitrary constants, provided that b_1, b_2, b_3 and c_1 are well defined.

We put (3.18) and (3.19) into (3.16) and (3.14) to obtain

$$\begin{aligned} & P(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t}) f \cdot f \\ = & -\frac{6c_3^2 f_{4x} f_{xx}}{c_4} + \frac{4c_3^2 f_{xxx}^2}{c_4} + 2c_2 f_{4x,y} f + 2c_3 f_{xxxz} f - 6c_3 f_{xxz} f_x \\ & - 2c_3 f_{xxx} f_z + 6c_3 f_{xx} f_{xz} + 2c_4 f_{zz} f - 2c_4 f_z^2 + 2c_5 f_{xt} f - 2c_5 f_x f_t, \end{aligned} \tag{3.20}$$

where c_2, c_3, c_4, c_5 are arbitrary constants, and the resonant solution

$$f = \sum_{i=1}^{N} \epsilon_i e^{k_i x + \frac{c_3^2 k_i^2 y}{c_4 c_2} + \frac{c_3 k_i^3 z}{c_4} - \frac{3c_3^2 k_i^5 t}{c_4 c_5}}, \tag{3.21}$$

where ϵ_i's and k_i's, $1 \le i \le N$, are arbitrary constants. On the other hand, we can get the corresponding (3+1)-dimentional nonlinear differential equation

$$\begin{aligned} & - c_3^2 u^6 + 2c_2 c_4 u^4 v - 18c_3^2 u^4 u_x + 16c_2 c_4 u^3 u_y - 32c_3^2 u^3 u_{xx} \\ & + 24c_2 c_4 u^2 u_x v - 36c_3^2 u^2 u_x^2 - 24c_3^2 u^2 u_{xxx} + 48c_2 c_4 u^2 u_{xy} \\ & + 96c_2 c_4 u u_x u_y + 32c_2 c_4 u u_{xx} v + 24c_2 c_4 u_x^2 v - 72c_3^2 u_x^3 \\ & + 64c_2 c_4 u u_{xxy} + 64c_2 c_4 u_{xx} u_y + 16c_2 c_4 u_{xxx} v - 48c_3^2 u_x u_{xxx} \\ & + 32c_3^2 u_{xx}^2 + 96c_2 c_4 u_x u_{xy} + 96c_3 c_4 u_x u_z + 32c_4 c_5 u_t \\ & + 32c_4^2 v_{zz} + 32c_2 c_4 u_{xxxy} + 32c_3 c_4 u_{xxz} = 0, \end{aligned} \tag{3.22}$$

under the transformation $u = 2(\ln f)_x$, where c_2, c_3, c_4, c_5 are arbitrary constants and $u_y = v_x$.

EXAMPLE 2 Let us introduce the weights of the independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 3, 4), \tag{3.23}$$

and take a general homogeneous polynomial in weight 7:

$$P = c_1x^7 + c_2x^5y + c_3x^4z + c_4x^3t + c_5xy^3 + c_6zt + c_7xyt + c_8y^2z. \tag{3.24}$$

Following the steps we mentioned above, we set the wave variables:

$$\eta_i = k_ix + b_1k_i^2y + b_2k_i^3z + b_3k_i^4t, 1 \leq i \leq N, \tag{3.25}$$

where the k_i's are arbitrary constants, but the constants b_1, b_2, b_3 are to be determined. Then we obtain the corresponding generalized bilinear differential equation with $p = 5$:

