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Radial Versus Othogonal and Minimal Projections onto Hyperplanes in l_4^3

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Radial Versus Orthogonal and Minimal Projections onto Hyperplanes in ℓ_4^3

by

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Dedication

To my wife, you made this possible.

To my son, you made me believe that it was.

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Abstract

In this thesis, we study the relationship between radial projections, and orthogonal and minimal projections in ℓ_4^3. Specifically, we calculate the norm of the maximum radial projection and we prove that the hyperplane constant, with respect to the radial projection, is not achieved by a minimal projection in this space. We will also show our numerical results, obtained using computer software, and use them to approximate the norms of the radial, orthogonal, and minimal projections in ℓ_4^3. Specifically, we show, numerically, that the maximum minimal projection is attained for $\ker\{1, 1, 1\}$ as well as compute the norms for the maximum radial and orthogonal projections.

Chapter 1

Introduction

A projection P is a bounded linear operator that maps a normed linear space X onto a closed subspace V such that $P^2 = P$. In approximation theory, the best approximation to $x \in X$ in V is given by $d(x, V) = inf\{||x - v|| : v \in V\}$. Now if V is complemented in X, there exists a projection from X onto V and we have $\forall x \in X$

$$||x - Px|| \leq ||I - P|| \cdot d(x, V) \leq (1 + ||P||) \cdot d(x, V)$$

Since Px is an approximation of x , this inequality naturally leads us to the desire to make $||P||$ as small as possible in order to acquire a quality approximation. We can see that if, in particular, $||I - P|| = 1$ then P will result in a best approximation. This is one motivation for studying minimal projections. [**?**, **?**]

The ***operator norm*** of a linear operator L is defined as

$$||L|| = \sup\{||L(x)|| : ||x|| = 1\}$$

and the ***relative projection constant*** of a projection P by

$$\lambda(V, X) = \inf\{||P|| : P \in \mathcal{P}(X, V), P(X) = V, P = P^2\}$$

If $\lambda(V, X) = ||P||$ then P is called a ***minimal projection***. Finding minimal projections is a difficult problem for obvious reasons. With regard to the relative projection constant, we would be looking for a minimum maximum. We note that a projection of norm 1 is minimal; although, a subspace is is not generally in the range of a projection with norm 1. This compels us to look at the minimal projections onto a range of subspaces and to examine the maximum minimum, i.e. the hyperplane constant. This maximum, over all subspaces, can be looked at as a worst case scenario with respect to a best approximation.

In this paper, we explore the radial projection

$$P_r = Id - f \otimes N(f)$$

where $N(f)$ is the norming functional, and relate it to the minimal projection P_m and the orthogonal projection P_o from the unit sphere onto hyperplanes in ℓ_4^3. Like the orthogonal projection, the radial projection has certain properties that makes it an interesting candidate for exploration. Most importantly, all minimal projections in $\mathbb{R}^2$ are also radial projections. Hence the motivation for determining whether radial projections in higher dimensional spaces are minimal. Since in our case we are looking only at P_r analytically, and not all $P \in \mathcal{P}(X, V)$, the task is not as daunting, yet still significant. In the next section we will define $N(f)$, but we note here that it is dependent on f, which reduces our analytic task to a max/max problem.

The minimal projection, in addition to being defined with the relative projection constant, has other properties that are necessary but not sufficient for their minimality. We will use one such characteristic in the results of this paper to determine if indeed the radial projection is minimal in our space. Much work has been done on the properties which characterize minimal projections from other projections, as well on questions of existence and uniqueness. Some notable examples of work done on the properties which characterize minimal projections, as well on questions of existence and uniqueness are are listed for the reader to explore. [**?**, **?**, **?**]

Chapter 2

Preliminaries

For the sake of completeness, we will use this chapter to define and prove the necessary ideas relating to the main focus of this paper.

2.1 Normed Spaces and Projections

Definition 2.1.1 A ***norm*** on a linear space X is a function $||\cdot|| : X \to \mathbb{R}^+$ with the following properties:

For $x, y \in X$,

1) $||x|| \geq 0$ and $||x|| = 0$ iff $x = 0$

2) for a scalar c and a vector x, $||cx|| = |c|||x||$

3) $||x + y|| \leq ||x|| + ||y||$ (*Triangle Inequality*)

Definition 2.1.2 For $p \geq 1$,the ***p-norm*** of x is defined by:

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

To prove the ***p-norm*** is indeed a norm, we need the following fact and theorem.

If two exponents $p \geq 1$ and $q \leq \infty$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

then p and q are said to be **dual** or **conjugate** exponents. Note that since $1/\infty = 0$ then the case $p = 1$ implies $q = \infty$ and $p = \infty$ implies $q = 1$.

Theorem 2.1.1 Hölder's Inequality: Suppose p, q are dual, then for $a_i, b_i \in \mathbb{R}^n, i = 1, ..., n$

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$$

Proof:

Part (*i*)

The proof is dependent on the weighted form of the Arithmetic-Geometric Mean Inequality which states, for $a_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$

$$\sum_{i=1}^{n} \lambda_i a_i \geq \prod_{i=1}^{n} {a_i}^{\lambda_i}$$

then for $n = 2$, $a_1 = \alpha$ and $a_2 = \beta$ we will show

$$\alpha^{\lambda} \beta^{1-\lambda} \leq \lambda\, \alpha + (1 - \lambda)\, \beta \tag{2.1}$$

with equality if and only if $\alpha = \beta$.

We can assume $\beta \neq 0$ and we let $x = a/b$.

$$(\beta x)^{\lambda} \leq \lambda\, (\beta x) + (1 - \lambda)\beta \tag{2.2}$$

Now if we divide by β we can see (2.1) is equivalent to

$$x^{\lambda} \leq \lambda x + (1 - \lambda) \tag{2.3}$$

We now let $f(x) = \lambda x + (1 - \lambda) - x^{\lambda}$ and we show that $f(x) \geq 0$. Taking the derivative, $df/dx = \lambda \left(1 - x^{\lambda - 1}\right)$ and we see that $f(x)$ is increasing on $[0, 1)$, decreasing on $(1, \infty)$ and since $f(1) = 0$ this gives the result.

Now, if in (2.1), we let $\alpha = a^p, \beta = b^q$, and $\lambda = 1/p$ such that p and q are dual, then (2.1) becomes Young's Inequality.

$$a\, b \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.4}$$

We will use this in part (*ii*).

Part (*ii*)

Claim: If the following statement holds, then the proof will be complete.

$$\left\{ \sum_{i=1}^{n} |x_i|^p = 1 \; and \; \sum_{i=1}^{n} |y_i|^q = 1 \right\} \Rightarrow \sum_{i=1}^{n} |x_i\, y_i| \leq 1 \tag{2.5}$$

Let $\alpha = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ and $\beta = \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$ be as in (4) and then let $\hat{x}_i = \frac{x_i}{\alpha}$ and $\hat{y}_i = \frac{y_i}{\beta}$.

Now $\left(\sum_{i=1}^{n} |\hat{x}_i|^p\right)^{1/p} = 1$ and $\left(\sum_{i=1}^{n} |\hat{y}_i|^q\right)^{1/q} = 1$ and it follows that $\sum_{i=1}^{n} |\hat{x}_i|^p = 1$ and $\sum_{i=1}^{n} |\hat{y}_i|^q = 1$.

By (5) we now have $\sum_{i=1}^{n} |\hat{x}_i\, \hat{y}_i| \leq 1$ and $\sum_{i=1}^{n} |x_i\, y_i| \leq \alpha\beta$. Therefore our claim is sufficient, and we will now show that it holds.

Let $\sum_{i=1}^{n} |x_i|^p = 1 = \sum_{i=1}^{n} |y_i|^q$ and let $a = |x_i|$ and $b = |y_i|$.

By Young's inequality,

$$ab = |x_i\, y_i| \leq \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q}$$

Now if we sum with respect to i we have

$$\sum_{i=1}^{n} |x_i\, y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

■

We can now show that a ***p-norm*** is indeed a norm. It is clear that conditions 1) and 2), from the definition of a norm hold for the *p-norm*. To show the triangle inequality holds for the *p-norm*, we will prove the following theorem.

