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Leonard Systems and their Friends

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by

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Abstract

Let V be a finite-dimensional vector space over a field $\mathbb{K}$, and let $\mathrm{End}(V)$ be the set of all $\mathbb{K}$-linear transformations from V to V. A *Leonard system* on V is a sequence

$$(A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d),$$

where A and B are multiplicity-free elements of $\mathrm{End}(V)$; $\{E_i\}_{i=0}^d$ and $\{E_i^*\}_{i=0}^d$ are orderings of the primitive idempotents of A and B, respectively; and for $0 \leq i, j \leq d$, the expressions E_iBE_j and $E_i^*AE_j^*$ are zero when $|i-j| > 1$ and nonzero when $|i-j| = 1$. Leonard systems arise in connection with orthogonal polynomials, representations of many nice algebras, and the study of some highly regular combinatorial objects. We shall use the construction of Leonard pairs of classical type from finite-dimensional modules of sl_2 and the construction of Leonard pairs of basic type from finite-dimensional modules of $U_q(sl_2)$.

Suppose $\Phi := (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. For $0 \leq i \leq d$, let

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + \cdots + E_dV).$$

Then $U_0, U_1, \ldots, U_d$ is the *split decomposition of V for Φ*. The split decomposition of V for Φ gives rise to canonical matrix representations of A and B in terms of useful parameters for the Leonard system.

In this thesis, we consider when certain Leonard systems share a split decomposition. We

say that Leonard systems $\Phi := (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ and $\hat{\Phi} := (\hat{A}; \hat{B}; \{\hat{E}_i\}_{i=0}^d; \{\hat{E}^*{}_i\}_{i=0}^d)$ are *friends* when $A = \hat{A}$ and Φ, $\hat{\Phi}$ have the same split decomposition. We obtain Leonard systems which share a split decomposition by constructing them from closely related module structures for either sl_2 or $U_q(sl_2)$ on V. We then describe friends by a parametric classification. In this manner we describe all pairs of friends of classical and basic types. In particular, friendship is not entirely a property of isomorphism classes.

1 Introduction

Leonard pairs and systems are linear algebraic objects. Our discussion assumes the reader is familiar with linear algebra at an undergraduate level. We postpone formal definitions until Chapter 2. The most concise description of a Leonard pair is as follows.

> Let A and B be diagonalizable linear operators on a finite-dimensional vector space. Then A, B is a *Leonard pair* when each is represented by an irreducible tridiagonal matrix with respect to some eigenbasis of the other.

In other words, when viewed with respect to the corresponding eigenbasis, the nonzero entries lie on the diagonal in one and on, above, or below the diagonal in the other. Moreover, the entries above and below the diagonal of the second are nonzero. See Figure 1.1.

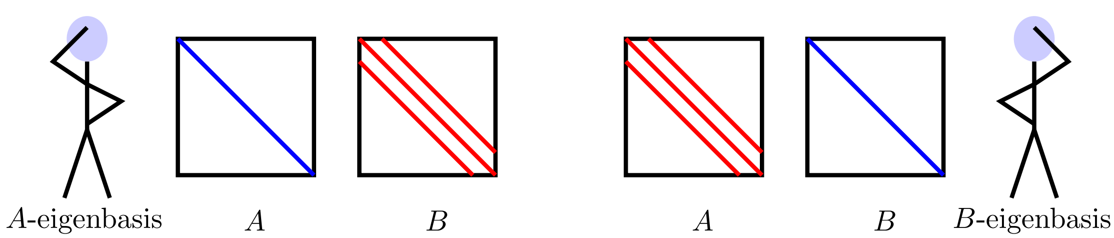

Figure 1.1: Two perspectives on a Leonard pair

This simple description of a Leonard pair belies the depth of the topic. The motivating result is an equivalence of Leonard pairs with the families of orthogonal polynomials in the terminating branch of the Askey scheme. This equivalence is essentially due to Doug

Leonard, for whom Leonard pairs are named. Although the connection between Leonard pairs and orthogonal polynomials is not the main thrust of this thesis, a little background is in order.

The Askey scheme and its q-analog [5], [6], [7], [8], [29] consist of the sequences of orthogonal polynomials which can be expressed with hypergeometric functions. Askey and Wilson showed that they are ${}_4F_3$ hypergeometric polynomials and ${}_4\phi_3$ basic hypergeometric polynomials, subject to some balancing conditions on the parameters, together with various limiting cases of each to simpler hypergeometric polynomials. The terminating branch consists of 11 families for which the sequences of polynomials are eventually zero. See Figures 1.2 and 1.3. Leonard's work [30] is related to the work of Askey and Wilson in the terminating branch of the Askey scheme.

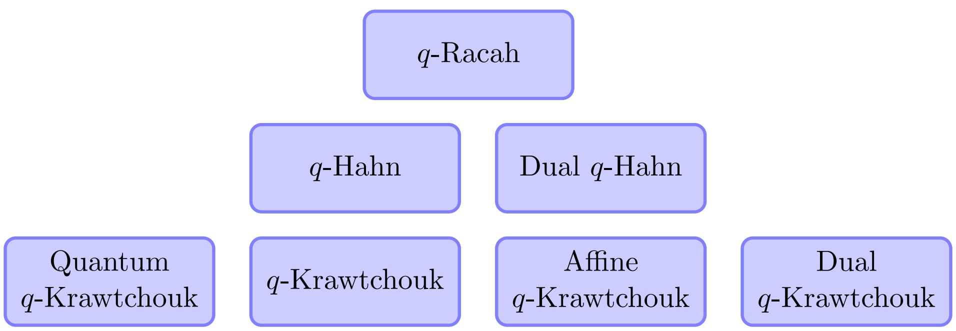

Figure 1.2: The Askey q-Scheme (terminating branch)
Basic types of Leonard pairs

We point out that both Askey and Leonard were inspired by the work of Delsarte [8], [15], [28], [30]. Delsarte's focus was the use of association schemes in the study of codes and designs (these are all highly regular combinatorial objects). His work included a description of the structure constants of many P- and Q-polynomial association schemes using (basic) hypergeometric series. The P- and Q-polynomial properties correspond to two related tridiagonal matrix representation, which define a three-term recurrence, which in turn defines a

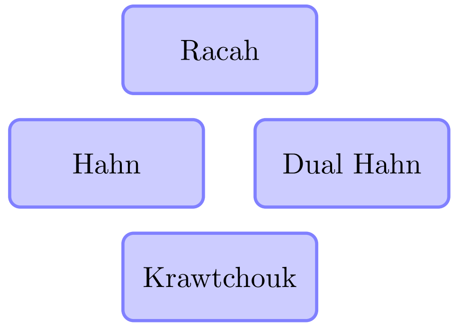

Figure 1.3: The classical Askey Scheme (terminating branch)
Classical types of Leonard pairs

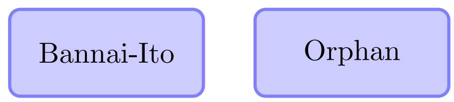

Figure 1.4: Missing from of the Askey Scheme
Missing types of Leonard pairs

terminating sequence of orthogonal polynomials. These hints of a connection between nice orthogonal polynomials and hypergeometric series influenced the work of Askey and Wilson and are the direct precursors to Leonard's results.

Leonard's result was a classification of the parameters of the P- and Q-polynomial association schemes into a number of families. By solving the various constraints, Leonard gave a classification of possible parameters. The families identified by Leonard correspond to the families of orthogonal polynomials in the terminating branch of the Askey scheme (a few families of the Askey scheme did not fit the combinatorial constraints, but arise from the computations prior to applying these constraints). A pair of matrices which form what we now call a Leonard pair appeared in his work. This was a very exciting result. Although a P- and Q-polynomial association scheme might be large and have many structure constants, they are all determined by at most 5 free parameters and fell into a handful of families. In Bannai and Ito's [9] presentation of Leonard's result, a missing family of solutions was identified (corresponding to $q = -1$).

Leonard pairs were introduced by Terwilliger to offer a purely linear algebraic understanding of Leonard's work and the terminating branch of the Askey scheme [31], [34], [32], [33], [46], [38], [50], [41], [49], [42], [40], [47], [48], [39], [43], [44], [45], [23], [24]. This was similar in spirit to Bannai and Ito's treatment of Leonard's work. The first result was that Leonard's results proceed without issue at this level, and in fact do so over any fields of sufficiently large characteristic. This established a (near) equivalence between Leonard pairs and the orthogonal polynomials in the terminating branch of the Askey scheme, plus the Bannai-Ito polynomials, plus a family that can arise over field of characteristic two (dubbed the "orphan" family). See Figures 1.2, 1.3, and 1.4. Given a Leonard pair, its *type* is the type of corresponding orthogonal polynomials.

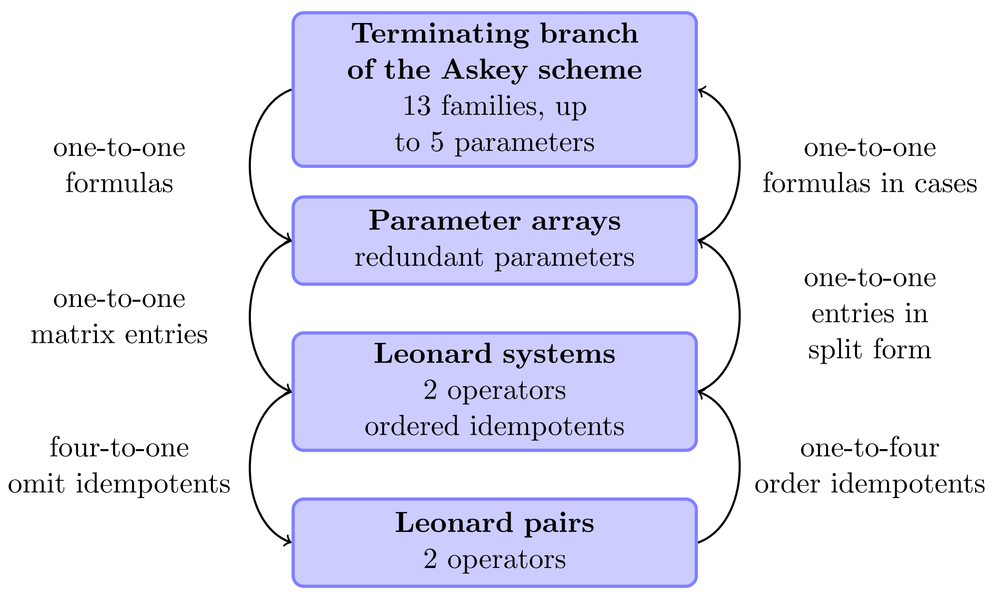

Figure 1.5: Orthogonal polynomials and Leonard pairs

At some point, it is inevitable that the 13 families of Leonard pairs must be treated separately to make full use of their individual characteristics. This was essentially the only approach prior to Terwilliger's introduction of Leonard pairs. Terwilliger's key observation was that by introducing an overdetermined set of parameters, referred to as the *parameter array* [44], many results could be proven for all Leonard pairs at once. The entries of the parameter array arise in connection with another pair of perspectives on Leonard pairs. There are "split bases" with respect to which A is lower-bidiagonal, B is upper-bidiagonal, and the entries on the subdiagonal of A are all 1. We are concerned with two of these bases, referred to as the "first split basis" and "second split basis," respectively. See Figure 1.6. The values on the diagonals and the nonzero superdigaonals of the matrices representing A and B with respect to these two split bases form a parameter array.

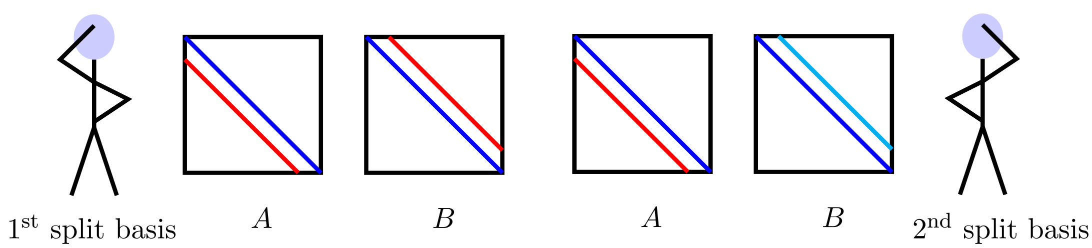

Figure 1.6: Split perspectives on Leonard pairs

Connections between Leonard pairs and the Lie algebra sl_2 and universal quantum enveloping algebra $U_q(sl_2)$ for sl_2 have been developed [1], [2], [3], [4], [10], [25]. We offer more details in Chapter 3. The existence of such connections should not be too surprising; indeed, the connections between the representation theory of these algebraic objects and Askey-Wilson polynomials are well-known. Additionally Zhedanov [51] showed that the families of orthogonal polynomials in the Askey scheme give rise to a pair of operators which satisfy certain algebraic relations. Both of the algebras sl_2 and $U_q(sl_2)$ have a triple of *equitable generators* x, y, z. They are so-named because the relations which they satisfy are invariant under cyclic shifts of these generators. Let V be a finite-dimensional irreducible module for either of these algebras. Then x and y act on an eigenbasis of V for z as lower-bidiagonal and upper-bidiagonal matrices, respectively (and similarly for cyclic shifts of x, y, z). See Figure 1.7. The shapes of these matrices are reminiscent of those arising in connection with split decompositions for Leonard pairs. In fact the relationship is far stronger.

> Up to isomorphism, every Leonard pair of classical/basic type is constructed on some irreducible sl_2-module/$U_q(sl_2)$-module V by restricting the action of some linear combinations
>
> $$A = \kappa I + \lambda y + \mu z + \nu yz \quad \text{and} \quad B = \kappa^* I + \lambda^* z + \mu^* x + \nu^* zx. \tag{1.0.1}$$

We say that a sequence $(d,\ q,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \kappa^*,\ \lambda^*,\ \mu^*,\ \nu^*)$ is *Leonine* when A, B in (1.0.1) form a Leonard pair, where $d+1$ is the dimension of the module and the scalar q is 1 if there is an sl_2-module structure and is the q of $U_q(sl_2)$ otherwise. This leads to a one-to-four correspondence between the isomorphism classes of a Leonard pair (type) and the collection of all Leonine parameters. We use this correspondence to describe our results. A key observation is this: By fixing the sl_2-module/$U_q(sl_2)$-module structure on V, we can control the "shape" of the isomorphism class representative of a Leonard pair. That is to say, there is a basis of V with respect to which A, B are respectively lower-bidiagonal and upper-bidiagonal. These match the split form of a Leonard pair. This offers another set of tools for the study of Leonard pairs. In this thesis, we use the fact that every Leonard pair can be constructed using nice generators of these algebras.

Our goal in this work is to describe when the following situation occurs.

> Two Leonard pairs A, B and $\hat{A}$, $\hat{B}$ are *friends* when $\hat{A} = A$ and the corresponding elements of the split basis for A, B and $\hat{A}$, $\hat{B}$ are scalar multiples of one another.

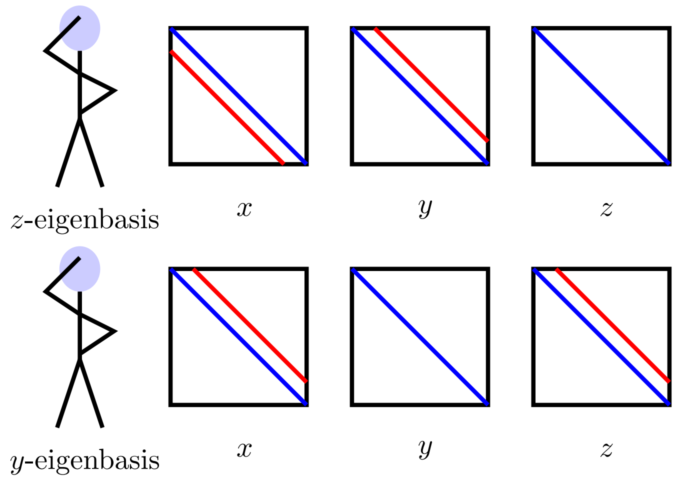

Figure 1.7: Two perspectives on the equitable generators

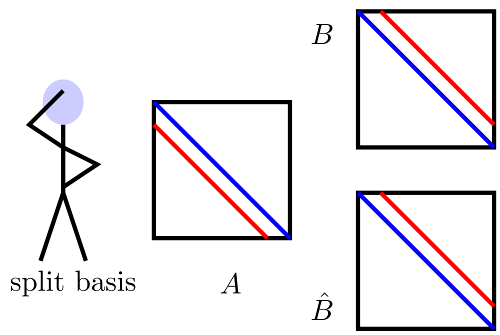

Figure 1.8: A perspective on friends

Friendship is an equivalence relation. In Chapter 4 we describe friends by giving the equivalence classes in the following way. Fix an irreducible module for either sl_2 or $U_q(sl_2)$ and a sequence of Leonine parameters corresponding to a particular Leonard pair. Any other sequence of Leonine parameters whose first six entries satisfy a few particular conditions corresponds to a Leonard pair belonging to the same class of friends. In Chapter 5 we examine friends type by type. We see that friendships may only result between certain types of Leonard pairs.