$$\begin{aligned}
&P(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t})f \cdot f \\
=& (c_1D_{5,x}^7 + c_2D_{5,x}^5D_{5,y} + c_3D_{5,x}^4D_{5,z} + c_4D_{5,x}^3D_{5,t} \\
&+ c_5D_{5,x}D_{5,y}^3 + c_6D_{5,z}D_{5,t} + c_7D_{5,x}D_{5,y}D_{5,t} + c_8D_{5,y}^2D_{5,z})f \cdot f \\
=& 2c_1f_{7x}f - 14c_1f_{6x}f_x + 42c_1f_{5x}f_{xx} + 20c_2f_{xx}f_{xxxy} - 20c_2f_{xxx}f_{xxy} \\
&+ 10c_2f_{4x}f_{xy} + 2c_3f_{4x,z}f + 2c_4f_{xxxt}f - 6c_4f_{xxt}f_x - 2c_4f_{xxx}f_t + 6c_4f_{xx}f_{xt} \\
&+ 2c_5f_{xyyy}f - 6c_5f_{yyx}f_y - 2c_5f_{yyy}f_x + 6c_5f_{yy}f_{xy} + 2c_6f_{zt}f - 2c_6f_zf_t.
\end{aligned} \tag{3.26}$$

In the generalized bilinear differential equation (3.26) we can see that

$$\begin{aligned}
c_7D_{5,x}D_{5,y}D_{5,t}f \cdot f = 0, \\
c_8D_{5,y}^2D_{5,z}f \cdot f = 0.
\end{aligned} \tag{3.27}$$

The corresponding resonant solution of N exponential waves is defined by

$$f = \sum_{i=1}^{N} \epsilon_i e^{\eta_i} = \sum_{i=1}^{N} \epsilon_i e^{k_ix + b_1k_i^2y + b_2k_i^3z + b_3k_i^4t}, \tag{3.28}$$

where the ϵ_i's and k_i's, $1 \leq i \leq N$, are arbitrary constants, but by a Maple computation, the proportional constants b_1, b_2, b_3 have to satisfy

$$\begin{cases}
5b_1c_2 + b_2c_3 + 15c_1 = 0, \\
b_1^3c_5 + 3b_3c_4 + 7c_1 = 0, \\
10b_1c_2 + 3b_3c_4 + 21c_1 - 3b_1^3c_5 = 0, \\
3b_1^3c_5 - b_2b_3c_6 - 5b_1c_2 - b_3c_4 = 0.
\end{cases} \tag{3.29}$$

Again, we can use the software Maple to get, and after calculation, derive the solution to (3.29):

$$\begin{cases} b_1 = -\dfrac{15c_1c_6 - 4c_3c_4}{5c_2c_6}, \\ b_2 = -\dfrac{4c_4}{c_6}, \\ b_3 = -\dfrac{3c_1c_6 + 2c_3c_4}{3c_4c_6}, \end{cases} \tag{3.30}$$

when the coefficients of the polynomial P satisfy

$$c_5 = \frac{250(2c_1c_6 - c_3c_4)c_2^3c_6^2}{(15c_1c_6 - 4c_3c_4)^3}, \tag{3.31}$$

where c_1, c_2, c_3, c_4, c_6 are arbitrary provided that b_1, b_2, b_3 and c_5 are well defined.

We put (3.30) and (3.31) into (3.28) and (3.26) to obtain

$$\begin{aligned} &P(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t})f \cdot f \\ =& 2c_1f_{7x}f - 14c_1f_{6x}f_x + 42c_1f_{5x}f_{xx} + 20c_2f_{xx}f_{xxxy} - 20c_2f_{xxx}f_{xxy} \\ &+ 10c_2f_{4x}f_{xy} + 2c_3f_{4x,z}f + 2c_4f_{xxxt}f - 6c_4f_{xxt}f_x - 2c_4f_{xxx}f_t + 6c_4f_{xx}f_{xt} \\ &+ \frac{500(2c_1c_6 - c_3c_4)c_2^3c_6^2f_{xyyy}f}{(15c_1c_6 - 4c_3c_4)^3} - \frac{1500(2c_1c_6 - c_3c_4)c_2^3c_6^2f_{yyx}f_y}{(15c_1c_6 - 4c_3c_4)^3} \\ &- \frac{500(2c_1c_6 - c_3c_4)c_2^3c_6^2f_{yyy}f_x}{(15c_1c_6 - 4c_3c_4)^3} + \frac{1500(2c_1c_6 - c_3c_4)c_2^3c_6^2f_{yy}f_{xy}}{(15c_1c_6 - 4c_3c_4)^3} \\ &+ 2c_6f_{zt}f - 2c_6f_zf_t, \end{aligned} \tag{3.32}$$