Theorem 2.1.2 Minkowski's Inequality: For $p \geq 1$

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$$

Proof:

For $p = 1$ the proof is trivial, so assume $p > 1$ and define q as dual to p. We have

$$\begin{aligned}\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i|\, |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \\ &\leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)\,q}\right)^{1/q} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)\,q}\right)^{1/q}\end{aligned}$$

by Hölder's inequality. Note $(p-1)q = (p-1)\dfrac{p}{(p-1)} = p$ so we can re-write as

$$\sum_{i=1}^{n} |x_i + y_i|^p \leq \left(\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{1/p} \right) \left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/q}$$

and then dividing by $\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/q}$ gives

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{(1-1/q)=p} \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p \right)^{1/p}$$

■

Thus the conditions on a norm hold for the ***p-norm***.

Definition 2.1.3 A **normed linear space** is a pair, $(V, ||\cdot||_V)$, where V is a linear space over a field of scalars, either the real or complex numbers, and $||\cdot||_V$ is a norm defined on that space.

Definition 2.1.4 Linear Operator: Given vector spaces V and W, over the same field of scalars F, an operator $L : V \to W$ is said to be linear if, for every pair of functions $f, g \in V$ and scalar $c \in F$, $L(f+g) = L(f) + L(g)$ and $L(cf) = cL(f)$.

Definition 2.1.5 Given a vector space X, a **projection** of X is a linear operator P such that $P^2 = P$

Proposition 2.1.1: A linear operator $P : X$ onto V is a projection if and only if $P|_V = I_V$.

Proof

Choose any $x \in X$ then $P(x) = v \in V$

$\Rightarrow P(P(x)) = P(v) = v = P(x)$

$\Rightarrow P^2 = P$

$\therefore P$ is a projection

Choose any $v \in V \Rightarrow \exists x \in X$ s.t. $P(x) = v$

Since P is a projection

$\Rightarrow P^2(x) = P(P(x)) = P(x) = v = P(v)$

$\therefore P|_V = I_V$

Definition 2.1.6 Let X be a normed space over a field $\mathbb{K}$, and let $V \subseteq X$ be a fixed subspace, then the ***relative projection constant*** of a projection P is given by

$$\lambda(V, X) = inf\{||P|| : P \in \mathcal{P}(X, V)\}$$

Definition 2.1.7 A projection is a ***minimal projection*** iff $||P|| = \lambda(X, V)$

Definition 2.1.8 Let X be a normed space over a field $\mathbb{K}$, a ***linear functional*** on X is a linear map $f : X \to \mathbb{K}$

Definition 2.1.9 Let X be a normed space over a field $\mathbb{K}$, then X^*, the space of all linear functionals $f : X \to \mathbb{K}$, is the ***dual space*** of X.

Theorem 2.1.3 Let X be a linear space, $P : X \to X$ be a linear opeator, and $f \in X^*$, then $P : X \to \ker f$ is a projection iff $\exists z$ where $f(z) = 1$ such that $P = Id - f \otimes z$.

Proof:

i) Assume $\exists z$ where $f(z) = 1$ such that $P = Id - f \otimes z$. We want to show that $P : X \to \ker f$ is a projection. Let $V = \ker f$ and take $x \in X$, then $Px \in V$ iff $f(Px) = 0$. Since $f(z) = 1$ we have $f(Px) = f(x - f(x)z) = f(x) - f(x)f(z) = 0$ implies $Px \in V$. Now if $x \in V$ then $f(x) = 0$ and $Px = x - f(x)z = x$ and from Proposition 2.1 $P|_{\ker f} = I_{\ker f}$ and P is a projection.

ii) Now assume $P : X \to V$ is a projection, then $I - P$ is also a projection since $P^2 = P$. Let $L = I - P$. Notice the $\ker(L) = im(P) = V$ so the dimension of $im(L) = 1$ since $dim(X) = n$ and $dim(V) = n - 1$. Now let $L : X \to W$ and since we now know L is an operator onto a one-dimensional space we can say $W = span\{w : w \neq 0\}$. So $\forall x \in X$, $Lx = g(x)z$ where $c : X \to \mathbb{R}$ is a linear functional. So $L = I - P = c(x)w$ and $P = I - c(x)w$. Since $P|_V = I_V$ then $\forall v \in V$ we have $v = v - c(v)w$ so $c(v) = 0$ and $\ker c = V$. Now $c(w) \neq 0$ otherwise $c|_X \cdot w = 0$ and P is the identity map on X. Let $c(w) = \alpha$, $\alpha \in \mathbb{K}$. Now we can write $c(x)w = \frac{c(x)}{\alpha}\alpha w$. If we let $\frac{c(x)}{\alpha} = f$ and $\alpha w = z$ gives the result. ■

Theorem 2.1.3 says that we can write any projection onto the $\ker f$ in the form $P = I - f \otimes z$. Geometrically speaking, we can project onto the $\ker f$ in any direction by varying z. For example, if we let $z = f$ then P is the orthogonal projection. We will now define and determine the norming functional of f which gives us the projection of interest in this paper.

Definition 2.1.10 Given a normed space X, a linear functional $f \in X^*$ such that $f(x) = ||x||_X$ and $||f||_{X^*} = 1$ is called a ***norming functional*** of x and will be denoted as $N(f)$.

Definition 2.1.11: Let $P : X \to V$ be a projection, then $x \in X : ||Px|| = ||P||$ is a ***norming point*** of P.

Proposition 2.1.2: Let $X = \ell^p$ and $S_p \in X$ be the unit sphere, then $N(f) = sgn(x_i)|x_i|^{q/p}$ $i = 1, 2, ..., n$.

Proof:

From our definition of a norming functional, we are looking for $f \in X^*$ such that$||f||_q = 1$ and $f(x) = 1$ since our space is the unit sphere. We can write

$$f(x) = \sum_{i=1}^{n} f_i\, x_i = f_1\, x_1 + f_2\, x_2 + ... + f_n\, x_n = 1 \tag{2.6}$$

Now consider Young's Inequality. (2.4)

$$a\, b \leq |a||b| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q} \quad \text{s.t. } \frac{1}{p} + \frac{1}{q} = 1\, ,\, p > 1 \tag{2.7}$$

and apply it to (2.6), representing each term on the RHS of the inequality.

$$\begin{aligned} f_1\, x_1 + f_2\, x_2 + ... + f_n\, x_n &\leq \left(\frac{|f_1|^q}{q} + \frac{|x_1|^p}{p}\right) + \left(\frac{|f_2|^q}{q} + \frac{|x_2|^p}{p}\right) + ... + \left(\frac{|f_n|^q}{q} + \frac{|x_n|^p}{p}\right) \\ &= \frac{1}{q}\left(|f_1|^q + ... + |f_n|^q\right) + \frac{1}{p}\left(|x_1|^p + ... + |x_n|^p\right) \\ &= \frac{1}{q} + \frac{1}{p} = 1 \end{aligned} \tag{2.8}$$

Note that equality holds in (2.8) only when $f_j\, x_j = \dfrac{f_j^q}{q} + \dfrac{x_j^p}{p}$ for $j = 1, 2..., n$ since $f(x) = 1$.

Since what we are looking for is f when equality holds in (2.8) we let

$$g(b) = \frac{a^p}{p} + \frac{b^q}{q} - xb \geq 0$$

Evaluating at the extremal points $g(0) = \dfrac{a^p}{p} > 0$ and $g(\infty) > 0$ and $\dfrac{\partial g}{\partial b} = b^{q-1} - a$ so $g'(b) = 0$ iff $b^{q-1} = a$ so $b = a^{p/q}$

Therefore $g(b)$ has a unique minimum when $b = a^{p/q}$. Now $g(a^{p/q}) = 0$, so $g(b) = 0$ iff $b = a^{p/q}$ and we can say equality holds in g iff $a^p = b^q$ when $a, b \geq 0$. We must, however, consider the case when $(-a)^p = (-b)^q$ then $b = -|a|^{p/q}$. So $b = sgn(a)|a|^{p/q}$ for all a, b.

2.2 Gröbner Bases

The results in Chapter 4 were obtained, in part, with the help of symbolic computation software. We use this section to define a Gröbner basis, which was calculated with the software, and provide a means to prove that the results obtained were indeed what they claimed to be. It is our aim to eliminate concerns brought on by any ambiguity of the software's methods by confirming the validity of the results. We first give the necessary foundation and then present Buchberger's Algorithm, by which Gröbner bases are calculated. The algorithm was first published in Buchnerger's 1965 Ph.D. thesis [**?**]

Definition 2.2.1: Let $\mathfrak{a}$ be a non zero ideal in $\mathbb{K}[x_1, ..., x_n]$, with respect to some monomial ordering, a set $G = \{g_1, ..., g_k\} \in \mathfrak{a}$ is a **Gröbner Basis** of $\mathfrak{a}$ iff G is finite and, $\prec$, for all $f \in \mathfrak{a}$ there exists some $g \in G$ whose leading monomial, $LM(g)$ divides $LM(f)$.