2 Preliminaries

In this chapter we recall basic properties of Leonard pairs and related notions. We begin with a formal definition of a Leonard pair and provide an example [38], [39], [40], [41]. We elaborate on the relationship between Leonard pairs and orthogonal polynomials [43], [21], [30], [44]. We extend the concept of a Leonard pair to a Leonard system, and describe this in depth [39], [43], [42], [40], [41]. We define the split decomposition of a vector space and emphasize its uniqueness [45], [32]. We describe the parameter array as an alternative method to describing a Leonard system using a sequence of scalars [44]. These scalars appear in the matrix representation of a Leonard system with respect to a split basis. We elaborate on the split form of these matrices and, in particular, their importance to the motivating problem of this paper. Lastly, we describe how the permutation of some parameters of a given Leonard system can lead to other distinct Leonard systems by inducing an action of D_4 on the set of all Leonard systems [42], [40], [39], [43].

2.1 Leonard pairs

We begin with some vocabulary.

Definition 2.1.1 [38], [39], [40], [41] Let X denote a square matrix.

(i) X is called *tridiagonal* whenever each nonzero entry lies on the diagonal, the subdiagonal, or the superdiagonal.

(ii) Assume X is tridiagonal. Then X is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

For the rest of the paper, let $\mathbb{K}$ denote a field. We now define a Leonard pair.

Definition 2.1.2 [38], [39], [40], [41] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V, we mean an ordered pair of linear operators $A : V \to V$ and $B : V \to V$ that satisfy both (i) and (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing B is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is irreducible tridiagonal.

2.2 An example

Following the work in [38], [39], [40], [41] we provide a straightforward example of a Leonard pair. Let $V = \mathbb{K}^5$ (column vectors), and let

$$A = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

View A and B as linear operators on V. Assume the characteristic of $\mathbb{K}$ is not 2 or 3, so A is irreducible tridiagonal. By construction B is diagonal. Therefore condition (i) in Definition 2.1.2 is satisfied by the basis for V consisting of the columns of the 5×5 identity matrix, I. Set

$$P = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

By matrix multiplication $P^2 = 16I$, so P is invertible. Also by matrix multiplication,

$$AP = PB. \tag{2.2.1}$$

Hence $P^{-1}AP$ is equal to B and is therefore diagonal. By (2.2.1) and since P^{-1} is a scalar multiple of P, we find $P^{-1}BP$ is equal to A and is therefore irreducible tridiagonal. Thus condition (ii) of Definition 2.1.1 is satisfied by the basis for V consisting of the columns of the matrix P. We conclude that A, B is a Leonard pair. In fact, it is just one member of the following infinite family of Leonard pairs.

Theorem 2.2.1 [39], [40], [42] *For any nonnegative integer d, the pair*

$$A = \begin{pmatrix} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix}, \qquad B = \operatorname{diag}(d, d-2, d-4, \ldots, -d)$$

is a Leonard pair on the vector space $\mathbb{K}^{d+1}$, provided the characteristic of $\mathbb{K}$ is zero or an odd prime greater than d.

Theorem 2.2.1 is verified in a manner similar to the above example. Indeed, by [40, Section 16], the matrix P with ij entry

$$P_{ij} = \binom{d}{j} {}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \,\middle|\, 2 \right) \qquad (0 \le i, j \le d).$$

satisfies $P^2 = 2^d I$ and $AP = PB$. Recall that ${}_2F_1$ is the hypergeometric series [21, p. 3] defined by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!},$$

where $(a)_n$ is the falling factorial defined by

$$(a)_n = \begin{cases} 1 & \text{if } n = 0, \\ a(a+1)\cdots(a+n-1) & \text{if } n > 0. \end{cases}$$

The entries of P are given by Krawtchouk polynomials, a family of orthogonal polynomials from the terminating branch of the Askey scheme. A similar phenomenon occurs for all Leonard pairs – we discuss this in the next section.

2.3 The 13 types of Leonard pairs

We sketch the connection between Leonard pairs and the orthogonal polynomials from the terminating branch of the Askey scheme. Further details can be found in [39], [40], [41], and [44]. Here we discuss only what is needed to frame our work.

Theorem 2.3.1 [44] *Let A, B be a Leonard pair. Let β be a basis with respect to which $[A]_\beta$ is irreducible tridiagonal with constant row sum and $[B]_\beta = \operatorname{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$. Let δ be a basis with respect to which $[A]_\delta = \operatorname{diag}(\theta_0, \theta_1, \ldots, \theta_d)$ and $[B]_\delta$ is irreducible tridiagonal with constant row sum. The change of basis matrix, P, from β to δ has entries*

$$P_{ij} = f_i(\theta_j) \qquad\qquad (0 \le i, j \le d),$$

where these f_i are orthogonal polynomials belonging to some family in the terminating branch of the Askey scheme.

The Askey scheme consists of polynomials arising from hypergeometric polynomials ${}_4F_3$, basic hypergeometric polynomials ${}_4\phi_3$, and limiting cases of each to simpler hypergeometric polynomials. For certain choices of parameters, the sequences are finite. There are 13 different families of orthogonal polynomials from the terminating branch of the Askey scheme, thus limiting the possible forms of P – and hence A, B – to 13 types.

The Askey scheme consists of sequences of orthogonal polynomials $\{f_i\}$ for which the dual polynomials $\{f_i^*\}$ (with respect to some weight) form a sequence of orthogonal polynomials also in the Askey scheme. Some polynomials of the Askey scheme are self-dual.

Corollary 2.3.2 [44] *With reference to Theorem 2.3.1, the matrix P^{-1} has entries*

$$P_{ij}^{-1} = \alpha f_i^*(\theta_j^*) \qquad\qquad (0 \leq i, j \leq d)$$

for some scalar α.

We do not rely heavily upon this connection between Leonard pairs and orthogonal polynomials. The Leonard pairs associated with different families of orthogonal polynomials behave slightly differently. To make full use of the literature we will need to distinguish these cases by name and by some properties, however, the orthogonal polynomials themselves make no further appearance in our work.

Definition 2.3.3 [43] The *type* of a Leonard pair is the name of the associated family of orthogonal polynomials from the terminating branch of the Askey scheme. The 13 types are Racah, Hahn, dual Hahn, Krawtchouk, q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, affine q-Krawtchouk, Bannai-Ito, and orphan.

For example, the Leonard pair from Theorem 2.2.1 is a Leonard pair of Krawtchouk type because the transition matrix P had entries determined by a polynomial belonging to the Krawtchouk family of polynomials. More details for each type of Leonard pair, including explicit formulas, can be found in [21], [30], and [44]. It will be useful to categorize Leonard pairs into somewhat broader families.

Definition 2.3.4 [2], [21] We say that a Leonard pair is of *classical* type if it is a Leonard pair of Racah, Hahn, dual Hahn, or Krawtchouk type. We say that a Leonard pair is of *basic* type if it is a Leonard pair of q-Racah, q-Hahn, q-Krawtchouk, quantum q-Krawtchouk, or affine q-Krawtchouk type.

The term *basic* refers to a scalar q which is called the *base*. In the basic case of Leonard pairs, the value for q has several restrictions but maintains some freedom. On the other hand, the classical and Bannai-Ito cases correspond precisely to $q = 1$ and $q = -1$, respectively. The latter case is less well-understood. The orphan case of Leonard pairs occurs only when dealing with a finite field. For these reasons, our study is concerned only with Leonard pairs of classical and basic types.

2.4 Leonard systems

We introduce the notion of a Leonard system which refines that of a Leonard pair. A Leonard system consists of a Leonard pair, together with information regarding the bases from Definition 2.1.2. Before defining a Leonard system explicitly, we must first lay some groundwork and make a few observations. We start by establishing some notation and recalling a few linear algebraic facts that we will use throughout the rest of the thesis.

Definition 2.4.1 Let d be a nonnegative integer. Let V be a vector space over the field $\mathbb{K}$ with positive dimension $d+1$. Let $\mathrm{End}(V)$ be the set of all linear operators on V. Let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices with entries in $\mathbb{K}$. Index the rows and columns of these matrices by $0, 1, \ldots, d$.

Lemma 2.4.2 *In the setting of Definition 2.4.1,* $\mathrm{End}(V) \cong \mathrm{Mat}_{d+1}(\mathbb{K})$.

This allows us to discuss Leonard pairs in the context of matrices without issue.

Definition 2.4.3 [42, p. 4] Let A be in $\mathrm{End}(V)$. Then A is *multiplicity-free* whenever it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$.

Lemma 2.4.4 [39, Lemma 1.3] *Let A, B denote a Leonard pair on V. Then A and B are both multiplicity-free.*

Recall that multiplicity-free implies diagonalizable, so there are eigenbases of V for A and B.

Definition 2.4.5 [40], [41], [42], Let A denote a multiplicity-free element of $\mathrm{End}(V)$. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \le i \le d$ let

$$E_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j},$$

where I denotes the identity of $\mathrm{End}(V)$. Then

(i) $AE_i = \theta_i E_i \qquad (0 \le i \le d)$;

(ii) $E_i E_j = \delta_{ij} E_i \qquad (0 \le i, j \le d)$;

(iii) $\sum_{i=0}^{d} E_i = I$;

(iv) $A = \sum_{i=0}^{d} \theta_i E_i$.

We call E_i the *primitive idempotent* of A associated with θ_i.

Lemma 2.4.6 [40, p. 6] *With reference to Definition 2.4.5,*

$$V = E_0 V \oplus E_1 V \oplus \cdots \oplus E_d V,$$

where E_iV is the one-dimensional eigenspace of A in V associated with the eigenvalue θ_i for $0 \le i \le d$. Note that E_i acts on V as the projection onto this eigenspace.

We are now ready to define a Leonard system.

Definition 2.4.7 [39, Definition 1.4] By a *Leonard system* on V of diameter d we mean a sequence

$$\Phi := (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

(i) Each of A, B is a multiplicity-free element in $\mathrm{End}(V)$.

(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of A.

(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is an ordering of the primitive idempotents of B.

(iv)

$$E_iBE_j = \begin{cases} 0 & \text{if } |\mathrm{i}-\mathrm{j}| > 1, \\ \neq 0 & \text{if } |\mathrm{i}-\mathrm{j}| = 1 \end{cases} \qquad (0 \le i, j \le d).$$

(v)

$$E_i^*AE_j^* = \begin{cases} 0 & \text{if } |\mathrm{i}-\mathrm{j}| > 1, \\ \neq 0 & \text{if } |\mathrm{i}-\mathrm{j}| = 1 \end{cases} \qquad (0 \le i, j \le d).$$

We record the relationship between Leonard pairs and Leonard systems in the following theorem.

Theorem 2.4.8 [42, Lemma 3.3] *Let V and $\mathrm{End}(V)$ be as in Definition 2.4.1. Let A and B denote elements of $\mathrm{End}(V)$. Then the pair A, B is a Leonard pair if and only if the following* (i), (ii) *hold.*

(i) *Each of A, B is multiplicity-free.*

(ii) *There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of A and there exists an ordering $E_0^*, E_1^*, \ldots, E_d^*$ of the primitive idempotents of B such that $(A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system on V.*

We recall the notion of isomorphism for Leonard systems.

Definition 2.4.9 [39, Definition 1.5] Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V and let σ denote an isomorphism of $\mathbb{K}$-algebras. Write

$$\Phi^\sigma = (A^\sigma; B^\sigma; \{E_i^\sigma\}_{i=0}^d; \{E_i^{*\sigma}\}_{i=0}^d)$$

and observe Φ^σ is a Leonard system over $\mathbb{K}$. Let Φ and Φ' denote any Leonard systems over $\mathbb{K}$. By an *isomorphism of Leonard systems* from Φ to Φ' we mean an isomorphism of $\mathbb{K}$-algebras such that $\Phi^\sigma = \Phi'$. We say Φ and Φ' are *isomorphic* whenever there exists an isomorphism of Leonard systems from Φ to Φ', and write $\Phi \cong \Phi'$.

Lemma 2.4.10 *Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ be a Leonard system on V, and let β be a basis for V. Define $[\Phi]_\beta = ([A]_\beta; [B]_\beta; \{[E_i]_\beta\}_{i=0}^d; \{[E_i^*]_\beta\}_{i=0}^d)$. Then $[\Phi]_\beta$ is a Leonard system on $\mathbb{K}^{d+1}$ and $[\Phi]_\beta \cong \Phi$. We refer to $[\Phi]_\beta$ as the representation of Φ with respect to the basis β.*

Proof. The map $X \mapsto [X]_\beta$ is a $\mathbb{K}$-algebra isomorphism. The result follows from Definition

2.4.9. □

2.5 The D_4 action

We describe how a permutation of the elements of a given Leonard system can result in an entirely new system.

Lemma 2.5.1 [40, p. 10] *Let Φ denote the Leonard system from Definition 2.4.7. Then each of the following three sequences is also a Leonard system in* $\mathrm{End}(V)$*:*

$$\begin{aligned}\Phi^* &:= (B; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d),\\ \Phi^\downarrow &:= (A; B; \{E_i\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d),\\ \Phi^\Downarrow &:= (A; B; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d).\end{aligned}$$

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{2.5.2}$$

$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow\Downarrow = \Downarrow\downarrow. \tag{2.5.3}$$

The group generated by the symbols $*, \downarrow, \Downarrow$ subject to the relations (2.5.2), (2.5.3) is the dihedral group D_4. We recall D_4 is the group of symmetries of a square, and has 8 elements. Therefore $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all Leonard systems [39], [40], [42], [44], [45]. This idea is summed up in the following theorem.

Theorem 2.5.2 *For any given Leonard system $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ the D_4 action* *induced by* $*, \downarrow, \Downarrow$ *produces the following eight distinct Leonard systems.*

$$
\begin{aligned}
\Phi &:= (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d), & \Phi^{\downarrow} &:= (A; B; \{E_i\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d), \\
\Phi^{\Downarrow} &:= (A; B; \{E_{d-i}\}_{i=0}^d; \{E_i^*\}_{i=0}^d), & \Phi^{\Downarrow\downarrow} &:= (A; B; \{E_{d-i}\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d), \\
\Phi^* &:= (B; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d), & \Phi^{*\downarrow} &:= (B; A; \{E_i^*\}_{i=0}^d; \{E_{d-i}\}_{i=0}^d), \\
\Phi^{*\Downarrow} &:= (B; A; \{E_{d-i}^*\}_{i=0}^d; \{E_i\}_{i=0}^d), & \Phi^{*\downarrow\Downarrow} &:= (B; A; \{E_{d-i}^*\}_{i=0}^d; \{E_{d-i}\}_{i=0}^d).
\end{aligned}
$$

Proof. Clear by Lemma 2.5.1 and our discussion above. □

We emphasize that with the D_4 action a Leonard system can be altered, in particular by a reordering of the primitive idempotents, to form another distinct Leonard system. We will refer to this observation later on in the paper.

2.6 The split decomposition

We recall the first and second split bases of a Leonard system. With respect to both split bases A is lower-bidiagonal and B is upper-bidiagonal. The split bases play a very important role in our work. We refer to the following setup throughout this section.

Definition 2.6.1 [45, Definition 2.1] Let V be a vector space over the field $\mathbb{K}$ with positive dimension $d+1$. Let $\mathrm{End}(V)$ be the set of all linear operators on V. Let A and B denote multiplicity-free elements in $\mathrm{End}(V)$. Let $E_0, E_1, \ldots, E_d$ denote an ordering of the primitive idempotents of A and let θ_i denote the eigenvalue of A for E_i $(0 \leq i \leq d)$. Similarly, let $E_0^*, E_1^*, \ldots, E_d^*$ denote an ordering of the primitive idempotents of B and let θ_i^* denote the eigenvalue of B for E_i^* $(0 \leq i \leq d)$.

By a *decomposition* of V we mean a sequence $U_0, U_1, \ldots, U_d$ consisting of 1-dimensional subspaces of V such that

$$V = U_0 \oplus U_1 \oplus \cdots \oplus U_d \qquad \text{(direct sum)}.$$

We have a comment.