where c_1, c_2, c_3, c_4, c_6 are arbitrary constants, and the resonant solution

$$f = \sum_{i=1}^{N} \epsilon_i e^{k_ix - \frac{(15c_1c_6 - 4c_3c_4)k_i^2y}{5c_2c_6} - \frac{4c_4k_i^3z}{c_6} - \frac{(3c_1c_6 + 2c_3c_4)k_i^4t}{3c_4c_6}}, \tag{3.33}$$

where ϵ_i's and k_i's, $1 \le i \le N$, are arbitrary constants.

3.2.2 Example with positive and negative weights

EXAMPLE 3 Let us set the weights of the independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, -1, -2, 3), \tag{3.34}$$

and introduce a homogeneous polynomial in weight 2:

$$P = c_1x^2 + c_2x^3y + c_3x^4z + c_4yt + c_5xy^2t + c_6x^4y^2. \tag{3.35}$$

Following the steps we mentioned above, we set the wave variables:

$$\eta_i = k_i x + b_1 k_i^{-1} y + b_2 k_i^{-2} z + b_3 k_i^3 t, 1 \le i \le N, \tag{3.36}$$

where the k_i's are arbitrary constants, but the constants b_1, b_2, b_3 are to be determined. Then we obtain the corresponding generalized bilinear differential equation with $p = 5$:

$$\begin{aligned}
&P(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t}) f \cdot f \\
&= (c_1 D_{5,x}^2 + c_2 D_{5,x}^3 D_{5,y} + c_3 D_{5,x}^4 D_{5,z} + c_4 D_{5,y} D_{5,t} + c_5 D_{5,x} D_{5,y}^2 D_{5,t} + c_6 D_{5,x}^4 D_{5,y}^2) f \cdot f \\
&= 2c_1 f_{xx} f - 2c_1 f_x^2 + 2c_2 f_{xxxy} f - 6c_2 f_{xxy} f_x - 2c_2 f_{xxx} f_y \\
&\quad + 6c_2 f_{xx} f_{xy} + 2c_3 f_{4x,z} f + 2c_4 f_{yt} f - 2c_4 f_y f_t + 2c_5 f_{xyyt} f \\
&\quad - 4c_5 f_{xyt} f_y - 2c_5 f_{xyy} f_t - 2c_5 f_{tyy} f_x + 4c_5 f_{xy} f_{yt} + 2c_5 f_{yy} f_{xt} \\
&\quad + 2c_6 f_{yy} f_{4x} + 12c_6 f_{xxyy} f_{xx} - 8c_6 f_{xyy} f_{xxx} + 16c_6 f_{xxxy} f_{xy} - 12c_6 f_{xxy}^2,
\end{aligned} \tag{3.37}$$

and the corresponding resonant solution of N exponential waves is defined by

$$f = \sum_{i=1}^{N} \epsilon_i e^{\eta_i} = \sum_{i=1}^{N} \epsilon_i e^{k_i x + b_1 k_i^{-1} y + b_2 k_i^{-2} z + b_3 k_i^3 t}, \tag{3.38}$$

where the ϵ_i's and k_i's, $1 \le i \le N$, are arbitrary constants, but by a Maple computation, the proportional constants b_1, b_2, b_3 have to satisfy

$$\begin{cases}
5b_1^2 c_6 + b_2 c_3 = 0, \\
b_1^2 b_3 c_5 + b_1^2 c_6 = 0, \\
b_1^2 b_3 c_5 + 6b_1^2 c_6 + 3b_1 c_2 + c_1 = 0, \\
3b_1^2 b_3 c_5 + 4b_1^2 c_6 + b_1 b_3 c_4 + b_1 c_2 = 0.
\end{cases} \tag{3.39}$$