Since G is required to be finite, it is unclear, at this point, whether a Gröbner basis exists. The following is one example that follows from the definition above.

Proposition 2.2.1: Let $\mathfrak{a} = (m_1, m_2, ..., m_s) \subset \mathbb{K}[x_1, ..., x_n]$ be the ideal generated by the monomials $m_1, m_2, ..., m_s$, then $G = \{m_1, m_2, ..., m_s\}$ is a Gröbner basis of $\mathfrak{a}$.
Proof: For all $f \in \mathfrak{a}$, $f = \beta m_1 + \beta m_2 + ... + \beta m_s$, where $\beta_i \in \mathbb{K}[x_1, ..., x_n]$. Therefore, every term of f is divisible by some m_i. ■

Definition 2.2.2: Polynomials that are obtained from variables by multiplication are **power products**. A **leading power product**, denoted $LT(p)$ is the largest power product appearing in a polynomial p, based on some monomial ordering $\prec$ on $\mathbb{K}[x_1, ..., x_n]$. [**?**]

An acceptable ordering $\prec$ has the following properties.

(*i*) For any pair of monomials m, n we have $m \prec n$ or $n \prec m$ or $m = n$

(*ii*) If $m_1 \prec m_2$ and $m_2 \prec m_3$ then $m_1 \prec m_3$

(*iii*) $1 \prec m$ for any monomila $m \neq 1$

(*iv*) If $m_1 \prec m_2$ then $mm_1 \prec mm_2$ for any monomial m.

Now if we have an ordering $\prec$ on $\mathbb{K}[x_1, ..., x_n]$, every $f \in \mathbb{K}[x_1, ..., x_n]$ has a $LT(f)$ with respect to the ordering. If $\mathfrak{a}$ is an ideal, we are able to distinguish the leading power product of all polynomials in the ideal.

Definition 2.2.3: An ideal $\mathfrak{a} \in \mathbb{K}[x_1, ..., x_n]$ has monomial order $\prec$, then the **initial ideal** of $\mathfrak{a}$

$$LT(\mathfrak{a}) = \langle\{LT(f) : f \subset \mathfrak{a}\}\rangle \in \mathbb{K}[x_1, ..., x_n]$$

is the ideal generated by the leading power product of every element in $\mathfrak{a}$

The previous definition allows us now to define a Gröbner basis in terms of $LT(\mathfrak{a})$

Definition 2.2.4 Let $\mathfrak{a}$ be an ideal in $\mathbb{K}[x_1, ..., x_n]$. A set $G = \{g_1, ..., g_s\} \subset \mathfrak{a}$ is a **Gröbner basis** of $\mathfrak{a}$ iff $\langle LT(g_1), ..., LT(g_s)\rangle = LT(\mathfrak{a})$ [**?**]

Theorem 2.2.1: Let $G = \{m_1, m_2, ..., m_s\}$ be a Gröbner basis for $\mathfrak{a} \in \mathbb{K}[x_1, ..., x_n]$, then for any $f \in \mathbb{K}[x_1, ..., x_n]$ there are polynomials $q_1, ...q_s, r \in \mathbb{K}[x_1, ..., x_n]$ such that $f = q_1 g_1 + ... + q_s g_s + r$. Where r is reduced with respect to G and is uniquely determined. and also $LT(f) \geq LT(q_i)LT(g_i)$ for $i = 1, ..., s$.

Proof: The division algorithm tells us that this r exists, the remainder of the proof addresses the uniqueness. Suppose $f = g_1 + r_1 = g_2 + r_2$, and we claim $r_1 = r_2$. Clearly $r_1 - r_2 = g_2 - g_1 \in \mathfrak{a}$ so $\exists g \in G$ such that $LM(g)$ divides $LM(r_1 - r_2)$. Since r_1 and r_2 are reduced with respect to G, $(r_1 - r_2)$ is also. This implies there is no monomial in $(r_1 - r_2)$ that is divisible by $LM(g)$ for any $g \in G$ unless $r_1 - r_2 = 0$. ■

We will denote the remainder of f after division by G as $\overline{f}^G$.

Corollary to Theorem 2.2.1: Let $G = \{m_1, m_2, ..., m_s\}$ be a Gröbner basis for $\mathfrak{a} \in \mathbb{K}[x_1, ..., x_n]$ and $f \in \mathbb{K}[x_1, ..., x_n]$. Then

(*i*) there is a unique r that is reduced with respect to G and so $f - r \in \mathfrak{a}$.

(*ii*) $f \in \mathfrak{a}$ iff $\overline{f}^G = 0$

If $f \in \mathfrak{a}$ then $r = 0$ satisfies part (*i*) then clearly part (*ii*) follows

The main element of Buchberger's Algorithm and, therefore, in determining a Gröbner basis is the notion of an S-polynomial.

Definition 2.2.5 Let $f \neq 0$ and $g \neq 0$ be elements in $\mathbb{K}[x_1, ..., x_n]$, then the **S-Polynomial** of (f, g) in $\prec$ is

$$S(f, g) = LCM(LM(f), LM(g))\left(\frac{f}{LT(f)} - \frac{g}{LT(g)}\right)$$

where $LCM(f, g)$ is the least common multiple of f and g.

Theorem 2.2.2 (Buchberger's Criterion): Let $G = \{g_1, ..., g_s\} \subset \mathbb{K}[x_1, ..., x_n]$ be a set of non-zero poloynomials, and let $I = \langle g_1, ..., g_s \rangle$. Then G is a Gröbner basis for I iff

$$\overline{S(g_i, g_j)}^G = 0$$

for all $1 \leq i < j \leq s$.

The proof of Buchberger's Criterion is lengthy and, therefore, omitted here; see [**?**]. Buchberger's Algorithm, Figure 1 below, follows from Theorem 2.2.2, which is used to compute a Gröbner basis.

```
Input: A finite polynomial set F = {f_1, ..., f_n}, that generates an ideal I
Output: A Gröbner basis G = {g_1, ..., g_t}, that generates the same ideal I
G := F
C := G × G;
while C ≠ ∅ do
  Choose a pair (f, g) from C;
  C := C \ {(f, g)};
  h := \overline{S(f, g)}^G;
  if h ≠ 0 then
     C := C ∪ (G × {h});
     G := G ∪ {h};
  end
  return G
end
```

Figure 1.: Buchberger's Algorithm

It is important to note that Buchberger's algorithm will terminate, and therefore confirms the existence of a Gröbner basis. This termination is supported by the following, well known, lemma and theorem.

Lemma 2.2.1 (Dickson's Lemma): Every monomial ideal in a polynomial ring, $\mathbb{K}[x_1, ..., x_n]$, is finitely generated.

Theorem 2.2.3 (Hilbert's Basis Theorem): If $I \subset \mathbb{K}[x_1, ..., x_n]$ is an ideal, then there are finitely many polynomials $f_1, ..., f_s \in I$ such that $I = \langle f_1, ..., f_s \rangle$.
In other words, every ideal $I \subset \mathbb{K}[x_1, ..., x_n]$ is finitely generated.

The proof of Hilbert's Basis Theorem follows directly from Dickson's Lemma, who's proof follows from induction on the number of variables, n. Let I be an ideal, then consider $LT(I)$. (recall definition 2.2.3). By Dickson's Lemma, $LT(I)$ is generated by finitley many of the monomials $LT(f) \in I$, so $LT(I) = \langle LT(f_1), ...LT(f_s) \rangle$ for $f_1, ..., f_s \in I$. Therefore $\{f_1, ..., f_s\}$ is a Gröbner basis, and in particular a basis, by definition. The following corollary demonstates existence of Gröbner bases.

A Gröbner basis G of I is reduced if

i) G is a minimal set of generators for $LT(I)$.

ii) The leading coefficient of $g_i \in G$ is 1

iii) No $LM(g_i)$ is in $\langle LM(g_j) \rangle \quad \forall j \neq i$

Although Gröbner bases are not unique, reduced Gröbner bases are unique for all non-zero ideals. [**?**, p. 53] This property is useful in determining whether two sets of polynomials generate the same ideal and can be used to verify if a basis, calculated by software, is Gröbner.