Lemma 2.6.2 [45, p. 5] *Let $u_0, u_1, \ldots, u_d$ denote a basis for V and let U_i denote the subspace of V spanned by u_i $(0 \le i \le d)$. Then the sequence $U_0, U_1, \ldots, U_d$ is a decomposition of V. Conversely, let $U_0, U_1, \ldots, U_d$ denote a decomposition of V. Let u_i denote a nonzero vector in U_i $(0 \le i \le d)$. Then $u_0, u_1, \ldots, u_d$ is a basis for V.*

Definition 2.6.3 [45, Definition 2.2] With reference to Definition 2.6.1, let $U_0, U_1, \ldots, U_d$ denote a decomposition of V. We say this decomposition is *split* with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$ whenever

$$\begin{aligned} (A - \theta_i I)U_i &= U_{i+1} \quad (0 \le i \le d-1), \qquad & (A - \theta_d I)U_d = 0, \\ (B - \theta_i^* I)U_i &= U_{i-1} \quad (1 \le i \le d), \qquad & (B - \theta_0^* I)U_0 = 0. \end{aligned}$$

We emphasize the uniqueness of the split decomposition.

Lemma 2.6.4 [45, Lemma 2.3] *With reference to Definition 2.6.1, the following* (i), (ii) hold.

(i) Assume there exists a decomposition $U_0, U_1, \ldots, U_d$ of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Then

$$U_i = \prod_{h=0}^{i-1}(A - \theta_h I)E_0^* V, \qquad U_i = \prod_{h=i+1}^{d}(B - \theta_h^* I)E_d V \qquad (0 \leq i \leq d).$$

(ii) There exists at most one decomposition of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$.

The next result is useful for finding an explicit expression for the split decomposition in terms of the primitive idempotents of the linear operators A and B.

Lemma 2.6.5 [45, Lemma 2.4(v)] *With reference to Definition 2.6.1, assume there exists a decomposition $U_0, U_1, \ldots, U_d$ of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Then*

$$U_i = (E_0^* V + E_1^* V + \cdots + E_i^* V) \cap (E_i V + E_{i+1} V + \cdots + E_d V) \qquad (0 \leq i \leq d).$$

With respect to a split decomposition, we elaborate on the relationship between Leonard pairs and Leonard systems.

Lemma 2.6.6 [45, Lemma 5.9] *Assume there exists a decomposition of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ of A and $E_0^*, E_1^*, \ldots, E_d^*$ of B. Then the following* (i), (ii) *are equivalent.*

(i) *The pair A, B is a Leonard pair.*

(ii) *The sequence $(A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system.*

Theorem 2.6.7 [45, Theorem 5.1] *With reference to Definition 2.6.1 the sequence $(A;\ B;\ \{E_i\}_{i=0}^d;\ \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if the following* (i) *and* (ii) *hold.*

(i) *There exists a decomposition of V which is split with respect to the orderings E_0, E_1, …, E_d of A and E_0^*, E_1^*, …, E_d^* of B.*

(ii) *There exists a decomposition of V which is split with respect to the orderings E_d, E_{d-1}, …, E_0 of A and E_0^*, E_1^*, …, E_d^* of B.*

2.7 The split decomposition and matrix representations

We now give matrix representations for the operators A and B with respect to a basis for the split decomposition.

Lemma 2.7.1 [45, p. 13] *Let $U_0, U_1, \ldots, U_d$ be a decomposition of V which is split with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$. Let u_i denote a nonzero vector in U_i $(0 \le i \le d)$ and recall $u_0, u_1, \ldots, u_d$ is a basis for V. We normalize the u_i so that $(A - \theta_i I)u_i = u_{i+1}$ $(0 \le i \le d-1)$. With respect to the basis $u_0, u_1, \ldots, u_d$ the matrices which represent A and B are as follows:*

$$A = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad B = \begin{pmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}.$$

The sequence of scalars $\{\varphi_i\}_{i=1}^d$ is called the split sequence *of A, B with respect to the orderings $E_0, E_1, \ldots, E_d$ and $E_0^*, E_1^*, \ldots, E_d^*$.*

We consider the converse to Lemma 2.7.1.

Definition 2.7.2 [39, Definition 4.1] Let d denote a nonnegative integer. Let A and B denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$ of the form

$$A = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad B = \begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix},$$

where

$$\theta_i \neq \theta_j, \qquad \theta^*_i \neq \theta^*_j \quad \text{if} \quad i \neq j \qquad\qquad (0 \leq i, j \leq d),$$

$$\varphi_i \neq 0 \qquad\qquad (1 \leq i \leq d).$$

We observe in Definition 2.7.2 that A and B are multiplicity-free, with eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$ and $\theta^*_0, \theta^*_1, \ldots,\ \theta^*_d$, respectively. For $0 \leq i \leq d$ we let E_i denote the primitive idempotent for A associated with θ_i, and E^*_i denote the primitive idempotent for B associated with θ^*_i.

Lemma 2.7.3 [45, Lemma 6.2] *With reference to Definition 2.7.2, the following* (i), (ii) *are equivalent.*

(i) *The pair A, B is a Leonard pair.*

(ii) *The sequence $(A; B; \{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ is a Leonard system.*

Example 2.7.4 [45, Theorem 6.3] Recall the Leonard system $\Phi^{\Downarrow}$ from Lemma 2.5.1,

$$\Phi^{\Downarrow} = (A; B; \{E_{d-i}\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d}).$$

By Theorem 2.6.7 there exists a unique split decomposition for $\Phi^{\Downarrow}$. With respect to some basis for this split decomposition the matrices which represent A and B are as follows:

$$A = \begin{pmatrix} \theta_d & & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & & \\ & 1 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_0 \end{pmatrix}, \qquad B = \begin{pmatrix} \theta_0^* & \phi_1 & & & & \mathbf{0} \\ & \theta_1^* & \phi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \phi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}.$$

We call the sequence of scalars $\{\phi_i\}_{i=1}^{d}$ the *split sequence* of A, B for $\Phi^{\Downarrow}$ and the *second split sequence* of A, B for $\Phi = (A; B; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$.

2.8 Normalizing bidiagonal matrices

We follow up on some of the results discussed in the previous section to state them in a form more directly applicable to our work. We emphasize that most of the results of this section will be refered to in our main results in Chapters 4 and 5. We begin by making clearer the normalization mentioned in Lemma 2.7.1.

Definition 2.8.1 Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ be a Leonard system on V with split decomposition $\{U_i\}_{i=0}^d$.

(i) By an *LB-UB basis for* Φ, we mean a basis $\beta = \{b_0, b_1, \ldots b_d\}$ for V where $b_i \in U_i$.

(ii) By a *split basis for* Φ, we mean an LB-UB basis γ such that the subdiagonal entries of $[A]_\gamma$ are all 1.

Lemma 2.8.2 *Let Φ be as in Definition 2.8.1. If β is an LB-UB basis for Φ, then $[A]_\beta$, $[B]_\beta$ are lower- and upper-bidiagonal, respectively.*

Proof. Straightforward from Definition 2.6.3. □

Lemma 2.8.3 *Let Φ be a Leonard system. Suppose $\beta = \{b_0, b_1, \ldots b_d\}$ is an LB-UB basis for Φ and the i^{th} subdiagonal entry of $[A]_\beta$ is t_i $(1 \leq i \leq d)$. Let $\alpha_i = \prod_{j=1}^{i} t_j$, with $\alpha_0 = 1$. Then $\gamma = \{b_0, \alpha_1 b_1, \ldots . \alpha_d b_d\}$ is a split basis for Φ.*

Proof. The change of basis matrix from β to γ is $P = \mathrm{diag}(1, \alpha_1, \alpha_2, \ldots, \alpha_d)$. Conjugating by this matrix gives the i^{th} subdiagonal entry of $[A]_\gamma$ to be $\alpha_{i-1}(\alpha_i)^{-1}t_i = \prod_{j=1}^{i-1} t_j \left(\prod_{j=1}^{i} t_j\right)^{-1} t_i = 1$ for $1 \leq i \leq d$. Thus the subdiagonal entries of $[A]_\gamma$ are all 1. □

Theorem 2.8.4 *Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ be a Leonard system on V where $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ are the eigenvalue and dual eigenvalue sequences of Φ, respectively. Let $\beta = \{b_0, b_1, \ldots b_d\}$ be a basis for V such that $[A]_\beta$ is lower-bidiagonal with (i,i)-entry θ_i and $[B]_\beta$ is upper-bidiagonal with (i,i)-entry θ_i^* for $0 \leq i \leq d$. Then β is an LB-UB basis for Φ. Moreover, $\{U_i\}_{i=0}^d$ is the split decomposition for Φ, where $U_i = \mathrm{span}\{b_i\}$.*

Proof. Suppose the i^{th} subdiagonal entry of $[A]_\beta$ is $t_i \neq 0$ $(1 \leq i \leq d)$. By Lemma 2.8.3 the change of basis matrix between an LB-UB basis for Φ and a split basis for Φ is $\mathrm{diag}(1, \alpha_1, \alpha_2, \ldots, \alpha_d)$ where $\alpha_i = \prod_{j=1}^{i} t_j$. Conjugation by this matrix will preserve diagonal entries, in particular, the eigenvalue and dual eigenvalue sequences for A and B respectively. The split basis is of the form $\{b_0, \alpha_1 b_1, \ldots . \alpha_d b_d\}$. By basic linear algebra we have $\mathrm{span}\{\alpha_i b_i\} = \mathrm{span}\{b_i\} = U_i$ $(0 \leq i \leq d)$ and the result follows. □

Merely requiring $[A]_\beta$ and $[B]_\beta$ to be lower- and upper-bidiagonal is insufficient to reach the conclusion in Theorem 2.8.4. We need the eigenvalues along the diagonal of each in order to claim that β is an LB-UB basis.

2.9 The parameter array

We end this chapter by discussing an alternative way of describing a Leonard system, that is, with a sequence of scalars called the parameter array. These scalars are easily described, and appear naturally in the matrix representations for the linear operators of a Leonard pair with respect to the split decomposition.

Definition 2.9.1 [44, Definition 1.1] Let d denote a nonnegative integer. By a *parameter array over* $\mathbb{K}$ *of diameter* d we mean a sequence of scalars

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$$

taken from $\mathbb{K}$ which satisfy the following conditions:

(PA1) $\theta_i \neq \theta_j, \qquad \theta_i^* \neq \theta_j^* \qquad \text{if} \quad i \neq j \qquad (0 \leq i, j \leq d).$

(PA2) $\varphi_i \neq 0, \qquad \phi_i \neq 0 \qquad\qquad (1 \leq i \leq d).$

(PA3) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d).$

(PA4) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \qquad (1 \leq i \leq d).$

(PA5) The expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{2.9.4}$$

are equal and independent of i for $2 \leq i \leq d-1$. We call the value of these expressions the *common value*.

For notational convenience we define $\varphi_0 = 0$, $\varphi_{d+1} = 0$, $\phi_0 = 0$, $\phi_{d+1} = 0$.

Definition 2.9.2 [44, p. 4] Let Φ denote a Leonard system. By the *parameter array of* Φ we mean the parameter array

$$(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$$

where $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$ are the eigenvalue and dual eigenvalue sequences of Φ, respectively. We call $\{\varphi_j\}_{j=1}^d$ the *first split sequence* of Φ, and $\{\phi_j\}_{j=1}^d$ the *second split sequence* of Φ.

We remark that the first and second split sequences get their names from their appearance in the matrix representations with respect to the split decomposition, as seen in Section 2.7. The significance of the parameter array is the following:

Theorem 2.9.3 [44, Theorem 2.1] *Two Leonard systems over* $\mathbb{K}$ *are isomorphic if and only if they have the same parameter array.*

The result reduces the linear algebraic problem of isomorphism to a problem of finding solutions to the constraining equations of a parameter array. We shall take advantage of this in the next several chapters.

We turn our attention now to the *type* of a parameter array. Recall from Section 2.3 our discussion of types of Leonard pairs. These are synonymous. In fact, given a parameter array all associated Leonard pairs and Leonard systems are said to be of the same type as the parameter array, and vice versa. In the following theorem we see that the type of a parameter array depends on its eigenvalue sequences.

Theorem 2.9.4 [2], [44] *Given a parameter array over* $\mathbb{K}$ *with* $d \geq 3$*, let* β *be the common value of (2.9.4) minus 1.*

(i) *If* $\beta = 2$ *and* $\mathrm{char}(\mathbb{K}) \neq 2$*, then the parameter array is of classical type.*

(ii) *If* $\beta \neq \pm 2$ *and* $\mathrm{char}(\mathbb{K}) \neq 2$*, then the parameter array is of basic type.*

(iii) *If* $\beta = -2$ *and* $\mathrm{char}(\mathbb{K}) \neq 2$*, then the parameter array is of Bannai-Ito type.*

(iv) *If* $\mathrm{char}(\mathbb{K}) = 2$ *then* $\beta = 0$ *and the parameter array is of orphan type.*

When $d \leq 2$, β is not defined by (2.9.4). However, in this case β may be taken to have any value and the parameter array can be expressed as several types. It is customary to take $\beta = 2$ when $d \leq 2$. When $d = 1$, we view the parameter array as being of Krawtchouk type [2]. When $d = 2$, the type will depend upon the spacing of the eigenvalue sequences.

In Chapter 3 we shall recall uniform constructions of all Leonard pairs of classical type from sl_2 and of all Leonard pairs of basic type from $U_q(sl_2)$.

3 Representations of sl_2 and $U_q(sl_2)$

In this chapter we recall from the literature a construction of Leonard pairs from the algebras sl_2 and $U_q(sl_2)$. First, it turns out that all Leonard pairs of classical and basic types are constructed uniformly from these respective algebras. Second, the construction returns the Leonard pair in a lower-bidiagonal - upper-bidiagonal form related to its representation relative to a split basis. These will be important to our description of Leonard systems which are friends.

3.1 The Lie algebra sl_2

There is an extensive theory behind the Lie algebra sl_2. We need only a few basic facts which can be found in most texts on the subject. See [20], [22], and [27] for example.

Definition 3.1.1 [10, p. 2] The Lie algebra sl_2 is the $\mathbb{K}$-algebra that has a basis e, f, h satisfying

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h,$$

where $[\cdot, \cdot]$ denotes the Lie bracket. The elements e, f, h are known as the *Chevalley generators* for sl_2.

It is more convenient to work with an alternate, symmetric presentation of sl_2 rather than the Chevalley presentation of Definition 3.1.1.

Lemma 3.1.2 [10, p. 537] *With reference to Definition 3.1.1, let*

$$x = 2e - h, \qquad y = -2f - h, \qquad z = h.$$

Then x, y, z is a basis for sl_2 and

$$\begin{aligned}
[x,y] &= 2x + 2y, \\
[y,z] &= 2y + 2z, \\
[z,x] &= 2z + 2x.
\end{aligned}$$

We call x, y, z the equitable basis *for the Lie algebra sl_2.*

One advantage of the equitable basis is the cyclic shift $x \mapsto y \mapsto z \mapsto x$ defines an automorphism of sl_2 [10]. We may apply this automorphism to each result involving the equitable basis.

Lemma 3.1.3 [10, p. 651] *With reference to Lemma 3.1.2, there is an irreducible finite-dimensional sl_2-module V_d with basis $v_0, v_1, \ldots, v_d$ and action*

$$\begin{aligned}
xv_0 &= -dv_0, \\
xv_i &= (2i-d)v_i + 2(d-i+1)v_{i-1} \qquad (1 \leq i \leq d), \\
yv_i &= (2i-d)v_i - 2(i+1)v_{i+1} \qquad (0 \leq i \leq d-1), \\
yv_d &= dv_d, \\
zv_i &= (d-2i)v_i \qquad (0 \leq i \leq d),
\end{aligned}$$

where I is the identity operator on V_d.

We represent this action pictorially in Figure 3.1.

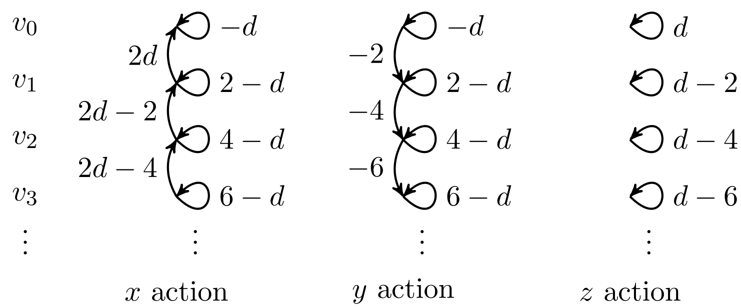

Figure 3.1: The action of x, y, z on the sl_2-module V_d

3.2 Leonard pairs of classical type from sl_2

In order to construct a Leonard pair on V_d using the equitable basis for sl_2, we restrict the action of the equitable basis elements x, y, z to V_d.