Again, we can use the software Maple to get, and after calculations, derive the solution to (3.39):

$$\begin{cases}
b_1 = -\dfrac{c_2 c_5 - c_4 c_6}{c_5 c_6}, \\
b_2 = -\dfrac{5(c_2 c_5 - c_4 c_6)^2}{c_3 c_5^2 c_6}, \\
b_3 = -\dfrac{c_6}{c_5},
\end{cases} \tag{3.40}$$

when the coefficients of the polynomial P satisfy

$$c_1 = -\frac{(c_2 c_5 - c_4 c_6)(2c_2 c_5 - 5c_4 c_6)}{c_5^2 c_6}, \tag{3.41}$$

where c_2, c_3, c_4, c_5, c_6 are arbitrary provided that b_1, b_2, b_3 and c_1 are well defined.

We put (3.40) and (3.41) into (3.38) and (3.37) to obtain

$$\begin{aligned} & P(D_{5,x}, D_{5,y}, D_{5,z}, D_{5,t}) f \cdot f \\ = & -\frac{2(c_2c_5 - c_4c_6)(2c_2c_5 - 5c_4c_6) f_{xx} f}{c_5^2 c_6} + \frac{2(c_2c_5 - c_4c_6)(2c_2c_5 - 5c_4c_6) f_x^2}{c_5^2 c_6} \\ & + 2c_2 f_{xxxy} f - 6c_2 f_{xxy} f_x - 2c_2 f_{xxx} f_y + 6c_2 f_{xx} f_{xy} + 2c_3 f_{4x,z} f \\ & + 2c_4 f_{yt} f - 2c_4 f_y f_t + 2c_5 f_{xyyt} f - 4c_5 f_{xyt} f_y - 2c_5 f_{xyy} f_t \\ & - 2c_5 f_{tyy} f_x + 4c_5 f_{xy} f_{yt} + 2c_5 f_{yy} f_{xt} + 2c_6 f_{yy} f_{4x} + 12c_6 f_{xxyy} f_{xx} \\ & - 8c_6 f_{xyy} f_{xxx} + 16c_6 f_{xxxy} f_{xy} - 12c_6 f_{xxy}^2, \end{aligned} \tag{3.42}$$

where c_2, c_3, c_4, c_5, c_6 are arbitrary constants, and the resonant solution

$$f = \sum_{i=1}^{N} \epsilon_i e^{k_i x - \frac{(c_2c_5 - c_4c_6)y}{c_5c_6k_i} - \frac{5(c_2c_5-c_4c_6)^2 z}{c_3c_5^2c_6k_i^2} - \frac{c_6k_i^3 t}{c_5}}, \tag{3.43}$$

where the ϵ_i's and k_i's, $1 \leq i \leq N$, are arbitrary constants.

Finally, we can similarly get the corresponding (3+1)-dimensional nonlinear differential equation

$$\begin{aligned} & 64c_5^3c_6u_yv_{yt} + 6c_5^2c_6^2u^4v_{y,y} + 8c_5^2c_6^2u_{xxx}v_y^2 \\ & \quad + 72c_5^2c_6^2u_x^2v_{yy} + 16c_5^2c_6^2u_{xxx}v_{yy} + 32c_4c_5^2c_6v_{yt} \\ & \quad + 32c_5^3c_6u_tv_{yy} + 64c_5^2c_6^2uu_{xxy}v_y + 16c_3c_5^2c_6u_{xxx}v_z \\ & \quad + 32c_5^2c_6^2uu_{xx}v_y + 24c_5^2c_6^2u^2u_xv_{yy} + 96c_5^2c_6^2u^2u_{xy}v_y \\ & \quad + 32c_3c_5^2c_6uu_{xx}v_z + 24c_3c_5^2c_6u^2u_xv_z + 96c_5^2c_6^2uu_xu_yv_y \\ & \quad + 32c_5^3c_6u_{yyt} - 96c_5^2c_6^2u_{xy}^2 + 24c_3c_5^2c_6u_x^2v_z + 36c_5^2c_6^2u^2u_xv_y^2 \\ & \quad + 48c_5^2c_6^2u^3u_yv_y + 2c_3c_5^2c_6u^4v_z + 48c_3c_5^2c_6u^2u_{xz} \\ & \quad + 64c_3c_5^2c_6uu_{xxz} + 64c_3c_5^2c_6u_zu_{xx} + 96c_3c_5^2c_6u_xu_{xz} + 5c_5^2c_6^2u^4v_y^2 \\ & \quad + 12c_5^2c_6^2u_x^2v_y^2 + 32c_5^2c_6^2u^3u_{yy} + 48c_5^2c_6^2u^2u_{xyy} + 96c_5^2c_6^2u_xu_{xyy} \\ & \quad - 64c_5^2c_6^2u_{xx}u_{yy} + 128c_5^2c_6^2u_yu_{xxy} + 32c_2c_5^2c_6u_{xxy} + 32c_3c_5^2c_6u_{xxxz} \\ & \quad + 16c_3c_5^2c_6u^3u_z + 224c_2c_4c_5c_6u_x + 96c_2c_5^2c_6u_xu_y - 64c_2^2c_5^2u_x \\ & \quad - 160c_4^2c_6^2u_x + 96c_3c_5^2c_6uu_xu_z + 48c_5^2c_6^2u^2u_y^2 + 288c_5^2c_6^2u_xu_y^2 = 0, \end{aligned} \tag{3.44}$$