Chapter 3

Numerial Analysis

3.1 Algorithm

In order to gain a better understanding of the relation between $||P_m||$, $||P_r||$, and $||P_o||$ initially, we performed a numerical analysis using a computer algebra system. In this chapter we describe our process and give the numerical results obtained. We will use the term ”minimal” and the notation $||P||$ loosely in this section.

In our case we define $P = I - f \otimes z$

$$P : S(\ell_4^3) \to \ker f$$

$$(x_1, x_2, x_3) \mapsto \Big(x_1 - (f_1x_1 + f_2x_2 + f_3x_3)z_1, x_2 - (f_1x_1 + f_2x_2 + f_3x_3)z_2, x_3 - (f_1x_1 + f_2x_2 + f_3x_3)z_3\Big)$$

to be all the projections from the unit sphere in ℓ_4^3 onto $\ker f$. Then the norm

$$||Px|| = \Big((x_1 - (f_1x_1 + f_2x_2 + f_3x_3)z_1)^4 + (x_2 - (f_1x_1 + f_2x_2 + f_3x_3)z_2)^4 + (x_3 - (f_1x_1 + f_2x_2 + f_3x_3)z_3)^4\Big)^{1/4} \tag{3.1}$$

The problem of calculating the norm of P is the same as maximizing $||P||$ for a fixed z and fixed f. However, we want to examine the norms over all z and f. Specifically, we are looking for

$$\eta = \sup_f \left\{ \inf_z \left\{ \sup_{x \in S(\ell_4^3)} \{||Px||\} \right\} \right\} \tag{3.2}$$

We will call this η the **hyperplane constant** of $S(\ell_4^3)$. We begin by first fixing the hyperplane with a functional f, then varying z over a desired range for each hyperplane. Then for each z we ask the software to maximize the equation using its own algorithm.We then record the norm for each projection, until the range of z is covered. We then sort the set of norms onto f by value, and then record the min norm, $\lambda\Big(\ker f, (S(\ell_4^3)\Big)$. The loop then returns to vary f and repeats until we have covered the desired range of f. When the process is completed we are left with a second list consisting of $\lambda\Big(\ker f, (S(\ell_4^3)\Big)$, f, and the

z for which the relative projection constant was attained. The list was then sorted by the norm as a means to discover the max/min/max projection for which η. For the radial and orthogonal cases z is dependent on f; therefore, we calculated and collected the max/max, $\lambda_r \left(\ker f, (S(\ell_4^3)\right)$ and $\lambda_o \left(\ker f, (S(\ell_4^3)\right)$ projection.

Of course the size of the increments by which we choose to vary the functionals is critical in the process. The method used was to run the algorithm at larger increments and then reduce the size around a set of minimal norm candidates until $||P||$ stabilized at a minimal value agreed upon by a significant number of trials. The total number of projections whose norms were calculated, to estimate $||P_m||$ alone, was in the hundreds of billions. Increments of the functionals f and z in this paper ranged from one to 0.00001 degrees.

Additionally, the necessary range and limits of z were considered as a means to reduce the number of iterations required for each fixed hyperplane when attempting to determine $||P_m||$. Geometrically speaking, we hypothesized that at some point, as the angle between f and z grew larger, the norms of the projections would grow larger. Since we were looking for the minimum projection, the range of z needed only to be within a neighborhood of f, which was determined to be no greater than 50 degrees. This hypothesis was tested and supported anecdotally.

The pseudo-code in Figure 2 represents the algorithm along with the parameterization used. Note that $||P||$ has been split into eight separate equations, each representing one octant of the unit sphere determined by θ and $r = (0, 1)$ in each in order to eliminate division by zero.

for Θ *from* 0 *to* 90 *by some increment* **do**

for Φ *from* 0 *to* 90 *by some increment* **do**

$f_1 = \cos\left(\frac{\Theta\pi}{180}\right)\sin\left(\frac{\Phi\pi}{180}\right);$

$f_2 = \sin\left(\frac{\Theta\pi}{180}\right)\sin\left(\frac{\Phi\pi}{180}\right);$

$f_3 = \cos\left(\frac{\Phi\pi}{180}\right);$

for A *from* Φ *to* $(\Phi + c)$ *by some increment* **do** *% set c for max angle between z and f*

for B *from* Θ *to* $(\Theta + c)$ *by some increment* **do**

$u = \left\langle \sin\left(\frac{A\pi}{180}\right)\cos\left(\frac{B\pi}{180}\right), \sin\left(\frac{A\pi}{180}\right)\sin\left(\frac{B\pi}{180}\right), \cos\left(\frac{A\pi}{180}\right)\right\rangle;$

$v = \langle f_1, f_2, f_3\rangle;$

$l = u.v;$

$z_1 = \frac{1}{l}\left(\cos\left(\frac{A\pi}{180}\right)\left(\sin\left(\frac{B\pi}{180}\right)\right)\right);$

$z_2 = \frac{1}{l}\left(\sin\left(\frac{A\pi}{180}\right)\left(\sin\left(\frac{B\pi}{180}\right)\right)\right);$

$z_3 = \frac{1}{l}\left(\cos\left(\frac{A\pi}{180}\right)\right);$

$$||P_{(i=1,\ldots,8)}(\theta, r)|| = \Bigg(\Big|\Big(\pm(r\cos\theta)^{1/2} - \Big(\pm f_1(r\cos\theta)^{1/2} \pm f_2(r\sin\theta)^{1/2} \pm f_3(1-r^2)^{1/4}\Big)z_1\Big)\Big|^4 + \Big|\Big(\pm(r\sin\theta)^{1/2} - \Big(\pm f_1(r\cos\theta)^{1/2} \pm f_2(r\sin\theta)^{1/2} \pm f_3(1-r^2)^{1/4}\Big)z_2\Big)\Big|^4 + \Big|\Big(\pm(1-r^2)^{1/4} - \Big(\pm f_1(r\cos\theta)^{1/2} \pm f_2(r\sin\theta)^{1/2} \pm f_3(1-r^2)^{1/4}\Big)z_3\Big)\Big|^4\Bigg)^{1/4}$$

% Maximize using software's algorithm for each z.

end

end

end

% Write $\min_{z} \max_{x\in S(\ell_3^4)} ||P||$ *with respect to fixed hyperplane to file*

end

% Sort desending to find max/min/max

Figure 2.: Norm Calculating Algorithm

3.2 Numerical Results

The algorithm, as written, in Figure 2 is such that it will look at all projections, within the specified range and intervals, onto each hyperplane; in other words, z does not depend on f. In this form the algorithm will return the max/min/max in (3.2). Of course we use the term "all" loosely here. The number of norms calculated depends on the range and increment size of z. To examine the radial and orthogonal projections specifically, we need to fix z.

From Proposition 2.1.2 we know that when we let $z = N(f) = sgn(f_i)|f_i|^{q/p}$ the projection will be radial. So by letting $f = (f_1^3, f_2^3, f_3^3)$ and $N(f) = (f_1, f_2, f_3) = z$ our algorithm will look only at the radial projection onto each hyperplane.

For the case of the orthogonal projection we note the following. Consider the projection $P : \mathbb{R}^{\ltimes} \rightarrow$ hyperplane. If we equip the space with the ℓ^2 norm, then for $f \in S(\ell^2)$, $f_1^2 + f_2^2 + ... + f_n^2 = 1$. Therefore the orthogonal projection is given by $P = I - f \otimes f$

Figures 3, 4, and 5 below show the results of our numerical analysis for the radial, orthogonal, and minimal projections, respectively. Listed are the largest, $\max_f$, seven norms for each type calculated. The number of unique projections displayed is given in order that the reader have a sense of the nature of the data gathered and is, therefore, arbitrary.