Definition 3.2.1 With reference to Lemmas 3.1.2 and 3.1.3, let X, Y, Z be the linear operators on V_d which act as x, y, z, respectively.

We now define operators A and B in terms of the identity operator I and X, Y, Z and describe their actions on V_d.

Definition 3.2.2 [2, Definitions 5.1, 5.2] Pick $A \in \text{span}\{I, Y, Z, YZ\}$ and $B \in \text{span}\{I, Z, X, ZX\}$. Write

$$A = \kappa I + \lambda Y + \mu Z + \nu YZ, \qquad B = \kappa^* I + \lambda^* Z + \mu^* X + \nu^* ZX$$

for scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, $\nu^* \in \mathbb{K}$. Define

$$\theta_i = \kappa - (\lambda - \mu)(d-2i) - (d-2i)^2\nu \qquad (0 \leq i \leq d), \tag{3.2.1}$$

$$\theta^*_i = \kappa^* + (\lambda^* - \mu^*)(d-2i) - (d-2i)^2\nu^* \qquad (0 \leq i \leq d). \tag{3.2.2}$$

Lemma 3.2.3 [2, Lemma 5.3] *Let $\{v_i\}_{i=0}^d$ be the basis of V_d from Lemma 3.1.3, and A, B be as in Definition 3.2.2. Then the pair A, B act on the sl_2-module V_d in the following way.*

$$\begin{aligned}
Av_i &= \theta_i v_i - 2(i+1)(\lambda + (d-2i)\nu)v_{i+1} && (0 \leq i \leq d-1),\\
Av_d &= \theta_d v_d,\\
Bv_0 &= \theta^*_0 v_0,\\
Bv_i &= 2(d-i+1)(\mu^* + (d-2(i-1))\nu^*)v_{i-1} + \theta^*_i v_i && (1 \leq i \leq d).
\end{aligned}$$

Observe that the matrix representations for A, B with respect to the basis $\{v_i\}_{i=0}^d$ are respectively lower-bidiagonal and upper-bidiagonal. This resemblance to a Leonard pair goes further. Up to isomorphism, every Leonard pair of classical type arises from this construction with an appropriate choice of parameters.

Theorem 3.2.4 [2, Theorem 6.10, 7.1] *With reference to Definition 3.2.2, the pair A, B*

acts on V_d as a Leonard pair of classical type if and only if

$$\begin{aligned}
\lambda - \mu + 2(d-i)\nu &\neq 0 \qquad (1 \leq i \leq 2d-1), && (3.2.3)\\
\lambda^* - \mu^* + 2(d-i)\nu^* &\neq 0 \qquad (1 \leq i \leq 2d-1), && (3.2.4)\\
\lambda - (d-2j)\nu &\neq 0 \qquad (1 \leq j \leq d), && (3.2.5)\\
\mu^* - (d-2j)\nu^* &\neq 0 \qquad (1 \leq j \leq d), && (3.2.6)\\
(\lambda + d\nu)(\mu^* + d\nu^*) &\neq -(\lambda - \mu + 2(j-1)\nu)(\lambda^* - \mu^* - 2(d-j)\nu^*) && (3.2.7)\\
&\qquad (1 \leq j \leq d), &&\\
\lambda^*\nu + \mu\nu^* + 2\nu\nu^* &= 0. && (3.2.8)
\end{aligned}$$

This construction of Leonard pairs of classical types gives rise to every such Leonard pair. This allows us to study Leonard pairs of classical type using the representation theory of sl_2.

Theorem 3.2.5 [2, Lemmas 9.1-9.4] *Assume $d \geq 2$. Let A, B be a Leonard pair of classical type on the vector space V. Then there exists an irreducible sl_2-module structure on V such that A and B act on V as linear combinations of $\{I, Y, Z, YZ\}$ and $\{I, Z, X, ZX\}$, respectively.*

We now recall the parameter restrictions which give rise to each of the classical types, namely Racah, Hahn, dual Hahn, and Krawtchouk.

Theorem 3.2.6 [2, Theorem 7.2, Lemmas 7.3 - 7.5] *Let A, B be a Leonard pair of classical type. In light of Theorem 3.2.5, we may assume A and B are as in Definition 3.2.2 and the conditions given in Theorem 3.2.4 hold.*

(i) *A, B is a Leonard pair of Krawtchouk type if and only if*

$$\mu \neq \lambda, \quad \mu^* \neq \lambda^*, \quad \lambda \neq 0, \quad \mu^* \neq 0, \quad \nu = 0, \quad \nu^* = 0, \quad \lambda\lambda^* - \mu\lambda^* + \mu\mu^* \neq 0.$$

(ii) *A, B is a Leonard pair of Hahn type if and only if*

$$\lambda \neq 0, \qquad \mu = 0, \qquad \nu = 0, \qquad \nu^* \neq 0,$$

$$\mu^* - (d-2i)\nu^* \neq 0, \qquad \lambda^* - (d-2i)\nu^* \neq 0 \qquad (1 \leq i \leq d),$$

$$\lambda^* - \mu^* + 2\nu^*(d-i) \neq 0, \qquad (1 \leq i \leq 2d-1).$$

(iii) *A, B is a Leonard pair of dual Hahn type if and only if*

$$\lambda^* = 0, \qquad \nu \neq 0, \qquad \mu^* \neq 0, \qquad \nu^* = 0,$$

$$\lambda - (d-2i)\nu \neq 0, \qquad \mu - (d-2i)\nu \neq 0 \qquad (1 \leq i \leq d),$$

$$\lambda - \mu + 2\nu(d-i) \neq 0, \qquad (1 \leq i \leq 2d-1).$$

(iv) *A, B is a Leonard pair of Racah type if and only if $\nu \neq 0$, $\nu^* \neq 0$ and (3.2.3) − (3.2.8) hold.*

We shall refer to Theorem 3.2.6 when further describing Leonard pairs of classical type in the subsequent chapters. Now, we present another way of describing a Leonard system using scalars from the sl_2 construction.

Lemma 3.2.7 *Given a Leonard system $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V of classical* *type with diameter $d \geq 3$, there exists an sl_2-module structure on V and unique scalars $(d, 1, \kappa, \lambda, \mu, \nu, \kappa^*, \lambda^*, \mu^*, \nu^*)$ such that A, B are as in Definition 3.2.2 and E_i, E_i^* are as in Definition 2.4.5.*

Proof. If A, B form a Leonard pair of classical type, then an sl_2-module structure is guaranteed by Theorem 3.2.5. In Definition 3.2.2 expressions for the eigenvalues of A and B are uniquely expressed in terms of the scalars d, κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^*. By Definition 2.4.5, the idempotents $\{E_i\}_{i=0}^d$ and $\{E_i^*\}_{i=0}^d$ are determined by the eigenvalues of A and B, respectively. □

Definition 3.2.8 With reference to Lemma 3.2.7 we call $(d,\ 1,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \kappa^*,\ \lambda^*,\ \mu^*,\ \nu^*)$ the *Leonine parameters for* Φ relative to the sl_2-module structure on V.

We can fix an sl_2-module structure on V and produce a representation of every isomorphism class of Leonard system of classical type by some appropriate choice of parameter. Our next goal is describe Leonard pairs of basic type using representations of $U_q(sl_2)$ in the same way.

3.3 The quantum algebra $U_q(sl_2)$

The quantum algebra $U_q(sl_2)$ has been heavily studied since its introduction by Michio Jimbo in the 1980s. For the purposes of this paper, we need only a few basic facts which can be found in most texts on the subject. See [12], [13], [14], and [26] for further study.

Definition 3.3.1 [25, Definition 1.1] Let q be a nonzero scalar in the field $\mathbb{K}$ such that q is not a root of unity of $\mathbb{K}$. Let $U_q(sl_2)$ denote the unital associative $\mathbb{K}$-algebra with generators

k, k^{-1}, e, f and the following relations:

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1, \\ ke &= q^2ek, \\ kf &= q^{-2}fk, \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}. \end{aligned}$$

The elements k, k^{-1}, e, f are known as the *Chevalley generators* for $U_q(sl_2)$.

Theorem 3.3.2 [25, Theorem 2.1] *The algebra $U_q(sl_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators x, x^{-1}, y, z and the following relations:*

$$\begin{aligned} xx^{-1} = x^{-1}x &= I, \\ \frac{qxy - q^{-1}yx}{q - q^{-1}} &= I, \\ \frac{qyz - q^{-1}zy}{q - q^{-1}} &= I, \\ \frac{qzx - q^{-1}xz}{q - q^{-1}} &= I. \end{aligned}$$

We call x, x^{-1}, y, z the equitable generators *for $U_q(sl_2)$.*

Lemma 3.3.3 [25, Lemma 3.1] *With reference to Theorem 3.3.2, up to isomorphism there are two irreducible finite-dimensional $U_q(sl_2)$-modules V_d^+, V_d^- of dimension $d + 1$. For*

$\varepsilon \in \{+,-\}$, V_d^ε *has a basis* $\{v_i^\varepsilon\}_{i=0}^d$ *with action*

$$
\begin{aligned}
\varepsilon x v_0^\varepsilon &= q^d v_0^\varepsilon, \\
\varepsilon x v_i^\varepsilon &= (q^d - q^{2i-2-d}) v_{i-1}^\varepsilon + q^{d-2i} v_i^\varepsilon \qquad (1 \le i \le d), \\
\varepsilon y v_i^\varepsilon &= q^{d-2i}(q^{-2i-2} - 1) v_{i+1}^\varepsilon + q^{d-2i} v_i^\varepsilon \qquad (0 \le i \le d-1), \\
\varepsilon y v_d^\varepsilon &= q^{-d} v_d^\varepsilon, \\
\varepsilon z v_i^\varepsilon &= q^{2i-d} v_i^\varepsilon \qquad (0 \le i \le d).
\end{aligned}
$$

We shall focus on V_d^+, and simply write V_d for this module. See Figure 3.2.

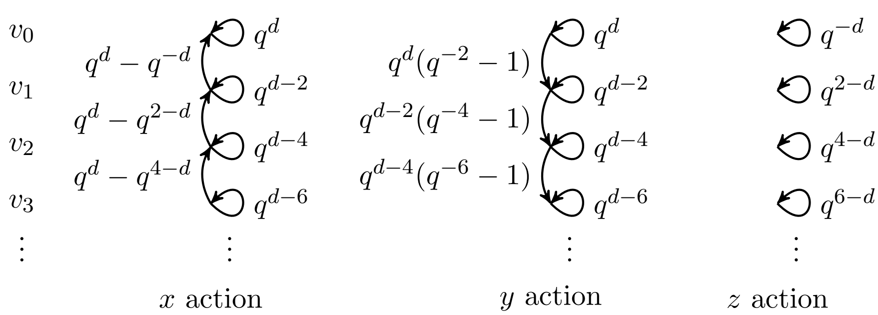

Figure 3.2: The action of x, y, z on the $U_q(sl_2)$-module V_d

3.4 Leonard pairs of basic type from $U_q(sl_2)$

We recall the construction of Leonard pairs of basic type from $U_q(sl_2)$. We proceed as we did in the classical case.

Definition 3.4.1 With reference to Theorem 3.3.2 and Lemma 3.3.3, let X, Y, Z be the linear operators on V_d which act as x, y, z, respectively.

Lemma 3.4.2 [25] *With reference to Definition 3.4.1, X, Y, Z are invertible elements of* $\mathrm{End}(V_d)$.

The cyclic shift $X \mapsto Y \mapsto Z \mapsto X$ of the equitable generators defines an automorphism of $\mathrm{End}(V_d)$. We may apply this automorphism to each result involving the equitable generators.

We now define A and B in terms of the identity operator I and X, Y, Z and describe their actions on V_d.

Definition 3.4.3 [1, Definition 5.1, Lemma 5.2] Pick $A \in \mathrm{span}\{I, Y, Z, YZ\}$ and $B \in \mathrm{span}\{I, Z, X, ZX\}$. Write

$$A = \kappa I + \lambda Y + \mu Z + \nu YZ, \qquad B = \kappa^* I + \lambda^* Z + \mu^* X + \nu^* ZX$$

for scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, $\nu^* \in \mathbb{K}$. We also define

$$\theta_i = \kappa + \nu + \lambda q^{d-2i} + \mu q^{2i-d} \qquad (0 \le i \le d), \tag{3.4.9}$$

$$\theta^*_i = \kappa^* + \nu^* + \mu^* q^{d-2i} + \lambda^* q^{2i-d} \qquad (0 \le i \le d). \tag{3.4.10}$$

Lemma 3.4.4 [1, Lemma 5.2] *Let $\{v_i\}_{i=0}^d$ be the basis of V_d from Lemma 3.3.3, and A, B be as in Definition 3.4.3. Then the pair A, B act on the $U_q(sl_2)$-module V_d in the following way.*

$$\begin{aligned}
Av_i &= \theta_i v_i + (q^{-2(i+1)} - 1)(\lambda q^{d-2i} + \nu) v_{i+1} && (0 \le i \le d-1),\\
Av_d &= \theta_d v_d,\\
Bv_0 &= \theta^*_0 v_0,\\
Bv_i &= \theta^*_i v_i - q^{2(i-1)}(q^{-2(d-i+1)} - 1)(\mu^* q^{d-2i+2} + \nu^*) v_{i-1} && (1 \le i \le d).
\end{aligned}$$

Observe that the matrix representations for A, B with respect to the basis $\{v_i\}_{i=0}^d$ are respectively lower-bidiagonal and upper-bidiagonal. This resemblance to a Leonard pair goes further. Up to isomorphism, every Leonard pair of basic type arises from this construction with an appropriate choice of parameters.

Theorem 3.4.5 [1, Theorem 4.1, 4.2] *With reference to Definition 3.4.3, the pair A, B acts on V_d as a Leonard pair of basic type if and only if*

$$\begin{aligned}
\mu q^{2(i-d)} - \lambda &\neq 0 \qquad (1 \leq i \leq 2d-1), && (3.4.11)\\
\lambda^* q^{2(i-d)} - \mu^* &\neq 0 \qquad (1 \leq i \leq 2d-1), && (3.4.12)\\
\lambda q^{d-2i} + \nu &\neq 0 \qquad (1 \leq i \leq d-1), && (3.4.13)\\
\mu^* q^{d-2i} + \nu^* &\neq 0 \qquad (1 \leq i \leq d-1), && (3.4.14)\\
q^{-2}(\lambda q^d + \nu)(\mu^* q^d + \nu^*) &\neq (\mu - \lambda q^{2(i-1)})(\lambda^* - \mu^* q^{2(d-i)}) && (3.4.15)\\
&\qquad (1 \leq i \leq d),\\
\mu\lambda^* q^2 - \nu\nu^* &= 0. && (3.4.16)
\end{aligned}$$

This construction of Leonard pairs of basic types gives rise to every such Leonard pair. This allows us to study Leonard pairs of basic type using the representation theory of $U_q(sl_2)$.

Theorem 3.4.6 [1, Theorem 4.2] *Assume $d \geq 2$. Let A, B be a Leonard pair of basic type on the vector space V. Then there exists an irreducible $U_q(sl_2)$-module structure on V such that A and B act on V as linear combinations of $\{I, Y, Z, YZ\}$ and $\{I, Z, X, ZX\}$, respectively.*

We now recall the parameter restrictions which give rise to each of the basic types, namely

q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, and affine q-Krawtchouk.