under the transformation $u = 2(\ln f)_x$, where c_2, c_3, c_4, c_5, c_6 are arbitrary constants and $u = v_x$.

Chapter 4

Conclusions

In this thesis, we applied the finding that the D_p-operators have resonant solutions. As a result, in order to further analyze these resonant solutions, we have attempted to generate a class of generalized bilinear differential equations suitable for the (3+1)-dimensional case with $p = 5$. One of our main fundamental components is to construct resonant solutions by using the linear superposition principle in juxtaposition with parameterizations of wave numbers and frequencies. Let us note that it is critical to create a multivariate polynomial $P = P(x_1, x_2, \cdots, x_l)$ such that

$$P(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) + P(k_{1,j} + \alpha k_{1,i}, \cdots, k_{l,j} + \alpha k_{l,i}) = 0,\ 1 \leq i \leq j \leq N, \quad (4.1)$$

where the powers of α obey the rule (2.18), and $k_{j,i}$'s are arbitrary constants. In conclusion of these priorities, we can get the corresponding generalized bilinear differential equation. By using the mathematical software Maple, we can compute the relationship between the coefficients of the generalized bilinear equation and the unique constants b_j's in the resonant solution we formulated. Therefore, we can be certain that the generalized bilinear differential equations we obtain are the ones desired, indeed. To further verify that the bilinear differential equations in question have resonant solutions, we can again use Maple to do additional substitutions.

In order to understand generalized bilinear differential equations clearly; it is imperative to provide background information on the Hirota bilinear operators – D-operators and emphasize specific equations found in the Hirota bilinear form. When the D-operators and the Hirota bilinear equations are compared simultaneously, differences in the D_p-operators and generalized bilinear differential equation under D_p-operators are made clear. From these observances, we can settle on the meaning that the class of bilinear differential equations that have resonant solutions can be enlarged, allowing us to have the resources to generate much more soliton equations.

In this thesis, we presented three illustrative examples to showcase the existence of resonant solutions when $p = 5$. It is common in mathematics to alter variables in order to achieve alternative answers, so it is obvious when we choose different values of p, different sets of generalized bilinear differential equations are derived. In accordance, the discovery of the D_p-operators has prominently changed the course and study of bilinear differential equations, especially by enlarging the magnitude of the scope of the study. Nevertheless, there are still many interesting questions, which remain open, in further conducting investigation of generalized bilinear differential equations, and a few of them are listed as follows:

1. Are there any exchange formulas for the bilinear D_p-operators, which lead to bilinear Bäcklund transformations?

2. Can any combination of two generalized bilinear differential equations with different values of p, which have resonant solutions, possess resonant solutions?

3. How can one construct any combined solutions of trigonometric function and exponential function solutions to generalized bilinear equations?

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