Radial						
f1	f2	f3	z1	z2	z3	||P||
0.5602631453	0.5602631453	0.1451093763	0.8243861465	0.8243861465	0.5254908501	1.21156503128450
0.1451096178	0.1451096178	0.8832870125	0.5254911417	0.5254911417	0.9594756296	1.21156503120850
0.1451185734	0.1451185734	0.8832772029	0.5255019518	0.5255019518	0.9594720777	1.21156503118837
0.0862825514	0.0862825514	0.9422465408	0.4418833721	0.4418833721	0.9803658709	1.21156503118359
0.0862904135	0.0862904135	0.9422394533	0.4418967933	0.4418967933	0.9803634128	1.21156503106640
0.5602589308	0.5602589308	0.1451225988	0.8243840794	0.8243840794	0.5255068107	1.21156503094451
0.1451006624	0.1451006624	0.8832968218	0.5254803313	0.5254803313	0.9594791814	1.21156503070066

Figure 3.: Numerical Results - Radial

Orthogonal						
f1	f2	f3	z1	z2	z3	||P||
0.6523350000	0.6523350000	0.3859059045	0.6523327059	0.6523327059	0.3859059045	1.09610933570639
0.6522350000	0.6522350000	0.3862301317	0.6522411706	0.6522411706	0.3862301317	1.09610775515816
0.6519050000	0.6519050000	0.3873549308	0.6519039141	0.6519039141	0.3873549308	1.09610735142192
0.6521400000	0.6521400000	0.3865517467	0.6521452207	0.6521452207	0.3865517467	1.09610711249640
0.6524750000	0.6524750000	0.3854236051	0.6524768372	0.6524768372	0.3854236051	1.09610701628306
0.6525700000	0.6525700000	0.3851016611	0.6525722302	0.6525722302	0.3851016611	1.09610700512136
0.6522850000	0.6522850000	0.3860673588	0.6522858253	0.6522858253	0.3860673588	1.09610685800900

Figure 4.: Numerical Results - Orthogonal

Minimal						
f1	f2	f3	z1	z2	z3	||P||
0.5773502692	0.5773502692	0.5773502692	0.5773502692	0.5773502692	0.5773502692	1.06416586284650
0.5823665531	0.5722900179	0.5773502692	0.5876250828	0.5670666744	0.5772236138	1.06415473498848
0.5722900179	0.5823665531	0.5773502692	0.5670666744	0.5876250828	0.5772236138	1.06415473498848
0.5873384875	0.5671861844	0.5773502692	0.5977248874	0.5568026403	0.5769849283	1.06408585184202
0.6067752092	0.5463428523	0.5773502692	0.6390400966	0.5174844668	0.5707494908	1.06403301474462
0.5922656938	0.5620391575	0.5773502692	0.6078497947	0.5463506688	0.5766360003	1.06394215108404
0.5971477968	0.5568493292	0.5773502692	0.6178822861	0.5359819193	0.5760312235	1.06377314719311

Figure 5.: Numerical Results - Minimal

Examining the functionals at which these relevant values were computed, reveals some interesting characteristics. For $||P_m||$ the max/min/max value was attained for functionals

$$f_1 = f_2 = f_3 = z_1 = z_2 = z_3$$

This is an orthogonal projection, $P = I - f \otimes f$, but was lost when calculating the orthogonal norms since the algorithm found a max/max, rather than the max/min/max, in that case. We can also see that this projection is radial, when we normalize the functionals.

For the radial and orthogonal cases the max/max value was attained and we note that the functionals were of the form

$$f_1 = f_2 \text{ and } z_1 = z_2$$

We will examine the consequence of these functional forms and compare our numerical results with the analytic in this paper's conclusion.

Chapter 4

Analytic Analysis

4.1 Radial Projection Norm Calculation

We now have our radial projection $P_r = I - f \otimes N(f)$ where $f \otimes N(f)$ denotes the one-dimensional operator from X to X such that $\big(f \otimes N(f)\big)(x) = \big(f(x)\big)N(f)$.

Let $S(\ell_4^3)$ be the unit sphere then $P_r : S(\ell_4^3) \to \ker f$. We let $f = (a^3, b^3, c^3)$ and
$N(f) = sgn(f_i)|f_i|^{q/p} = (a, b, c)$

Finding the norm of this projection is equivalent to finding the maximum of

$$
\begin{aligned}
&||P_r||_4 \\
&= \Big((x - a(a^3x + b^3y + c^3z))^4 + (y - b(a^3x + b^3y + c^3z))^4 + (z - c(a^3x + b^3y + c^3z))^4\Big)^{1/4} \quad (4.1)
\end{aligned}
$$

with the constraints

$$S\big(\ell_4^3\big) = x^4 + y^4 + z^4 = 1 \quad \text{and} \quad f\big(N(f)\big) = a^4 + b^4 + c^4 = 1$$

We apply the method of Lagrange Multipliers to maximize $||P_r||_4$.

$$
\begin{aligned}
\Lambda = (x - a(a^3x + b^3y + c^3z))^4 + (y - b(a^3x + b^3y + c^3z))^4 + (z - c(a^3x + b^3y + c^3z))^4 \\
- \mu(a^4 + b^4 + c^4 - 1) - \lambda(x^4 + y^4 + z^4 - 1) \quad (4.2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \Lambda}{\partial x} = 4\left(x - a\left(a^3x + b^3y + c^3z\right)\right)^3\left(-a^4 + 1\right) - 4\left(y - b\left(a^3x + b^3y + c^3z\right)\right)^3 ba^3 \\
- 4\left(z - c\left(a^3x + b^3y + c^3z\right)\right)^3 ca^3 - 4\,\lambda x^3 = 0 \quad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \Lambda}{\partial y} = -4\left(x - a\left(a^3x + b^3y + c^3z\right)\right)^3 ab^3 + 4\left(y - b\left(a^3x + b^3y + c^3z\right)\right)^3\left(-b^4 + 1\right) \\
- 4\left(z - c\left(a^3x + b^3y + c^3z\right)\right)^3 cb^3 - 4\,\lambda y^3 = 0 \quad (4.4)
\end{aligned}
$$

$$\frac{\partial \Lambda}{\partial z} = -4\,\left(x - a\left(a^3x + b^3y + c^3z\right)\right)^3 ac^3 - 4\,\left(y - b\left(a^3x + b^3y + c^3z\right)\right)^3 bc^3 + 4\,\left(z - c\left(a^3x + b^3y + c^3z\right)\right)^3\left(-c^4 + 1\right) - 4\,\lambda z^3 = 0 \quad (4.5)$$

$$\frac{\partial \Lambda}{\partial a} = 4\,\left(x - a\left(a^3x + b^3y + c^3z\right)\right)^3\left(-4\,a^3x - b^3y - c^3z\right) - 12\,\left(y - b\left(a^3x + b^3y + c^3z\right)\right)^3 ba^2x - 12\,\left(z - c\left(a^3x + b^3y + c^3z\right)\right)^3 ca^2x - 4\,\mu a^3 = 0 \quad (4.6)$$

$$\frac{\partial \Lambda}{\partial b} = -12\,\left(x - a\left(a^3x + b^3y + c^3z\right)\right)^3 ab^2y + 4\,\left(y - b\left(a^3x + b^3y + c^3z\right)\right)^3\left(-a^3x - 4\,b^3y - c^3z\right) - 12\,\left(z - c\left(a^3x + b^3y + c^3z\right)\right)^3 cb^2y - 4\,\mu b^3 = 0 \quad (4.7)$$

$$\frac{\partial \Lambda}{\partial c} = -12\,\left(x - a\left(a^3x + b^3y + c^3z\right)\right)^3 ac^2z - 12\,\left(y - b\left(a^3x + b^3y + c^3z\right)\right)^3 bc^2z + 4\,\left(z - c\left(a^3x + b^3y + c^3z\right)\right)^3\left(-a^3x - b^3y - 4\,c^3z\right) - 4\,\mu c^3 = 0 \quad (4.8)$$

$$\frac{\partial \Lambda}{\partial \lambda} = -x^4 - y^4 - z^4 + 1 = 0 \quad (4.9)$$

$$\frac{\partial \Lambda}{\partial \mu} = -a^4 - b^4 - c^4 + 1 = 0 \quad (4.10)$$

We make some substitutions to simplify the derivatives and let $\Omega = a^3x + b^3y + c^3z$ and $M = (-\Omega\,a + x)^3\,a + (-\Omega\,b + y)^3\,b + (-\Omega\,c + z)^3\,c$ in equations (4.3) through (4.8).