Lemma 3.4.7 [1, Lemmas 7.2 - 7.5] *Let A, B be a Leonard pair of basic type. In light of Theorem 3.4.6, we may assume A and B are as in Definition 3.4.3.*

(i) *A, B is a Leonard pair of q-Racah type if and only if $\nu\nu^* \neq 0$ and (3.4.11) $-$ (3.4.16) hold.*

(ii) *A, B is a Leonard pair of q-Hahn or q-Krawtchouk type if and only if*

$$\mu = 0, \qquad \lambda \neq 0, \qquad \nu\nu^* = 0,$$

$$\begin{aligned} q^{d-2i}\lambda + \nu &\neq 0, & q^{d-2i}\mu^* + \nu^* &\neq 0 & (0 \leq i \leq d-1), \\ \mu^* - q^{2(i-d)}\lambda^* &\neq 0, & \lambda\nu^* + \mu^*\nu + q^{2i-d}\lambda\lambda^* &\neq 0 & (1 \leq i \leq d). \end{aligned}$$

(iii) *A, B is a Leonard pair of dual q-Hahn or dual q-Krawtchouk type if and only if*

$$\lambda^* = 0, \qquad \mu^* \neq 0, \qquad \nu\nu^* = 0,$$

$$\begin{aligned} q^{d-2i}\lambda + \nu &\neq 0, & q^{d-2i}\mu^* + \nu^* &\neq 0 & (0 \leq i \leq d-1), \\ \lambda - q^{2(i-d)}\mu &\neq 0, & \lambda\nu^* + \mu^*\nu + q^{d-2i+2}\mu\mu^* &\neq 0 & (1 \leq i \leq d). \end{aligned}$$

(iv) *A, B is a Leonard pair of quantum q-Krawtchouk type if and only if*

$$\lambda = 0, \qquad \lambda^* = 0, \qquad \mu \neq 0, \qquad \mu^* \neq 0, \qquad \nu \neq 0, \qquad \nu^* = 0,$$

$$\mu q^{d-2i+2} + \nu \neq 0 \qquad (1 \leq i \leq d).$$

(v) *A, B is a Leonard pair of affine q-Krawtchouk type if and only if*

$$\mu = 0, \qquad \lambda^* = 0, \qquad \mu^* \neq 0, \qquad \lambda \neq 0, \qquad \nu\nu^* = 0, \qquad \lambda\nu^* + \mu^*\nu \neq 0,$$

$$q^{d-2i}\lambda + \nu \neq 0, \qquad q^{d-2i}\mu^* + \nu^* \neq 0 \qquad (0 \leq i \leq d-1).$$

We shall refer to Lemma 3.4.7 when further describing Leonard pairs of basic type in the subsequent chapters. Now, we present another way of describing a Leonard system using scalars from the $U_q(sl_2)$ construction.

Lemma 3.4.8 *Given a Leonard system $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V of basic type with diameter $d \geq 3$, there exists a $U_q(sl_2)$-module structure on V and unique scalars $(d, q, \kappa, \lambda, \mu, \nu, \kappa^*, \lambda^*, \mu^*, \nu^*)$ such that A, B are as in Definition 3.4.3 and E_i, E_i^* are as in Definition 2.4.5.*

Proof. If A, B form a Leonard pair of basic type, then a $U_q(sl_2)$-module structure is guaranteed by Theorem 3.4.6. In Definition 3.4.3 expressions for the eigenvalues of A and B are uniquely expressed in terms of the scalars $(d,\ q,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \kappa^*,\ \lambda^*,\ \mu^*,\ \nu^*)$. By Definition 2.4.5 the idempotents $\{E_i\}_{i=0}^d$ and $\{E_i^*\}_{i=0}^d$ are determined by the eigenvalues of A and B, respectively. $\square$

Definition 3.4.9 With reference to Lemma 3.4.8 we call $(d,\ q,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \kappa^*,\ \lambda^*,\ \mu^*,\ \nu^*)$ the *Leonine parameters for* Φ relative to the $U_q(sl_2)$-module structure on V.

In the next section we unify Lemmas 3.2.7 and 3.4.8.

3.5 A unified approach

Lemmas 3.2.7 and 3.4.8 give an alternative way of describing Leonard systems in the context of the equitable generators for sl_2 and $U_q(sl_2)$. We shall lean heavily upon the constructions of Leonard pairs of classical and basic type from sl_2 and $U_q(sl_2)$ in the sequel. These constructions allow us to control both the isomorphism class and the shape of a Leonard pair. We begin by giving a single definition to subsume Definitions 3.2.8 and 3.4.9.

Definition 3.5.1 Given a sequence $(d, q, \kappa, \lambda, \mu, \nu, \kappa^*, \lambda^*, \mu^*, \nu^*)$ with $d \in \mathbb{Z}^+$ and q, κ, λ, μ, ν, κ^*, λ^*, μ^*, $\nu^* \in \mathbb{K}$, we say that it is *Leonine* when there exists an irreducible $sl_2/U_q(sl_2)$-module V with dimension $d+1$, and the linear combinations

$$A = \kappa I + \lambda Y + \mu Z + \nu YZ, \quad B = \kappa^* I + \lambda^* Z + \mu^* X + \nu^* ZX$$

involving the equitable generators X, Y, Z of sl_2 or $U_q(sl_2)$ form a Leonard pair. We say this Leonard pair *corresponds* with the Leonine parameters $(d, q, \kappa, \lambda, \mu, \nu, \kappa^*, \lambda^*, \mu^*, \nu^*)$.

Now Lemmas 3.2.7 and 3.4.8 are restated as follows.

Theorem 3.5.2 *The sequence of scalars $(d, q, \kappa, \lambda, \mu, \nu, \kappa^*, \lambda^*, \mu^*, \nu^*)$ is Leonine if and only if the scalars satisfy Lemma 3.2.7 or Lemma 3.4.8.*

Leonine parameters provide an alternative description of Leonard systems.

Corollary 3.5.3 *There is a bijection between the set of Leonine parameters and the isomorphism classes of Leonard systems of classical and basic type.*

Proof. Lemma 3.2.7 gives all Leonard pairs of classical type. Lemma 3.4.8 gives all Leonard systems of basic type. $\square$

Definition 3.5.4 Let Φ be a Leonard system on V. We say that Φ and $(d,\ q,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \kappa^*,\ \lambda^*,\ \mu^*,\ \nu^*)$ *correspond* when $(d,\ q,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \kappa^*,\ \lambda^*,\ \mu^*,\ \nu^*)$ are the Leonine parameters for Φ, relative to some fixed module structure on V.

We can now control the isomorphism class of a Leonard system with the Leonine parameters, and the shape by the choice of module structure.

4 Friendship

In this chapter we discuss our own research results. In Section 4.1 we present the motivating problem for the paper, derived from a problem originally posed by Terwilliger in [46]. We work to restate the problem, defining a relationship between two Leonard systems called *friendship* and equating the problem to a search for friends. We say two Leonard systems are *friends* if they share a linear operator, an ordering of the primitive idempotents for that operator, and split decomposition.

The following section discusses friendship in a bit more detail. We see that friendship is an equivalence relation among Leonard systems and that any affine transformation of a given Leonard system will result in friendship.

In Section 4.4 we acknowledge that friendship depends on the choice of representation. We say two Leonard systems are *acquaintances* if they share a linear operator and an ordering of the primitive idempotents for that operator. We investigate conditions for acquaintances to be friends. We see that if we fix the shape of the matrix representations for all three linear operators in a particular way, that friendship will result. These required shapes are inherently provided by using the equitable representations for sl_2 and $U_q(sl_2)$.

4.1 Motivation

Our motivation is a problem posed by Paul Terwilliger. Throughout, we assume V is a $(d+1)$-dimensional vector space over the field $\mathbb{K}$.

Problem 4.1.1 [46, Problem 36.102] Given a Leonard system $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V, find all Leonard systems $\hat{\Phi} = (A, \hat{B}, \{E_i\}_{i=0}^d, \{\hat{E}_i^*\}_{i=0}^d)$ on V such that

$$E_0^*V + E_1^*V + \cdots + E_i^*V = \hat{E}_0^*V + \hat{E}_1^*V + \cdots + \hat{E}_i^*V. \tag{4.1.1}$$

To avoid degenerate situations, we shall assume $d \geq 3$ when solving problem 4.1.1. We emphasize that Φ and $\hat{\Phi}$ share the same linear operator A and ordering of primitive idempotents $\{E_i\}_{i=0}^d$. With this we can see that condition (4.1.1) has implications for the split decompositions of the two Leonard systems. Recall from Lemma 2.6.5 that a Leonard system $(A, B, \{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d)$ has unique split decomposition

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + \cdots + E_dV) \qquad (0 \leq i \leq d). \tag{4.1.2}$$

Lemma 4.1.2 *Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ be a Leonard system on V. Let $\{U_i\}_{i=0}^d$ be the split decomposition for Φ. Then for $0 \leq i \leq d$,*

$$U_0 + \cdots + U_i = E_0^*V + \cdots + E_i^*V.$$

Proof. Using (4.1.2) and Lemma 2.4.6 observe that

$$U_0 = E_0^*V \cap (E_0V + \cdots + E_dV) = E_0^*V \cap V = E_0^*V.$$

Suppose by induction that for $0 < i \leq d$, $U_0 + \cdots + U_{i-1} = E_0^*V + \cdots + E_{i-1}^*V$. Then

$$
\begin{aligned}
U_0 + \cdots + U_i &= E_0^*V + \cdots + E_{i-1}^*V + U_i \\
&= E_0^*V + \cdots + E_{i-1}^*V + (E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV) \\
&= E_0^*V + \cdots + E_{i-1}^*V + E_i^*V \cap (E_iV + E_{i+1}V + \cdots + E_dV).
\end{aligned}
$$

We note that the left-hand side has dimension $i+1$ since each U_i has dimension 1 by Definition 2.6.1. Hence the right-hand side has dimension $i+1$. Since the E_j^*V $(0 \leq j \leq d)$ are distinct with dimension 1 (Lemma 2.4.6), it must be that $E_i^*V \cap (E_iV + E_{i+1}V + \cdots + E_dV)$ has dimension 1. This space is contained in the 1-dimensional subspace E_i^*V. Therefore any generator of $E_i^*V \cap (E_iV + E_{i+1}V + \cdots + E_dV)$ will generate the entire space E_i^*V, so the spaces are equal. Thus $U_0 + \cdots + U_i = E_0^*V + \cdots + E_i^*V$. □

Theorem 4.1.3 *Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ and $\hat{\Phi} = (A, \hat{B}, \{E_i\}_{i=0}^d, \{\hat{E}_i^*\}_{i=0}^d)$ be two Leonard systems on V. The following are equivalent.*

(i) *Φ and $\hat{\Phi}$ satisfy $E_0^*V + E_1^*V + \cdots + E_i^*V = \hat{E}_0^*V + \hat{E}_1^*V + \cdots + \hat{E}_i^*V$.*

(ii) *Φ and $\hat{\Phi}$ have the same split decomposition.*

Proof. Suppose (i) holds. By assumption, Φ and $\hat{\Phi}$ have the same orderings of the primitive idempotents for A. Thus these Leonard systems have the same unique split decomposition by Lemma 2.6.5. Now suppose (ii) holds; say Φ and $\hat{\Phi}$ have same split decomposition $\{U_i\}_{i=0}^d$. Then by Lemma 4.1.2, $E_0^*V + \cdots + E_i^*V = U_0 + \cdots + U_i = \hat{E}_0^*V + \hat{E}_1^*V + \cdots + \hat{E}_i^*V$ $(0 \leq i \leq d)$.

□

Definition 4.1.4 Any two Leonard systems that satisfy the hypotheses and equivalent conditions of Theorem 4.1.3 are called *friends*. Given Leonard systems Φ and $\hat{\Phi}$ which are friends, we say $\hat{\Phi}$ is a *friend of* Φ.

Using Theorem 4.1.3 and Definition 4.1.4, we restate problem 4.1.1.

Corollary 4.1.5 *Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ be a Leonard system on V. The following sets are equal.*

(i) $\{\hat{\Phi} = (A, \hat{B}, \{E_i\}_{i=0}^d, \{\hat{E}_i^*\}_{i=0}^d)$ *is a Leonard system on* $V \mid$ (4.1.1) holds$\}$.

(ii) $\{\hat{\Phi} = (A, \hat{B}, \{E_i\}_{i=0}^d, \{\hat{E}_i^*\}_{i=0}^d)$ *is a Leonard system on* $V \mid \Phi, \hat{\Phi}$ *have the same split decomposition*$\}$.

(iii) $\{\hat{\Phi} = (A, \hat{B}, \{E_i\}_{i=0}^d, \{\hat{E}_i^*\}_{i=0}^d)$ *is a Leonard system on* $V \mid \Phi, \hat{\Phi}$ *are friends*$\}$, *i.e., the set of all friends of* Φ.

Part (ii) of Corollary 4.1.5, in conjunction with the module-theoretic constructions of Leonard pairs, allows us to further refine our search for friends.

Lemma 4.1.6 *Let Φ and $\hat{\Phi}$ be Leonard systems with diameter $d \geq 3$. If Φ, $\hat{\Phi}$ are friends then Φ and $\hat{\Phi}$ are both of classical, both of basic, both of Bannai-Ito, or both of orphan type.*

Proof. Recall that if Φ, $\hat{\Phi}$ are friends then they share a linear operator, say A, and an ordering of the primitive idempotents for A. They also share the sequence of eigenvalues $\{\theta_i\}_{i=0}^d$ for A. By Theorem 2.9.4 the type of a Leonard system depends on the common value, which is determined by the eigenvalue sequence for A. Since Φ and $\hat{\Phi}$ share an eigenvalue sequence, they must be the same type. □

Table 4.1: Possible friendships by type

Type	Classical	Basic	Bannai-Ito	Orphan
Classical	✓			
Basic		✓		
Bannai-Ito			✓	
Orphan				✓

We shall appeal to the material in Sections 2.8, 3.2, and 3.3 to show that Leonard pairs of classical/basic type share the same split decomposition precisely when they are constructed from the same sl_2 or $U_q(sl_2)$-module. This in turn allows us to describe friends using Leonine parameters.

4.2 Common split decompositions

We describe when Leonard systems of classical and basic types share a split decomposition.

Definition 4.2.1 Two irreducible sl_2-module structures on V as in Theorem 3.2.5 are *diagonally similar* whenever the vectors with the same index in the corresponding bases of Lemma 3.1.3 are scalar multiples of one another.

The matrices representing each equitable generator with respect to diagonally similar bases are similar via conjugation by a diagonal matrix. Diagonal similarity captures the idea of normalization in Lemma 2.8.3 and Theorem 2.8.4 for the specific construction of Leonard systems from sl_2-modules.

Lemma 4.2.2 *Two Leonard systems of classial on V have the same split decomposition if and only if they can be constructed from diagonally similar irreducible sl_2-module structures on V as in Theorem 3.2.5.*

Proof. For all Leonard systems constructed from a given irreducible sl_2-module structure on V as in Theorem 3.2.5, the split decomposition is $\{U_i = \text{span}(v_i)\}_0^d$. Since the module structures are diagonally similar, the corresponding v_i span the same subspace, so the two Leonard systems have the same split decomposition. Conversely, suppose classical Leonard systems Φ, $\hat{\Phi}$ have split decomposition $\{U_i\}_{i=0}^d$. For each of the sl_2-module structures as in Theorem 3.2.5, the corresponding bases of Lemma 3.1.3 satisfy v_i, $v_i' \in U_i$ $(0 \leq i \leq d)$. Since U_i is 1-dimensional, the two module structures are diagonally similar. □

We now give a similar treatment for Leonard systems of basic type.

Definition 4.2.3 Two irreducible $U_q(sl_2)$- and $U_{\hat{q}}(sl_2)$-module structures on V as in Theorem 3.4.6 are *reminiscent* whenever the vectors with the same index in the corresponding bases of Lemma 3.3.3 are scalar multiples of one another.

Lemma 4.2.4 *Two basic Leonard systems Φ, $\hat{\Phi}$ on V have the same split decomposition if and only if they can be constructed from respective irreducible $U_q(sl_2)$- and $U_{\hat{q}}(sl_2)$-module structures on V which are reminiscent.*

Proof. Similar to that of Lemma 4.2.2. □

4.3 Friendship

We comment on friends. Write $\Phi \sim_{\mathrm{fr}} \hat{\Phi}$ whenever Φ, $\hat{\Phi}$ are friends. We note that $\sim_{\mathrm{fr}}$ is an equivalence relation, which we shall call *friendship*.

Theorem 4.3.1 *Suppose Φ, $\hat{\Phi}$, and $\tilde{\Phi}$ are Leonard systems on V.*

(i) $\Phi \sim_{\mathrm{fr}} \Phi$.

(ii) *If $\Phi \sim_{\mathrm{fr}} \hat{\Phi}$, then $\hat{\Phi} \sim_{\mathrm{fr}} \Phi$.*

(iii) *If $\Phi \sim_{\mathrm{fr}} \hat{\Phi}$ and $\hat{\Phi} \sim_{\mathrm{fr}} \tilde{\Phi}$, then $\Phi \sim_{\mathrm{fr}} \tilde{\Phi}$.*

Proof. Recall that two Leonard systems are friends when they share the same unique split decomposition. From this definition, (i) and (ii) follow routinely. To show (iii), let $\{U_i\}_{i=0}^d$, $\{\hat{U}_i\}_{i=0}^d$, and $\{\tilde{U}_i\}_{i=0}^d$ be the split decompositions of Φ, $\hat{\Phi}$, and $\tilde{\Phi}$ respectively. Since Φ, $\hat{\Phi}$ are friends, by definition $U_i = \hat{U}_i$ $(0 \leq i \leq d)$. Similarly since $\hat{\Phi}$, $\tilde{\Phi}$ are friends $\hat{U}_i = \tilde{U}_i$ $(0 \leq i \leq d)$. Thus $U_i = \tilde{U}_i$ $(0 \leq i \leq d)$ and Φ, $\tilde{\Phi}$ are friends. $\square$

We describe friends by describing friendship equivalence classes. We recall affine transformations which allow us to construct some friends of a given Leonard system.