Simplifying gives:

$$\frac{\partial \Lambda}{\partial x} = (-\Omega\,a + x)^3 - a^3M - \lambda x^3 = 0 \quad (4.11)$$

$$\frac{\partial \Lambda}{\partial y} = (-\Omega\,b + y)^3 - b^3M - \lambda y^3 = 0 \quad (4.12)$$

$$\frac{\partial \Lambda}{\partial x} = (-\Omega\,c + z)^3 - c^3M - \lambda z^3 = 0 \quad (4.13)$$

$$\frac{\partial \Lambda}{\partial a} = (-\Omega\,a + x)^3\,\Omega - 3\,a^2xM - \mu a^3 = 0 \quad (4.14)$$

$$\frac{\partial \Lambda}{\partial b} = (-\Omega\,b + y)^3\,\Omega - 3\,b^2yM - \mu b^3 = 0 \quad (4.15)$$

$$\frac{\partial \Lambda}{\partial c} = (-\Omega\,c + z)^3\,\Omega - 3\,c^2zM - \mu c^3 = 0 \quad (4.16)$$

With another simplifying substitution we now let $p = \frac{x}{a}$, $s = \frac{y}{b}$, and $t = \frac{z}{c}$ in equations (4.11) through (4.16)

Simplifying gives:

$$\frac{\partial \Lambda}{\partial x} = (p - \Omega)^3 - M - \lambda p^3 = 0 \tag{4.17}$$
$$\frac{\partial \Lambda}{\partial y} = (s - \Omega)^3 - M - \lambda s^3 = 0 \tag{4.18}$$
$$\frac{\partial \Lambda}{\partial z} = (t - \Omega)^3 - M - \lambda t^3 = 0 \tag{4.19}$$
$$\frac{\partial \Lambda}{\partial a} = (p - \Omega)^3 \Omega + 3\,Mp + \mu = 0 \tag{4.20}$$
$$\frac{\partial \Lambda}{\partial b} = (s - \Omega)^3 \Omega + 3\,Ms + \mu = 0 \tag{4.21}$$
$$\frac{\partial \Lambda}{\partial c} = (t - \Omega)^3 \Omega + 3\,Mt + \mu = 0 \tag{4.22}$$

Let

$$g_1 = \frac{\partial \Lambda}{\partial x} - \frac{\partial \Lambda}{\partial y} = (p - \Omega)^3 - \lambda p^3 - (s - \Omega)^3 + \lambda s^3 = 0 \tag{4.23}$$
$$g_2 = \frac{\partial \Lambda}{\partial x} - \frac{\partial \Lambda}{\partial z} = (p - \Omega)^3 - \lambda p^3 - (t - \Omega)^3 + \lambda t^3 = 0 \tag{4.24}$$

and

$$g_3 = \frac{\partial \Lambda}{\partial a} - \frac{\partial \Lambda}{\partial b} = (p - \Omega)^3 \Omega + 3\,Mp - (s - \Omega)^3 \Omega - 3\,Ms = 0 \tag{4.25}$$
$$g_4 = \frac{\partial \Lambda}{\partial a} - \frac{\partial \Lambda}{\partial c} = (p - \Omega)^3 \Omega + 3\,Mp - (t - \Omega)^3 \Omega - 3\,Mt = 0 \tag{4.26}$$

With the help of computer algebra software, we calculate the Gröbner basis for $\{g_1, g_2\}$:

$$\Big\{ (s - t)\left(-\lambda s^2 - \lambda st - \lambda t^2 + 3\,\Omega^2 - 3\,\Omega\,s - 3\,\Omega\,t + s^2 + st + t^2\right), \tag{4.27}$$
$$(p - t)\left(-\lambda p^2 - \lambda pt - \lambda t^2 + 3\,\Omega^2 - 3\,\Omega\,p - 3\,\Omega\,t + p^2 + pt + t^2\right), \tag{4.28}$$
$$\Omega\,(s - t)(p - t)(p - s)(\Omega\,p + \Omega\,s + \Omega\,t - ps - pt - st)\Big\} \tag{4.29}$$

and

the Gröbner basis for $\{g_3, g_4\}$:

$$\Big\{ (p-s)\left(3\,\Omega^3 - 3\,p\Omega^2 - 3\,s\Omega^2 + \Omega\,p^2 + sp\Omega + \Omega\,s^2 + 3\,M\right), \tag{4.30}$$

$$(p-t)\left(3\,\Omega^3 - 3\,p\Omega^2 - 3\,t\Omega^2 + \Omega\,p^2 + pt\Omega + t^2\Omega + 3\,M\right), \tag{4.31}$$

$$M\,(s-t)\,(p-t)\,(p-s)\,(-p-s-t+3\,\Omega)\Big\} \tag{4.32}$$

With the methods shown in Section 2.2 these Gröbner bases can be verified.

In an attempt to simplify $||P_r||$ even further, we prove the following proposition.

Proposition 4.1: In equations (4.28) through (4.33) $p = s$ or $p = t$ or $s = t$.

Proof:

Claim 1: $\Omega \neq 0$

Rewrite $\parallel \phi_r \parallel$, substituting Ω for $\left(a^3x + b^3y + c^3z\right)$

$$\left((x - a\,\Omega)^4 + (y - b\,\Omega)^4 + (z - c\,\Omega)^4\right)^{1/4} = \left(x^4 + y^4 + z^4\right)^{1/4} = 1 \tag{4.33}$$

Clearly the maximum norm is not 1; therefore, $\Omega \neq 0$

Claim 2: $M \neq 0$

Assume $M = 0$ and consider the derivatives (4.17), (4.18), and (4.19).

$$(p-\Omega)^3 = \lambda p^3$$
$$(s-\Omega)^3 = \lambda s^3$$
$$(t-\Omega)^3 = \lambda s^t$$
$$\Rightarrow \frac{\Omega}{p} = \frac{\Omega}{s} = \frac{\Omega}{t} \Rightarrow \Omega = 0 \text{ or } p = s = t$$

If $p = s = t$ the proof is complete; therefore, we assume $M \neq 0$.

Now let us assume

$$(s-t) \neq 0,\ (p-t) \neq 0, \text{ and } (p-s) \neq 0$$

Multiplying equations (4.27) and (4.28) by $\frac{1}{(s-t)}$ and $\frac{1}{(p-t)}$ respectively, then subtracting (4.27) from (4.28) gives

$$(p-s)(\lambda(p+s+t) + 3\Omega - (p+s+t)) = 0$$

Since $(p-s) \neq 0$ we have

$$\lambda(p+s+t) + 3\Omega - (p+s+t) = 0 \tag{4.34}$$

Since $M \neq 0$ and we are assuming $(s-t) \neq 0$, $(p-t) \neq 0$, and $(p-s) \neq 0$, equation (4.32) becomes $3\Omega = p+s+t$ and now (4.34) becomes $\lambda(p+s+t) = 0$. Now if $p+s+t = 0$ then $\Omega = 0$, but, again $\Omega = 0 \Rightarrow \| \phi_r \| = 1$, which is not maximal; therefore, $\lambda = 0$.

Now consider equations (4.11), (4.12), and (4.13).

$$0 = \left[(-\Omega\, a + x)^3 - a^3 M - \lambda x^3\right] x + \left[(-\Omega\, b + y)^3 - b^3 M - \lambda y^3\right] y + \left[(-\Omega\, c + z)^3 - c^3 M - \lambda z^3\right] z$$

$$= (-\Omega\, a + x)^3\; x - a^3\, M\, x - \lambda\, x^4 + (-\Omega\, b + y)^3\; y - b^3\, M\, y - \lambda\, y^4 + (-\Omega\, c + z)^3\; z - c^3\, M\, z - \lambda\, z^4$$

$$= (-\Omega\, a + x)^3\; x - a^3\, M\, x - \lambda\, x^4 + (-\Omega\, b + y)^3\; y - b^3\, M\, y - \lambda\, y^4 + (-\Omega\, c + z)^3\; z - c^3\, M\, z - \lambda\, z^4$$

$$= -M\, \left(a^3\, x + b^3\, y + c^3\, z\right) + (-\Omega\, a + x)^3\; x + (-\Omega\, b + y)^3\; y + (-\Omega\, c + z)^3\; z - \lambda\, \left(x^4 + y^4 + z^4\right)$$

$$= -\left[(-\Omega\, a + x)^3\, a + (-\Omega\, b + y)^3\, b + (-\Omega\, c + z)^3\, c\right]\, \Omega + (-\Omega\, a + x)^3\; x + (-\Omega\, b + y)^3\; y + (-\Omega\, c + z)^3\; z = \lambda$$

$$= -\,(-\Omega\, a + x)^3\; a\,\Omega - (-\Omega\, b + y)^3\; b\,\Omega - (-\Omega\, c + z)^3\; c\,\Omega + (-\Omega\, a + x)^3\; x + (-\Omega\, b + y)^3\; y + (-\Omega\, c + z)^3\; z$$

$$= (-\Omega\, a + x)^3\; (-\Omega\, a + x) + (-\Omega\, b + y)^3\; (-\Omega\, b + y) + (-\Omega\, c + z)^3\; (-\Omega\, c + z)$$

$$= (-\Omega\, a + x)^4 + (-\Omega\, b + y)^4 + (-\Omega\, c + z)^4 = \lambda = 0 \tag{4.35}$$

Note that (4.35) is the norm of the radial projection in the form of equation (4.33), which implies $||P_r||_4 = 0$, which is not maximal; therefore, $(p - s) = 0$ which contradicts our assumption. So $p = s$ or $p = t$ or $s = t$. ■

Now we must consider each of the three cases, and we rewrite $||P_r||_4^4$ as

$$a^4\ (p - \Omega)^4 + b^4\ (s - \Omega)^4 + c^4\ (t - \Omega)^4 \tag{4.36}$$

and consider the case $p = s$.

Then from (4.36)

$$(p - \Omega) = c^4\,(p - t)$$
$$(s - \Omega) = c^4\,(p - t)$$
$$(t - \Omega) = (a^4 + b^4)\,(p - t)$$

and we rewrite again

$$\|\ \phi_r\ \|^4 = \left(a^4 + b^4\right)(c^4)^4\,(p - t)^4 + c^4\left(a^4 + b^4\right)^4(p - t)^4$$

Let $\alpha_1 = \left(a^4 + b^4\right)$ and $\beta_1 = c^4$ then when $p = s$

$$||P_r||_4^4 = (\alpha_1{\beta_1}^4 + \beta_1{\alpha_1}^4)(p - t)^4 \tag{4.37}$$

Now we consider the case $p = t$.

Then from (4.36)

$$(p - \Omega) = b^4\,(p - s)$$
$$(s - \Omega) = \left(a^4 + c^4\right)(p - s)$$
$$(t - \Omega) = b^4\,(p - s)$$

and we rewrite again

$$\|\ \phi_r\ \|^4 = \left(a^4 + c^4\right)(b^4)^4\,(p - s)^4 + b^4\left(a^4 + c^4\right)^4(p - s)^4$$

Let $\alpha_2 = \left(a^4 + c^4\right)$ and $\beta_2 = b^4$ then when $p = t$

$$||P_r||_4^4 = (\alpha_2 {\beta_2}^4 + \beta_2 {\alpha_2}^4)(p - s)^4 \tag{4.38}$$

And finally, the case $s = t$.

Then from (4.36)

$$(p - \Omega) = \left(b^4 + c^4\right)(p - s)$$
$$(s - \Omega) = a^4\,(p - s)$$
$$(t - \Omega) = a^4\,(p - s)$$

and we rewrite again

$$\parallel \phi_r \parallel^4 = a^4\,\left(b^4 + c^4\right)^4\,(p - s)^4 + \left(b^4 + c^4\right)\,\left(a^4\right)^4\,(p - s)^4$$

Let $\alpha_3 = \left(b^4 + c^4\right)$ and $\beta_3 = a^4$ then when $s = t$

$$||P_r||_4^4 = (\alpha_3 {\beta_3}^4 + \beta_3 {\alpha_3}^4)(p - s)^4 \tag{4.39}$$

Note that α_i and β_i are fixed by the hyperplane; therefore, without loss of generality, from equations (4.37),(4.38), and (4.39) we can maximize

$$g(u, v) = (u - v)^4 \tag{4.40}$$

with the constraint

$$\alpha_i\, u^4 + \beta_i\, v^4 = 1 \tag{4.41}$$

We use the method of Lagrange multipliers

$$h(u, v) = (u - v)^4 - \lambda(\alpha_i\, u^4 + \beta_i\, v^4 - 1) \tag{4.42}$$

$$\frac{\partial h}{\partial u} = 4\,(u - v)^3 - 4\,\lambda\, u^3 \alpha_i = 0 \tag{4.43}$$

$$\frac{\partial h}{\partial v} = 4\,(u - v)^3 - 4\,\lambda\, v^3 \beta_i = 0 \tag{4.44}$$

$$\Rightarrow\;\; u^3 \alpha_i = v^3 \beta_i \tag{4.45}$$

$$\Rightarrow \quad v = u\left(\frac{\alpha_i}{\beta_i}\right)^{1/3} \tag{4.46}$$

Substituting into (4.41)

$$\alpha_i\, u^4 + \beta_i\, u^4 \left(\frac{\alpha_i}{\beta_i}\right)^{4/3} = 1 \tag{4.47}$$

$$\Rightarrow \quad u^4 = \frac{{\beta_i}^{1/3}}{\alpha_i\left({\beta_i}^{1/3} + {\alpha_i}^{1/3}\right)} \tag{4.48}$$

$$\Rightarrow \quad u = \pm\frac{\left({\beta_i}^{1/3}\right)^{1/4}}{\alpha_i^{1/4}\left({\alpha_i}^{1/3} + {\beta_i}^{1/3}\right)^{1/4}} \tag{4.49}$$

and from equation (4.46)

$$v = \pm\frac{\left({\alpha_i}^{1/3}\right)^{1/4}}{\beta_i^{1/4}\left({\alpha_i}^{1/3} + {\beta_i}^{1/3}\right)^{1/4}} \tag{4.50}$$

Substituting back into (4.41)

$$g(u,v) = \left(\frac{\left({\beta_i}^{1/3}\right)^{1/4}}{\alpha_i^{1/4}\left({\alpha_i}^{1/3} + {\beta_i}^{1/3}\right)^{1/4}} + \frac{\left({\alpha_i}^{1/3}\right)^{1/4}}{\beta_i^{1/4}\left({\alpha_i}^{1/3} + {\beta_i}^{1/3}\right)^{1/4}}\right)^4 \tag{4.51}$$

$$= \frac{\left(\alpha_i^{1/3} + {\beta_i}^{1/3}\right)^3}{\alpha_i\beta} \tag{4.52}$$

Now we can rewrite $||P||_4^4$ in terms of α and β

$$||P_r||_4^4 = (\alpha_i{\beta_i}^4 + \beta_i\alpha{\alpha_i}^4)\frac{\left(\alpha_i^{1/3} + {\beta_i}^{1/3}\right)^3}{\alpha_i\beta} \tag{4.53}$$

$$= \left(\alpha_i^3 + {\beta_i}^3\right)\left(\alpha_i^{1/3} + \beta^{1/3}\right)^3 \tag{4.54}$$

Since $\alpha_i + \beta_i = 1$, we can rewrite (4.54) in one variable as

$$\Phi(\omega) = \left(\omega^3 + (1-\omega)^3\right)\left(\omega^{1/3} + (1-\omega)^{1/3}\right)^3 \tag{4.55}$$

We note that

$$p = s \implies \Phi(c^4) = \left({\alpha_1}^3 + {\beta_1}^3\right)\left({\alpha_1}^{1/3} + {\beta_1}^{1/3}\right)^3$$

$$p = t \implies \Phi(b^4) = \left({\alpha_2}^3 + {\beta_2}^3\right)\left({\alpha_2}^{1/3} + {\beta_2}^{1/3}\right)^3$$

$$s = t \implies \Phi(a^4) = \left({\alpha_3}^3 + {\beta_3}^3\right)\left({\alpha_3}^{1/3} + {\beta_3}^{1/3}\right)^3$$

and

$$||P_r||_4^4 \leq max\left\{\Phi\left(a^4\right), \Phi\left(b^4\right), \Phi\left(c^4\right)\right\}$$

Now we maximize $\Phi(\omega)$

$$\frac{d\Phi}{d\omega} = \left(3\,\omega^2 - 3\,\left(1-\omega\right)^2\right)\left(\omega^{1/3} + 1 - \omega^{1/3}\right)^3 + 3\,\left(\omega^3 + \left(1-\omega\right)^3\right)\left(\omega^{1/3} + 1 - \omega^{1/3}\right)^2\left(\frac{1}{3\omega^{2/3}} - \frac{1}{3\left(1-\omega\right)^{2/3}}\right) = 0$$

$$= \left(\omega^{1/3} + \left(1-\omega\right)^{1/3}\right)^2\left(3\,\omega^{4/3} + 3\,\omega\left(1-\omega\right)^{1/3} - 2\,\omega^{1/3} - \left(1-\omega\right)^{1/3}\right) \cdot \left(3\,\omega^{4/3} - 3\,\omega\left(1-\omega\right)^{1/3} - 2\,\omega^{1/3} + \left(1-\omega\right)^{1/3}\right)\frac{1}{\omega^{2/3}\left(1-\omega\right)^{2/3}}$$