Lemma 4.3.2 [31, Lemma 5.1] *Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a Leonard system over the field $\mathbb{K}$ and let δ, γ denote scalars in $\mathbb{K}$ with $\delta \neq 0$. Then*

$$\hat{\Phi} = (A; \delta B + \gamma I; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$$

is a Leonard system.

Lemma 4.3.3 *The Leonard systems Φ, $\hat{\Phi}$ of Lemma 4.3.2 are friends.*

Proof. Note that Φ and $\hat{\Phi}$ share both sequences of primitive idempotents, so they have the same split decomposition. Thus they are friends by Definition 4.1.4. $\square$

4.4 Acquaintances

We show that friendship is not a purely intrinsic property of two Leonard systems, as it depends upon the choice of representation.

Lemma 4.4.1 *Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ be a Leonard system on V and β be a split basis for Φ. Then the split decomposition for $[\Phi]_\beta$ consists of the subspaces spanned by the standard basis elements for V.*

Proof. By Lemma 2.7.1, with respect to the split basis β the matrices which represent A and B are

$$[A]_\beta = \begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \theta_d \end{pmatrix} \qquad \text{and} \qquad [B]_\beta = \begin{pmatrix} \theta_0^* & \varphi_1 & & & 0 \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ 0 & & & & \theta_d^* \end{pmatrix}$$

respectively. By Theorem 2.8.4 the standard basis for V is a split basis for $[\Phi]_\beta$, and the result follows. $\square$

Theorem 4.4.2 *If $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ and $\Psi = (A; \hat{B}; \{E_i\}_{i=0}^d; \{\hat{E}_i^*\}_{i=0}^d)$ are two Leonard systems on the vector space V, then there exists a Leonard system $\hat{\Phi}$ isomorphic to Ψ such that Φ and $\hat{\Phi}$ are friends.*

Proof. Let β and $\hat{\beta}$ be respective split bases for Φ and Ψ, as described in Lemma 2.7.1. By Lemma 2.4.10, Φ and Ψ are respectively isomorphic to $[\Phi]_\beta$ and $[\Psi]_{\hat{\beta}}$. By Lemma 4.4.1, $[\Phi]_\beta$ and $[\Psi]_{\hat{\beta}}$ both have a split decomposition consisting of the subspaces spanned by the standard basis elements for $\mathbb{K}^{d+1}$. Thus $[\Phi]_\beta$ and $[\Psi]_{\hat{\beta}}$ are friends. Apply the inverse of the coordinate mapping $[\cdot]_\beta$ to $[\Psi]_{\hat{\beta}}$ to produce the desired $\hat{\Phi}$.

□

From Theorem 4.4.2, we see that friendship is not merely a product of isomorphism classes. We define another relationship among Leonard systems that emphasizes the importance of the shape of friends.

Definition 4.4.3 Two Leonard systems

$$\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \quad \text{and} \quad \Psi = (\hat{A}; \hat{B}; \{F_i\}_{i=0}^d; \{F_i^*\}_{i=0}^d)$$

on V will be said to be *acquaintances* whenever $A = \hat{A}$ and $E_i = F_i$ $(0 \le i \le d)$.

Friends are acquaintances, but not all acquaintances are friends. Consider the following.

Example 4.4.4 Suppose $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system. Let $P = f(A)$ for some polynomial $f \in \mathbb{K}$, and suppose P is invertible. Apply the automorphism σ of $\mathrm{End}(V)$ corresponding to conjugation by P, i.e., $\sigma(X) = P^{-1}XP$. Consider

$$\Phi^\sigma = (A^\sigma; B^\sigma; \{E_i^\sigma\}_{i=0}^d; \{(E_i^*)^\sigma\}_{i=0}^d) = (A; B^\sigma; \{E_i\}_{i=0}^d; \{(E_i^*)^\sigma\}_{i=0}^d).$$

Clearly Φ and Φ^σ are acquaintances. However, in general $E_0^*V \neq (E_0^*)^\sigma V$, so they are not friends.

Example 4.4.5 Recall the D_4 action of Theorem 2.5.2. Consider Leonard systems

$$\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) \text{ and } \Phi^{\downarrow} = (A; B; \{E_i\}_{i=0}^d; \{E_{d-i}^*\}_{i=0}^d).$$

Then Φ and $\Phi^{\downarrow}$ are acquaintances, but not friends.

The next two examples are known from the study of P- and Q-polynomial association schemes. For more information on these two examples see [9] and some work by G. Dickie in [16], [17], [18], and [19].

Example 4.4.6 For certain Leonard systems $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$, there is a permutation π of the primitive idempotents of B such that $\Phi^\pi = (A; B; \{E_i\}_{i=0}^d; \{E_{\pi(i)}^*\}_{i=0}^d)$ is also a Leonard system. Such Leonard systems are acquaintances. This is a very special situation. Some examples of such permutations are $\{E_0^*, E_{d-1}^*, E_2^*, \ldots, E_{d-2}^*, E_1^*, E_d^*\}$ and $\{E_0^*, E_d^*, E_1^*, E_{d-1}^*, \ldots\}$. In several instances, the *types* of the Leonard systems Φ and Φ^π are different. We will need such a possibility when looking for friends.

Example 4.4.7 For certain Leonard systems $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$, there is a permutation π of the primitive idempotents of A such that $\Phi_\pi = (A; B; \{E_\pi(i)\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is also a Leonard system. Note that Φ and Φ_π are not acquaintances. However, we prevent this situation from arising by fixing an ordering of the primitive idempotents of A in our definition of friends.

In light of the discussion in this section, we see that we will need to control the shape of the Leonard systems when seeking friends. We shall do so by using finite-dimensional representations of sl_2 and $U_q(sl_2)$. Before we do this we need one more result regarding when acquaintances have the same shape as friends.

Theorem 4.4.8 *Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ and $\hat{\Phi} = (A; \hat{B}; \{E_i\}_{i=0}^d; \{\hat{E}_i^*\}_{i=0}^d)$ be acquaintances on V. Then Φ and $\hat{\Phi}$ are friends if and only if there exists a basis β so that $[A]_\beta$ is lower-bidiagonal with (i,i)-entry θ_i and $[B]_\beta$, $[\hat{B}]_\beta$ are simultaneously upper-bidiagonal with respective (i,i)-entries θ_i^* and $\hat{\theta}_i^*$.*

Proof. If Φ and $\hat{\Phi}$ are friends then such a basis β exists by definition. Conversely, given such a basis $\beta = \{b_0, b_1, \ldots, b_d\}$, let $U_i = \mathrm{span}\{b_i\}$. By Theorem 2.8.4, $\{U_i\}_{i=1}^d$ is the split decomposition for both Φ and $\hat{\Phi}$. $\square$

In order to find friends, we must first find acquaintances and a basis that gives the form required by Theorem 4.4.8. We may do the former by finding Leonard pairs with an appropriate ordering of idempotents. The latter is achieved by using finite-dimensional representations of sl_2 and $U_q(sl_2)$.

4.5 Friends from Leonine parameters

To find Leonard systems it is natural to start by finding Leonard pairs. In this section we show that finding a pair of friends is equivalent to finding two Leonard pairs along with a decomposition inducing a special basis for both. This corresponds to finding two sets of Leonine parameters and an application of some appropriate module structure. We start with a remark.

By an LB-UB basis for a Leonard pair A, B, we mean an LB-UB basis for some Leonard system associated with A, B.

Theorem 4.5.1 *The following sets are equal.*

(i) *The set of all pairs of Leonard systems on V which are friends.*

(ii) *The pairs of Leonard pairs on V sharing the first operator together with a decomposition of V which induces an LB-UB basis for both pairs.*

Proof. Let $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ and $\hat{\Phi} = (A; \hat{B}; \{E_i\}_{i=0}^d; \{\hat{E}_i^*\}_{i=0}^d)$ be a pair of Leonard systems on V which are friends. The collection A, B and A, $\hat{B}$ and split decomposition $\{U_i\}_{i=0}^d$ shared by Φ and $\hat{\Phi}$ are contained in the second set.

A typical element of the second set consists of Leonard pairs A, B and A, $\hat{B}$ on V and a decomposition $\{U_i\}_{i=0}^d$ of V inducing an LB-UB basis β for each. To specify an element of the first set, we order the primitive idempotents of A consistent with the LB-UB decomposition: $AE_i = ([A]_\beta)(i,i)E_i$. Then $\{E_i\}_{i=0}^d$ is the standard ordering of primitive idempotents for A. We do likewise for the primitive idempotents of B and $\hat{B}$. Now $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ and $\hat{\Phi} = (A; \hat{B}; \{E_i\}_{i=0}^d; \{\hat{E}_i^*\}_{i=0}^d)$ are Leonard systems on V. Moreover, Φ and $\hat{\Phi}$ are friends by Theorem 4.4.8. □

We achieve the decomposition by using a construction from sl_2 or $U_q(sl_2)$. We then give scalars (d, q, κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^*) that uniquely determine a Leonard pair/system (by Lemmas 3.2.7 and 3.4.8). By an appropriate choice of scalars we can produce two Leonard pairs with the same first operator. We recall that friendship is an equivalence relation on Leonard systems.

Theorem 4.5.2 *Classical Leonard systems Φ and $\hat{\Phi}$ on V are friends if and only if they have corresponding Leonine parameters satisfying $\kappa = \hat{\kappa}$, $\lambda - \mu = \hat{\lambda} - \hat{\mu}$, $\nu = \hat{\nu}$ for respective sl_2-module structures on V having bases of Lemma 3.1.3 related by $\hat{v}_i = \alpha_i v_i$, where $0 \neq \alpha_0 \in \mathbb{K}$ is arbitrary and $\alpha_{i+1} = \alpha_i(\lambda + (d-2i)\nu)/(\hat{\lambda} + (d-2i)\nu)$ $(0 \leq i \leq d-1)$.*

Proof. Suppose Φ and $\hat{\Phi}$ are friends. The corresponding sl_2-module structures from Theorem

3.2.5 are diagonally similar, so the bases of V from Lemma 3.1.3 satisfy $\hat{v}_i = \alpha_i v_i$ $(0 \le i \le d)$. Since $A = \hat{A}$, we compute $\hat{\theta}_i \alpha_i v_i - 2(i+1)(\hat{\lambda} + (d-2i)\hat{\nu})\alpha_{i+1} v_{i+1} = \hat{\theta}_i \hat{v}_i - 2(i+1)(\hat{\lambda} + (d-2i)\hat{\nu})\hat{v}_{i+1} = \hat{A}\hat{v}_i = A\alpha_i v_i = \theta_i \alpha_i v_i - 2(i+1)(\lambda + (d-2i)\nu)\alpha_i v_{i+1}$ $(0 \le i \le d-1)$. Thus $\hat{\theta}_i = \theta_i$ and $(\hat{\lambda} + (d-2i)\hat{\nu})\alpha_{i+1} = (\lambda + (d-2i)\nu)\alpha_i$ $(0 \le i \le d)$ since the v_i form a basis. Note that the coefficients of α_{i+1} and α_i are nonzero since the parameters are Leonine. Now $\kappa = \hat{\kappa}$, $\lambda - \mu = \hat{\lambda} - \hat{\mu}$, $\nu = \hat{\nu}$ by (3.2.1) at $i = 0, 1, 2$ since $\theta_i = \hat{\theta}_i$. The converse is straightforward from the construction. $\square$

Theorem 4.5.3 *Basic Leonard systems Φ and $\hat{\Phi}$ on V are friends if and only if they have corresponding Leonine parameters satisfying one of the following*

(i) $\hat{q} = q$, $\hat{\lambda} = \lambda$, $\hat{\mu} = \mu$, $\hat{\kappa} + \hat{\nu} = \kappa + \nu$;

(ii) $\hat{q} = q^{-1}$, $\hat{\lambda} = \mu$, $\hat{\mu} = \lambda$, $\hat{\kappa} + \hat{\nu} = \kappa + \nu$;

(iii) $\hat{q} = -q$, $\hat{\lambda} = (-1)^d \lambda$, $\hat{\mu} = (-1)^d \mu$, $\hat{\kappa} + \hat{\nu} = \kappa + \nu$;

(iv) $\hat{q} = -q^{-1}$, $\hat{\lambda} = (-1)^d \mu$, $\hat{\mu} = (-1)^d \lambda$, $\hat{\kappa} + \hat{\nu} = \kappa + \nu$

for $U_q(sl_2)$- and $U_{\hat{q}}(sl_2)$-module structures with bases of Lemma 3.3.3 related by $\hat{v}_i = \alpha_i v_i$, where $0 \ne \alpha_0 \in \mathbb{K}$ is arbitrary and for $0 \le i \le d-1$, $\alpha_{i+1} = \alpha_i (q^{-2(i+1)} - 1)(\lambda q^{d-2i} + \nu)/(\hat{q}^{-2(i+1)} - 1)(\lambda \hat{q}^{d-2i} + \hat{\nu})$.

Proof. Suppose Φ and $\hat{\Phi}$ are friends. Then $\hat{\theta}_i = \theta_i$ $(0 \le i \le d)$. We claim that $\hat{q} \in$

$\{q, -q, q^{-1}, -q^{-1}\}$. Use (3.4.9) to expand the equations

$$
\begin{aligned}
0 &= q^{-2}(\theta_0 - \hat{\theta}_0) - (q^{-2}(\hat{q}^2+1)+1)(\theta_1 - \hat{\theta}_1) + (\hat{q}^2(q^{-2}+1)+1)(\theta_2 - \hat{\theta}_2) - \hat{q}^2(\theta_3 - \hat{\theta}_3) \\
&= \hat{q}^{-d} q^{-d-2} \left(\hat{q}^2 q^2 - 1\right) \left(\mu \hat{q} \left(\hat{q}^2-1\right)^2 \left(\hat{q}^2+1\right) q^d - \mu \left(q^2-1\right)^2 \left(q^2+1\right) \hat{q}^d\right), \\
0 &= q^2(\theta_0 - \hat{\theta}_0) - (q^2(\hat{q}^{-2}+1)+1)(\theta_1 - \hat{\theta}_1) + (\hat{q}^{-2}(q^2+1)+1)(\theta_2 - \hat{\theta}_2) - \hat{q}^{-2}(\theta_3 - \hat{\theta}_3) \\
&= -\frac{\left(\hat{q}^2 q^2 - 1\right)\left(\lambda \hat{q} \left(\hat{q}^2-1\right)^2 \left(\hat{q}^2+1\right) q^6 \hat{q}^d - \lambda \hat{q}^6 \left(q^2-1\right)^2 \left(q^2+1\right) q^d\right)}{\hat{q}^8 q^6}.
\end{aligned}
$$

Thus either $\hat{q}^2 q^2 = 1$ or both $\hat{\mu} = \mu \frac{\hat{q}^d (q^2-1)(q^4-1)}{q^d (\hat{q}^2-1)(\hat{q}^4-1)}$ and $\hat{\lambda} = \lambda \frac{q^{d-6}(q^2-1)(q^4-1)}{\hat{q}^{d-6}(\hat{q}^2-1)(\hat{q}^4-1)}$. In the latter case,

$$
\begin{aligned}
0 &= (\theta_0 - \hat{\theta}_0) - 2(\theta_1 - \hat{\theta}_1) + (\theta_2 - \hat{\theta}_2) \\
&= \frac{\left(q^2-1\right)^2 q^{-d-6} \left(q^2 - \hat{q}^2\right)\left(\lambda q^{2d} - \mu q^6\right)}{\hat{q}^2+1}.
\end{aligned}
$$

This implies that $\hat{q}^2 = q^2$ since $\mu q^{2(3-d)} - \lambda \neq 0$ by the Leonine condition. Thus the claim holds.

When $\hat{q} = q$, the above gives $\hat{\mu} = \mu$, $\hat{\lambda} = \lambda$. When $\hat{q} = -q$, the above gives $\hat{\mu} = (-1)^d \mu$, $\hat{\lambda} = (-1)^d \lambda$. To treat the cases $\hat{q} = \pm q^{-1}$ we expand the equations

$$
\begin{aligned}
0 &= q^{-2}(\theta_0 - \hat{\theta}_0) - (2q^{-2}+1)(\theta_1 - \hat{\theta}_1) + (q^{-2}+2)(\theta_2 - \hat{\theta}_2) - (\theta_3 - \hat{\theta}_3) \\
&= \frac{\left(\hat{q}^2-1\right)^2 q^d \left(\mu q \hat{q}^6 \left(\hat{q}^2 q^2 - 1\right) - \lambda \hat{q} \hat{q}^{2d} \left(\hat{q}^2 - q^2\right)\right) - \mu \left(q^2-1\right)^3 \left(q^2+1\right) \hat{q}^{d+6}}{\hat{q}^{d+6} q^{d+2}}, \\
0 &= \hat{q}^{-2}(\theta_0 - \hat{\theta}_0) - (2\hat{q}^{-2}+1)(\theta_1 - \hat{\theta}_1) + (\hat{q}^{-2}+2)(\theta_2 - \hat{\theta}_2) - (\theta_3 - \hat{\theta}_3) \\
&= \frac{q^d \left(\mu \hat{q} \left(\hat{q}^2-1\right)^3 \left(\hat{q}^2+1\right) q^6 - \lambda \left(q^2-1\right)^2 \hat{q}^d q^d \left(\hat{q}^2 - q^2\right)\right) - \mu q^6 \left(q^2-1\right)^2 \hat{q}^d \left(\hat{q}^2 q^2 - 1\right)}{\hat{q}^{d+2} q^{d+6}}.
\end{aligned}
$$

When $\hat{q} = q^{-1}$, we find $\hat{\mu} = \lambda$, $\hat{\lambda} = \mu$. When $\hat{q} = -q^{-1}$, we find $\hat{\lambda} = (-1)^d \mu$, $\hat{\mu} = (-1)^d \lambda$. In all four cases, $\hat{\kappa} + \hat{\nu} = \kappa + \nu$ since $(\theta_0 - \hat{\theta}_0) = 0$.

By Lemma 4.2.4, the corresponding $U_q(sl_2)$- and $U_{\hat{q}}(sl_2)$-module structures are such that the bases of V from Lemma 3.3.3 satisfy $\hat{v}_i = \alpha_i v_i$ $(0 \leq i \leq d)$. Arguing as in the proof of Theorem 4.5.2, we find

$$(\hat{q}^{-2(i+1)} - 1)(\lambda \hat{q}^{d-2i} + \hat{\nu})\alpha_{i+1} = (q^{-2(i+1)} - 1)(\lambda q^{d-2i} + \nu)\alpha_i \qquad (0 \leq i \leq d-1).$$

The converse is straightforward from the construction. $\square$

From Lemma 3.3.3, we note the following. For d even, $U_q(sl_2)$ and $U_{-q}(sl_2)$ have the same action on the respective V_d^+ and V_d^-. For d odd, $U_q(sl_2)$ and $U_{-q}(sl_2)$ swap actions on the respective V_d^+ and V_d^-. Any Leonard system constructed on the irreducible $U_q(sl_2)$ module V_d^- can also be constructed on V_d^+ by changing the signs of the coeffients of X, Y, and Z, provided that $q^{-2}(-\lambda q^d + \nu)(-\mu^* q^d + \nu^*) \neq (\mu - \lambda q^{2(i-1)})(\lambda^* - \mu^* q^{2(d-i)})$ $(1 \leq i \leq d)$. Thus in most cases, any Leonard system constructed from $U_{-q}(sl_2)$ can be constructed from $U_q(sl_2)$. Our construction yields essentially the same result on both $U_q(sl_2)$ and $U_{-q}(sl_2)$.

There is an antiautomorphism † of $\mathrm{End}(V_d^+)$ which swaps X and Y and fixes Z. This turns the $U_{q^{-1}}(sl_2)$-module V_d^+ into the $U_q(sl_2)$-module V_d^+

We are able to uniquely describe any Leonard system using the scalars (d, q, κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^*) and an appropriate sl_2 or $U_q(sl_2)$ construction. With the shape and type determined by the construction, and the first six scalars satisfying the conditions of Theorem 4.5.2 or Theorem 4.5.3, any working values of κ^*, λ^*, μ^*, ν^* (that is, those satisfying scalar requirements such as in Theorem 3.2.6 and Lemma 3.4.7) will result in a Leonard system that is a friend of the first. In this way we find all friends of a Leonard system and solve the problem of our thesis. In the next chapter we give more restrictions on the scalars (d, q, κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^*) so that we may classify possible friendships by type.

5 Classification of friends

In this chapter we reformulate the Leonine parameters on a type-by-type basis. Here we use eigenvalues as much as possible as the primary parameters. This allows one to construct Leonard pairs and Leonard systems from irreducible sl_2- or $U_q(sl_2)$-modules as in Chapter 3. By fixing the module structure the shape is guaranteed to be lower-bidiagonal/upper-bidiagonal. By comparing coefficients for the first operator we get all members of an equivalence class of friends.

5.1 Leonard systems of classical type

Given a Leonard system $\Phi = (A; B; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ on V, the eigenvalues are more immediately determined than the Leonine parameters. Our strategy is to describe Leonine parameters κ, λ, μ, ν, κ^*, λ^*, μ^*, $\nu^* \in \mathbb{K}$ in terms of the eigenvalues of A and B. When we write $\lambda = \lambda$ we mean λ is free, subject to any further restrictions listed. When we write $\lambda \neq a$ we mean λ is free but cannot be a, and is also subject to any further restrictions listed.

Theorem 5.1.1 *Let A and B be linear operators on the $(d+1)$-dimensional vector space V $(d \geq 2)$. Assume there exists an irreducible sl_2-module structure on V such that A and B act on V as the $\mathbb{K}$-linear combinations $\kappa I + \lambda Y + \mu Z + \nu YZ$ and $\kappa^* I + \lambda^* Z + \mu^* X + \nu^* ZX$, respectively. Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ be as in Definition 3.2.2. Then the following hold.*

(i) *A, B is a Leonard pair of Krawtchouk type if and only if $\theta_0 \neq \theta_1$, $\theta^*_0 \neq \theta^*_1$, and the scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy*

$$\begin{aligned} \kappa &= \theta_0 - \frac{d(\theta_0-\theta_1)}{2}, & \kappa^* &= \theta^*_0 - \frac{d(\theta^*_0-\theta^*_1)}{2}, \\ \lambda &\neq 0, & \lambda^* &\neq \frac{\theta^*_0-\theta^*_1}{2}, \\ \mu &= \frac{\theta_0-\theta_1}{2} + \lambda, & \mu^* &= \lambda^* - \frac{\theta^*_0-\theta^*_1}{2}, \\ \nu &= 0, & \nu^* &= 0, \end{aligned}$$

$$\lambda\lambda^* - \mu\lambda^* + \mu\mu^* \neq 0.$$

(ii) *A, B is a Leonard pair of Hahn type if and only if $\theta_0 \neq \theta_1$, $\theta^*_0 \neq \theta^*_1$, and the scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy*

$$\begin{aligned} \kappa &= \theta_0 - \frac{d(\theta_0-\theta_1)}{2}, & \kappa^* &= \frac{\theta^*_0(2-d) + d\theta^*_1 - (2d-4)d\nu^*}{2}, \\ \lambda &= \frac{\theta_1-\theta_0}{2}, & \lambda^* &= \lambda^*, \\ \mu &= 0, & \mu^* &= \frac{\theta^*_1 - \theta^*_0 + 2\lambda^* - (4d-4)\nu^*}{2}, \\ \nu &= 0, & \nu^* &\neq 0, \end{aligned}$$

$$\mu^* - (d-2i)\nu^* \neq 0, \qquad \lambda^* - (d-2i)\nu^* \neq 0 \qquad (1 \leq i \leq d),$$

$$\lambda^* - \mu^* + 2\nu^*(d-i) \neq 0 \qquad (1 \leq i \leq 2d-1).$$

(iii) *A, B is a Leonard pair of dual Hahn type if and only if $\theta_0 \neq \theta_1$, $\theta^*_0 \neq \theta^*_1$, and the scalars*

κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy

$$\begin{aligned} \kappa &= \frac{\theta_0(2-d)+d\theta_1+(4-2d)d\nu}{2}, & \kappa^* &= \theta_0^* - \frac{d(\theta_0^*-\theta_1^*)}{2}, \\ \lambda &= \lambda, & \lambda^* &= 0, \\ \mu &= \frac{\theta_0-\theta_1+2\lambda-(4-4d)\nu}{2}, & \mu^* &= \frac{\theta_1^*-\theta_0^*}{2}, \\ \nu &\neq 0, & \nu^* &= 0, \end{aligned}$$

$$\mu-(d-2i)\nu \neq 0, \qquad \lambda-(d-2i)\nu \neq 0 \qquad (1 \leq i \leq d),$$

$$\lambda-\mu+2\nu(d-i) \neq 0 \qquad (1 \leq i \leq 2d-1).$$

(iv) *A, B is a Leonard pair of Racah type if and only if $\theta_0 \neq \theta_1$, $\theta_0^* \neq \theta_1^*$, and the scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy (3.2.3) – (3.2.8) and*

$$\begin{aligned} \kappa &= \frac{\theta_0(2-d)+d\theta_1+(4-2d)d\nu}{2}, & \kappa^* &= \frac{\theta_0^*(2-d)+d\theta_1^*-(2d-4)d\nu^*}{2}, \\ \lambda &= \lambda, & \lambda^* &= \lambda^*, \\ \mu &= \frac{\theta_0-\theta_1+2\lambda-(4-4d)\nu}{2}, & \mu^* &= \frac{\theta_1^*-\theta_0^*+2\lambda^*-(4d-4)\nu^*}{2}, \\ \nu &\neq 0, & \nu^* &\neq 0. \end{aligned}$$

Proof. Recall from Definition 3.2.2 that

$$\begin{aligned} \theta_i &= \kappa-(\lambda-\mu)(d-2i)-(d-2i)^2\nu & (1 \leq i \leq d), \\ \theta_i^* &= \kappa^*+(\lambda^*-\mu^*)(d-2i)-(d-2i)^2\nu^* & (1 \leq i \leq d), \end{aligned}$$

so

$$\begin{aligned}
\theta_0 &= \kappa - (\lambda - \mu)d - d^2\nu, && (5.1.1)\\
\theta_0^* &= \kappa^* + (\lambda^* - \mu^*)d - d^2\nu^*, && (5.1.2)\\
\theta_1 &= \kappa - (\lambda - \mu)(d-2) - (d-2)^2\nu, && (5.1.3)\\
\theta_1^* &= \kappa^* + (\lambda^* - \mu^*)(d-2) - (d-2)^2\nu^*. && (5.1.4)
\end{aligned}$$

It follows that

$$\begin{aligned}
\theta_0 - \theta_1 &= -2(\lambda - \mu) + (4-4d)\nu, && (5.1.5)\\
\theta_0^* - \theta_1^* &= 2(\lambda^* - \mu^*) + (4-4d)\nu^*. && (5.1.6)
\end{aligned}$$

(i): Suppose A, B is a Leonard pair of Krawtchouk type so that the parameters are subject to the conditions given in Theorem 3.2.6(i). In particular $\nu = \nu^* = 0$. Take λ and λ^* to be free. Substituting these values into equations (5.1.5) and (5.1.6), and solving for μ and μ^* gives

$$\mu = \frac{\theta_0 - \theta_1}{2} + \lambda, \qquad \mu^* = \lambda^* - \frac{\theta_0^* - \theta_1^*}{2}.$$

In (5.1.1) and (5.1.2) these values give

$$\kappa = \theta_0 - \frac{d(\theta_0 - \theta_1)}{2}, \qquad \kappa^* = \theta_0^* - \frac{d(\theta_0^* - \theta_1^*)}{2}.$$

Conversely, these values satisfy the conditions of Theorem 3.2.6(i) so A, B is a Leonard pair of Krawtchouk type.

(ii): Suppose A, B is a Leonard pair of Hahn type so that the parameters are subject to the conditions given in Theorem 3.2.6(ii).

In particular $\nu = \mu = 0$. From (5.1.5) we get $\lambda = \dfrac{\theta_1 - \theta_0}{2}$, and then from (5.1.1) we get $\kappa = \theta_0 - \dfrac{d(\theta_0 - \theta_1)}{2}$. Take λ^* and $\nu^* \neq 0$ to be free. Then by (5.1.6) we get $\mu^* = \dfrac{\theta_1^* - \theta_0^* + 2\lambda^* - (4d-4)\nu^*}{2}$. Finally, from (5.1.2) we get $\kappa^* = \dfrac{\theta_0^*(2-d) + d\theta_1^* - (2d-4)d\nu^*}{2}$.

Conversely, these values satisfy the conditions of Theorem 3.2.6(ii) so A, B is a Leonard pair of Hahn type.

(iii): Suppose A, B is a Leonard pair of dual Hahn type so that the parameters are subject to the conditions given in Theorem 3.2.6(iii).

In particular $\lambda^* = \nu^* = 0$. From (5.1.6) we get $\mu^* = \dfrac{\theta_1^* - \theta_0^*}{2}$, and then by (5.1.2) we have $\kappa^* = \theta_0^* - \dfrac{d(\theta_0^* - \theta_1^*)}{2}$. Take λ and $\nu \neq 0$ to be free. Then from (5.1.5) we get $\mu = \dfrac{\theta_0 - \theta_1 + 2\lambda - (4-4d)\nu}{2}$. And finally, from (5.1.1) we get $\kappa = \dfrac{\theta_0(2-d) + d\theta_1 + (4-2d)d\nu}{2}$.

Conversely, these values satisfy the conditions of Theorem 3.2.6(iii) so A, B is a Leonard pair of dual Hahn type.

(iv): Suppose A, B is a Leonard pair of Racah type so that the parameters are subject to the conditions given in Theorem 3.2.6(iv). It can be verified that the values of κ, λ, μ, ν will be the same as those in the dual Hahn case and the values of κ^*, λ^*, μ^*, ν^* will be the same as those in the Hahn case.

Conversely, these values satisfy the conditions of Theorem 3.2.6(iv) so A, B is a Leonard pair of Racah type.

$\square$

We summarize Theorem 5.1.1 in Table 5.1.

Table 5.1: Scalar restrictions on Leonard pairs of classical type

Scalar	Krawtchouk	Hahn	dual Hahn	Racah
κ	$\theta_0 - \frac{d(\theta_0-\theta_1)}{2}$	$\theta_0 - \frac{d(\theta_0-\theta_1)}{2}$	$\frac{\theta_0(2-d)+d\theta_1+(4-2d)d\nu}{2}$	$\frac{\theta_0(2-d)+d\theta_1+(4-2d)d\nu}{2}$
λ	$\neq 0$	$\frac{\theta_1-\theta_0}{2}$	λ	λ
μ	$\frac{\theta_0-\theta_1}{2}+\lambda$	0	$\frac{\theta_0-\theta_1+2\lambda-(4-4d)\nu}{2}$	$\frac{\theta_0-\theta_1+2\lambda-(4-4d)\nu}{2}$
ν	0	0	$\neq 0$	$\neq 0$
κ^*	$\theta^*_0 - \frac{d(\theta^*_0-\theta^*_1)}{2}$	$\frac{\theta^*_0(2-d)+d\theta^*_1-(2d-4)d\nu^*}{2}$	$\theta^*_0 - \frac{d(\theta^*_0-\theta^*_1)}{2}$	$\frac{\theta^*_0(2-d)+d\theta^*_1-(2d-4)d\nu^*}{2}$
λ^*	$\neq \frac{\theta^*_0-\theta^*_1}{2}$	λ^*	0	λ^*
μ^*	$\lambda^* - \frac{\theta^*_0-\theta^*_1}{2}$	$\frac{\theta^*_1-\theta^*_0+2\lambda^*-(4d-4)\nu^*}{2}$	$\frac{\theta^*_1-\theta^*_0}{2}$	$\frac{\theta^*_1-\theta^*_0+2\lambda^*-(4d-4)\nu^*}{2}$
ν^*	0	$\neq 0$	0	$\neq 0$

From Theorem 5.1.1 we are able to see the possible friendships that may result between Leonard systems of different classical types. We do this by comparing the values for κ, λ, μ, and ν. Table 5.2 summarizes the different possible friendships that can result between Leonard systems of classical type.

5.2 Leonard systems of basic type

We now turn our attention to the basic case. We use the same conventions concerning free parameters as in the previous section. For simplicity, we focus on the case where $\hat{q} = q$ and the module structures coincide.

Theorem 5.2.1 *Let A and B be linear operators on the $(d+1)$-dimensional vector space V $(d \geq 2)$. Assume there exists an irreducible $U_q(sl_2)$-module structure on V such that A and B act on V as the $\mathbb{K}$-linear combinations $\kappa I + \lambda Y + \mu Z + \nu YZ$ and $\kappa^* I + \lambda^* Z + \mu^* X + \nu^* ZX$, respectively. Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ be as in Definition 3.4.3. Then the following hold.*

(i) *A, B is a Leonard pair of q-Racah type if and only if $\theta_0 \neq \theta_1$, $\theta_0^* \neq \theta_1^*$, and the scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy (3.4.11) – (3.4.16) and*

$$
\begin{aligned}
\kappa &= \kappa, & \kappa^* &= \kappa^*, \\
\lambda &= \frac{\theta_0 - \theta_1}{q^{d-2}(q^2-1)} + \mu q^{-2d+2}, & \lambda^* &= \frac{q^d(\theta_0^* - \theta_1^*)}{1-q^2} + \mu^* q^{2d-2}, \\
\mu &= \mu, & \mu^* &= \mu^*, \\
\nu &= \theta_0 - \kappa - \frac{q^2(\theta_0 - \theta_1)}{q^2-1} - \mu q^{-d}(q^2+1), & \nu^* &= \theta_0^* - \kappa^* - \frac{\theta_0^* - \theta_1^*}{1-q^2} - \mu^* q^{d-2}(1+q^2).
\end{aligned}
$$

(ii) *A, B is a Leonard pair of q-Hahn or q-Krawtchouk type if and only if $\theta_0 \neq \theta_1$, $\theta_0^* \neq \theta_1^*$,*

Table 5.2: Possible friendships by classical type

Type	Krawtchouk	Hahn	dual Hahn	Racah
Krawtchouk	✓	✓		
Hahn	✓	✓		
dual Hahn			✓	✓
Racah			✓	✓

and the scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy

$$\begin{aligned} \kappa &= \kappa, & \kappa^* &= \kappa^*, \\ \lambda &= \frac{\theta_0-\theta_1}{q^{d-2}(q^2-1)}, & \lambda^* &= \frac{q^d(\theta^*_0-\theta^*_1)}{1-q^2}+\mu^* q^{2d-2}, \\ \mu &= 0, & \mu^* &= \mu^*, \\ \nu &= \theta_0-\kappa-\frac{q^2(\theta_0-\theta_1)}{q^2-1}, & \nu^* &= \theta^*_0-\kappa^*-\frac{\theta^*_0-\theta^*_1}{1-q^2}-\mu^* q^{d-2}(1+q^2), \end{aligned}$$

$$\nu\nu^*=0, \qquad \nu \neq -q^{d-2i}\lambda, \qquad \nu^* \neq -q^{d-2i}\mu^* \qquad (0 \le i \le d-1),$$

$$\mu^* \neq q^{2(i-d)}\lambda^*, \qquad \lambda\nu^*+\mu^*\nu+q^{2i-d}\lambda\lambda^* \neq 0 \qquad (1 \le i \le d).$$

(iii) *A, B is a Leonard pair of dual q-Hahn or dual q-Krawtchouk type if and only if $\theta_0 \neq \theta_1$, $\theta^*_0 \neq \theta^*_1$, and the scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy*

$$\begin{aligned} \kappa &= \kappa, & \kappa^* &= \kappa^*, \\ \lambda &= \frac{\theta_0-\theta_1}{q^{d-2}(q^2-1)}+\mu q^{-2d+2}, & \lambda^* &= 0, \\ \mu &= \mu, & \mu^* &= \frac{\theta^*_0-\theta^*_1}{q^{d-2}(q^2-1)}, \\ \nu &= \theta_0-\kappa-\frac{q^2(\theta_0-\theta_1)}{q^2-1}-\mu q^{-d}(q^2+1), & \nu^* &= \theta^*_0-\kappa^*-\frac{q^2(\theta^*_0-\theta^*_1)}{q^2-1}, \end{aligned}$$

$$\nu\nu^*=0, \qquad \nu \neq -q^{d-2i}\lambda, \qquad \nu^* \neq -q^{d-2i}\mu^* \qquad (0 \le i \le d-1),$$

$$\lambda \neq q^{2(i-d)}\mu, \qquad \lambda\nu^*+\mu^*\nu+q^{2i-d}\mu\mu^* \neq 0 \qquad (1 \le i \le d).$$

(iv) *A, B is a Leonard pair of quantum q-Krawtchouk type if and only if $\theta_0 \neq \theta_1$, $\theta^*_0 \neq \theta^*_1$,* *and the scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy*

$$\begin{aligned}
\kappa &= \kappa, & \kappa^* &= \theta_0^* - \frac{q^2(\theta_0^* - \theta_1^*)}{q^2-1}, \\
\lambda &= 0, & \lambda^* &= 0, \\
\mu &= \frac{q^d(\theta_0 - \theta_1)}{1-q^2}, & \mu^* &= \frac{\theta_0^* - \theta_1^*}{q^{d-2}(q^2-1)}, \\
\nu &= \theta_0 - \kappa - \frac{\theta_0 - \theta_1}{1-q^2}, & \nu^* &= 0,
\end{aligned}$$

$$\nu \neq -q^{d-2i+2}\mu \qquad (1 \leq i \leq d).$$

(v) *A, B is a Leonard pair of affine q-Krawtchouk type if and only if $\theta_0 \neq \theta_1$, $\theta_0^* \neq \theta_1^*$, and the scalars κ, λ, μ, ν, κ^*, λ^*, μ^*, ν^* satisfy*

$$\begin{aligned}
\kappa &\neq \theta_0, & \kappa^* &\neq \theta_0^*, \\
\lambda &= \frac{\theta_0 - \theta_1}{q^{d-2}(q^2-1)}, & \lambda^* &= 0, \\
\mu &= 0, & \mu^* &= \frac{\theta_0^* - \theta_1^*}{q^{d-2}(q^2-1)}, \\
\nu &= \theta_0 - \kappa - \frac{q^2(\theta_0 - \theta_1)}{q^2-1}, & \nu^* &= \theta_0^* - \kappa^* - \frac{q^2(\theta_0^* - \theta_1^*)}{q^2-1},
\end{aligned}$$

$$\nu\nu^* = 0, \qquad \lambda\nu^* + \mu^*\nu \neq 0,$$

$$\nu \neq -q^{d-2i}\lambda, \qquad \nu^* \neq -q^{d-2i}\mu^* \qquad (0 \leq i \leq d-1).$$

Proof. Recall from Definition 3.4.3 that

$$\begin{aligned}
\theta_i &= \kappa + \nu + \lambda q^{d-2i} + \mu q^{2i-d} & (1 \leq i \leq d), \\
\theta_i^* &= \kappa^* + \nu^* + \mu^* q^{d-2i} + \lambda^* q^{2i-d} & (1 \leq i \leq d),
\end{aligned}$$

so

$$\begin{aligned}
\theta_0 &= \kappa + \nu + \lambda q^d + \mu q^{-d}, && (5.2.7)\\
\theta_0^* &= \kappa^* + \nu^* + \mu^* q^d + \lambda^* q^{-d}, && (5.2.8)\\
\theta_1 &= \kappa + \nu + \lambda q^{d-2} + \mu q^{2-d}, && (5.2.9)\\
\theta_1^* &= \kappa^* + \nu^* + \mu^* q^{d-2} + \lambda^* q^{2-d}. && (5.2.10)
\end{aligned}$$

It follows that

$$\begin{aligned}
\theta_0 - \theta_1 &= (q^2-1)(\lambda q^{d-2} - \mu q^{-d}), && (5.2.11)\\
\theta_0^* - \theta_1^* &= (q^2-1)(\mu^* q^{d-2} - \lambda^* q^{-d}). && (5.2.12)
\end{aligned}$$

(i): Suppose A, B is a Leonard pair of q-Racah type so that the parameters are subject to the conditions given in Lemma 3.4.7(i). Take κ and μ to be free. Then by (5.2.11) we get $\lambda = \dfrac{\theta_0 - \theta_1}{q^{d-2}(q^2-1)} + \mu q^{2-2d}$. Along with (5.2.7) this yields $\nu = \theta_0 - \kappa - \mu q^{-d}(1+q^2) - \dfrac{q^2(\theta_0-\theta_1)}{q^2-1}$. Now take κ^* and μ^* to be free. Then by (5.2.12) we get $\lambda^* = \dfrac{q^d(\theta_0^* - \theta_1^*)}{1-q^2} + \mu^* q^{2d-2}$. And finally by (5.2.8) we have $\nu^* = \theta_0^* - \kappa^* - \mu^* q^{d-2}(1+q^2) - \dfrac{\theta_0^* - \theta_1^*}{1-q^2}$.

Conversely, these values satisfy the conditions of Lemma 3.4.7(i) so A, B is a Leonard pair of q-Racah type.

(ii): Suppose A, B is a Leonard pair of q-Hahn or q-Krawtchouk type so that the parameters are subject to the conditions given in Lemma 3.4.7(ii). In particular $\mu = 0$. Then by (5.2.11) we get $\lambda = \dfrac{q^{2-d}(\theta_0-\theta_1)}{q^2-1}$. Take κ to be free. Then by (5.2.7) we have $\nu = \theta_0 - \kappa - \dfrac{q^2(\theta_0-\theta_1)}{q^2-1}$. It can be verified that the values of $\kappa^*, \lambda^*, \mu^*, \nu^*$ will be the same as those in the q-Racah case.

Conversely, these values satisfy the conditions of Lemma 3.4.7(ii) so A, B is a Leonard pair of q-Hahn or q-Krawtchouk type.

(iii): Suppose A, B is a Leonard pair of dual q-Hahn or dual q-Krawtchouk type so that the parameters are subject to the conditions given in Lemma 3.4.7(iii). In particular $\lambda^* = 0$. Take κ^* to be free. By (5.2.12) we get $\mu^* = \dfrac{q^{2-d}(\theta_0^* - \theta_1^*)}{q^2 - 1}$. Along with (5.2.8) this yields $\nu = \theta_0^* - \kappa^* - \dfrac{q^2(\theta_0^* - \theta_1^*)}{q^2 - 1}$. It can be verified that the values of $\kappa, \lambda, \mu, \nu$ will be the same as those in the q-Racah case.

Conversely, these values satisfy the conditions of Lemma 3.4.7(iii) so A, B is a Leonard pair of dual q-Hahn or dual q-Krawtchouk type.

(iv): Suppose A, B is a Leonard pair of quantum q-Krawtchouk type so the parameters are subject to the conditions given in Lemma 3.4.7(iv). In particular $\lambda = \lambda^* = \nu^* = 0$. Then by (5.2.12) we get $\mu^* = \dfrac{q^{2-d}(\theta_0^* - \theta_1^*)}{q^2 - 1}$. Along with (5.2.8) this yields $\kappa^* = \theta_0^* - \dfrac{q^2(\theta_0^* - \theta_1^*)}{q^2 - 1}$. Substituting into (5.2.11) gives $\mu = \dfrac{q^d(\theta_0 - \theta_1)}{1 - q^2}$. Takes κ to be free. Then from (5.2.7) we get $\nu = \theta_0 - \kappa - \dfrac{\theta_0 - \theta_1}{1 - q^2}$.

Conversely, these values satisfy the conditions of Lemma 3.4.7(iv) so A, B is a Leonard pair of quantum q-Krawtchouk type.

(v): Suppose A, B is a Leonard pair of affine q-Krawtchouk type so that the parameters are subject to the conditions given in Lemma 3.4.7(v). In particular $\mu = \lambda^* = 0$. It can be verified that the values of $\kappa, \lambda, \mu, \nu$ will be the same as those in (ii). Similarly, the values of $\kappa^*, \lambda^*, \mu^*, \nu^*$ will be the same as those in (iii).

Conversely, these values satisfy the conditions of Lemma 3.4.7(v) so A, B is a Leonard pair of affine q-Krawtchouk type. $\square$

We summarize Theorem 5.2.1 in Table 5.3.

From Theorem 5.2.1 we are able to see the possible friendships that may result between Leonard systems of different basic types. We do this by comparing the values for κ, λ, μ, and ν. Table 5.4 summarizes the different possible friendships that can result between Leonard systems of basic type.

Table 5.3: Scalar restrictions on Leonard pairs of basic type

Scalar	q-Racah	q-Hahn or q-Krawtchouk	dual q-Hahn or dual q-Krawtchouk
κ	κ	$\neq \theta_0$	κ
λ	$\frac{\theta_0 - \theta_1}{q^{d-2}(q^2-1)} + \mu q^{-2d+2}$	$\frac{\theta_0 - \theta_1}{q^{d-2}(q^2-1)}$	$\frac{\theta_0 - \theta_1}{q^{d-2}(q^2-1)} + \mu q^{-2d+2}$
μ	μ	0	μ
ν	$\theta_0 - \kappa - \frac{q^2(\theta_0 - \theta_1)}{q^2-1} - \mu q^{-d}(q^2+1)$	$\theta_0 - \kappa - \frac{q^2(\theta_0 - \theta_1)}{q^2-1}$	$\theta_0 - \kappa - \frac{q^2(\theta_0 - \theta_1)}{q^2-1} - \mu q^{-d}(q^2+1)$
κ^*	κ^*	$\neq \theta^*_0$	κ^*
λ^*	$\frac{q^d(\theta^*_0 - \theta^*_1)}{1-q^2} + \mu^* q^{2d-2}$	$\frac{q^d(\theta^*_0 - \theta^*_1)}{1-q^2} + \mu^* q^{2d-2}$	0
μ^*	μ^*	μ^*	$\frac{\theta^*_0 - \theta^*_1}{q^{d-2}(q^2-1)}$
ν^*	$\theta^*_0 - \kappa^* - \frac{\theta^*_0 - \theta^*_1}{1-q^2} - \mu^* q^{d-2}(1+q^2)$	$\theta^*_0 - \kappa^* - \frac{\theta^*_0 - \theta^*_1}{1-q^2} - \mu^* q^{d-2}(1+q^2)$	$\theta^*_0 - \kappa^* - \frac{q^2(\theta^*_0 - \theta^*_1)}{q^2-1}$

Table 5.3: Scalar restrictions on Leonard pairs of basic type (continued)

Scalar	Quantum q-Krawtchouk	Affine q-Krawtchouk
κ	κ	$\neq \theta_0$
λ	0	$\dfrac{\theta_0 - \theta_1}{q^{d-2}(q^2-1)}$
μ	$\dfrac{q^d(\theta_0 - \theta_1)}{1-q^2}$	0
ν	$\theta_0 - \kappa - \dfrac{q^2(\theta_0 - \theta_1)}{q^2-1}$	$\theta_0 - \kappa - \dfrac{q^2(\theta_0 - \theta_1)}{q^2-1}$
κ^*	$\theta^*_0 - \dfrac{q^2(\theta^*_0 - \theta^*_1)}{q^2-1}$	$\neq \theta^*_0$
λ^*	0	0
μ^*	$\dfrac{\theta^*_0 - \theta^*_1}{q^{d-2}(q^2-1)}$	$\dfrac{\theta^*_0 - \theta^*_1}{q^{d-2}(q^2-1)}$
ν^*	$\theta^*_0 - \kappa^* - \dfrac{q^2(\theta^*_0 - \theta^*_1)}{q^2-1}$	$\theta^*_0 - \kappa^* - \dfrac{q^2(\theta^*_0 - \theta^*_1)}{q^2-1}$

Table 5.4: Possible friendships by basic type

Type	q-Racah	q-Hahn	dual q-Hahn	q-Krawtchouk	dual q-Krawtchouk	affine q-Kraw.	quantum q-Kraw.
q-Racah	✓		✓		✓		
q-Hahn		✓		✓		✓	
dual q-Hahn	✓		✓		✓		
q-Krawtchouk		✓		✓		✓	
dual q-Krawtchouk	✓		✓		✓		
affine q-Kraw.		✓		✓		✓	
quantum q-Kraw.							✓

5.3 Concluding remarks

This thesis describes Leonard systems which are friends, that is, which share certain operators and a split decomposition. Friendship is not purely intrinsic, but depends on the choice of representation (shape). This leads to a two-part description of friends. First, we achieve the desired shape (split decomposition) by applying an sl_2- or $U_q(sl_2)$-module construction in terms of the equitable generators. The expressions involving the equitable generators give rise to the Leonine parameters that define a Leonard system of either classical or basic type. By taking two sets of Leonine parameters whose first six entries satisfy particular conditions we obtain a description of two Leonard systems that are friends.

We leave it as a problem to investigate friendship more thoroughly for Leonard systems of Bannai-Ito and orphan types. These cases do not have a well-developed algebraic construction to rely on. We expect that the Bannai-Ito case may be split according to whether the diameter is even or odd, once appropriate algebras are described. The orphan case may be small enough and special enough to be treated directly.

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