Since $\left(\omega^{1/3} + \left(1-\omega\right)^{1/3}\right)^2 \neq 0$ we are left with

$$3\,\omega^{4/3} - 2\omega^{1/3} - \left(3\omega - 1\right)\left(1-\omega\right)^{1/3} = 0 \tag{4.56}$$

or

$$3\,\omega^{4/3} - 2\omega^{1/3} + \left(3\omega - 1\right)\left(1-\omega\right)^{1/3} = 0 \tag{4.57}$$

Solving for ω in (4.56)

$$3\,\omega^{4/3} - 2\omega^{1/3} = \left(3\omega - 1\right)\left(1-\omega\right)^{1/3}$$

$$\left(3\,\omega - 2\right)^3\omega = \left(3\,\omega - 1\right)^3\left(1-\omega\right)$$

$$54\,\omega^4 - 108\,\omega^3 + 72\,\omega^2 - 18\,\omega + 1 = 0$$

To remove cubed term, let $\omega = \gamma + \frac{1}{2}$

$$54\,\left(\gamma + \frac{1}{2}\right)^4 - 108\,\left(\gamma + \frac{1}{2}\right)^3 + 72\,\left(\gamma + \frac{1}{2}\right)^2 - 18\,\left(\gamma + \frac{1}{2}\right) - 1 = 0$$

$$54\,\gamma^4 - 108\,\gamma^3 - 153\,\gamma^2 - 81\,\gamma - \frac{107}{8} + 108\,\gamma^3 + 162\,\gamma^2 + 81\,\gamma + \frac{27}{2} = 0$$

$$54\,\gamma^4 - 9\,\gamma^2 - \frac{1}{8} = 0$$

$$\gamma^4 - \frac{1}{6}\gamma^2 - \frac{1}{432} = 0$$

Now, let $\gamma = \sqrt{\nu}$

$$\nu^2 - \frac{1}{6}\nu - \frac{1}{432} = 0$$

Complete the square

$$\left(\nu - \frac{1}{12}\right)^2 - \frac{1}{108} = 0$$

$$\nu = \frac{1}{12} \pm \frac{1}{6\sqrt{3}}$$

Substitute back for ν

$$\gamma^2 = \frac{1}{12} \pm \frac{1}{6\sqrt{3}}$$

$$\gamma = \pm\frac{1}{6}\sqrt{3 + 2\sqrt{3}}$$

Substitute back for γ

$$\omega - \frac{1}{2} = \pm\frac{1}{6}\sqrt{3 + 2\sqrt{3}}$$

$$\omega_1 = \frac{1}{2} + \frac{1}{6}\sqrt{3 + 2\sqrt{3}} \tag{4.58}$$

and

$$\omega_2 = \frac{1}{2} - \frac{1}{6}\sqrt{3 + 2\sqrt{3}} \tag{4.59}$$

Therefore

$$\Phi(\omega_1) = \Phi(\omega_2) \approx 2.15$$

Solving similarly for equation (4.57)

$$\omega_3 = \frac{1}{2}$$

$$\Phi(\omega_3) = 1$$

By inspection we have

$$max\, \Phi(\omega) = \Phi(\omega_1) = \Phi(\omega_2)$$

and

$$min\, \Phi(\omega) = \Phi(\omega_3)$$

Let $\omega_1 = \omega_o$ and $\omega_2 = (1 - \omega_o)$, then

$$\Gamma = max\, \Phi(\omega) = \Phi\left(\omega_o\right) = \Phi\left(1 - \omega_o\right)$$
$$= \frac{1}{1296}\left(3 + \sqrt{3}\right)\left(\sqrt[3]{108 + 36\sqrt{3 + 2\sqrt{3}}} + \sqrt[3]{-36\sqrt{3 + 2\sqrt{3}} + 108}\right)^3 \approx 1.21156503127726^4 \tag{4.60}$$

Now we can state

$$||P_r||_4^4 \leq max\left\{\Phi\left(a^4\right), \Phi\left(b^4\right), \Phi\left(c^4\right)\right\} \leq \Gamma \tag{4.61}$$

and observe, for the equality to hold, at least one of the following must hold.

$$\begin{array}{llll} a^4 = \omega_o & \text{or} & a^4 = 1 - \omega_o & \text{or} \\ b^4 = \omega_o & \text{or} & b^4 = 1 - \omega_o & \text{or} \\ c^4 = \omega_o & \text{or} & c^4 = 1 - \omega_o & \end{array}$$

Therefore, there are three cases we need to examine for which the maximum may be attained.

Case 1: All three terms take the values ω_o or $(1 - \omega_o)$
This is not possible since $a^4 + b^4 + c^4 = 1$

Case 2: Two terms take the values ω_o or $(1 - \omega_o)$
Consider, without loss of generality, the case $s = t$.
Then $a^4 = \omega_o$ and let $b^4 = 1 - \omega_o$
then $c^4 = 0 \implies ||P_r|| = 1$, which is not maximum.

So we are left with the following two possibilities for Case 2.

Case 2(a) Let

$$a^4 = \omega_o, b^4 = \omega_o, c^4 = (1 - 2\,\omega_o) \tag{4.62}$$

$$a^4 = \omega_o \implies s = t \text{ and } b^4 = \omega_o \implies p = t$$

$$\implies p = s = t \implies \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \text{ Let } \frac{x}{a} = n$$

Let $\frac{x}{a} = n$

Then $x = a\,n$ and since $x^4 + y^4 + z^4 = 1$ we can write

$$
\begin{aligned}
(a\,n)^4 + (b\,n)^4 + (c\,n)^4 &= 1\\
n^4 \left(a^4 + b^4 + c^4\right) &= 1\\
n &= 1\\
\frac{x}{a} &= 1\\
x &= a
\end{aligned}
$$

Similarly $y = b$ and $z = c$

Case 2(b) Let

$$a^4 = (1 - \omega_o)\,, b^4 = (1 - \omega_o)\,, c^4 = (2\,\omega_o - 1) \tag{4.63}$$

In the same way as 1, this leaves us with $x = a$, $y = b$, and $z = c$

So the two norming points for case 2 are $(x = a, y = b, z = c)$ or $(x = -a, y = -b, z = -c)$ Again, this is not possible. If $(x = a, y = b, z = c)$ then $||P_r||_4 = 0$.

Case 3: Only one of the terms takes the value ω_o or $(1 - \omega_o)$

Without loss of generality, we use

$$p = s \implies \alpha = \left(a^4 + b^4\right) \text{ and } \beta = c^4$$

to calculate the norming points for case 3.

$$p = \pm \frac{\left(\beta^{1/3}\right)^{1/4}}{\alpha^{1/4}\left(\alpha^{1/3} + \beta^{1/3}\right)^{1/4}} \quad \text{and} \quad t = \pm \frac{\left(\alpha^{1/3}\right)^{1/4}}{\beta^{1/4}\left(\alpha^{1/3} + \beta^{1/3}\right)^{1/4}} \quad \text{from (4.50) and (4.51)}$$

Substitute for α and β gives:

$$p = \pm \frac{c^{1/3}}{\left(a^4 + b^4\right)^{1/4}\left[\left(a^4 + b^4\right)^{1/3} + c^{4/3}\right]^{1/4}} \quad \text{and} \quad t = \pm \frac{\left(a^4 + b^4\right)^{1/12}}{c\left[\left(a^4 + b^4\right)^{1/3} + c^{4/3}\right]}$$

Substitute $c^4 = \omega_o \implies \left(a^4 + b^4\right) = 1 - \omega_o$

and

$$x = p\,a$$
$$y = s\,b = p\,b$$
$$z = c\,t$$

gives norming points

$$x_1 = a \cdot \frac{2^{1/4} \cdot 3^{1/4} \left(3 + \sqrt{3 + 2\sqrt{3}}\right)^{1/12}}{\left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/4} \left(\left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/3} + \left(3 + \sqrt{3 + 2\sqrt{3}}\right)^{1/3}\right)^{1/4}} = \gamma_x$$

$$y_1 = b \cdot \frac{2^{1/4} \cdot 3^{1/4} \left(3 + \sqrt{3 + 2\sqrt{3}}\right)^{1/12}}{\left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/4} \left(\left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/3} + \left(3 + \sqrt{3 + 2\sqrt{3}}\right)^{1/3}\right)^{1/4}} = \gamma_y$$

$$z_1 = \frac{3^{5/12} \cdot 2^{5/12} \left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/12}}{\left(6 \cdot 2^{2/3} \cdot 3^{2/3} \left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/3} + 6 \left(108 + 36\sqrt{3 + 2\sqrt{3}}\right)^{1/3}\right)^{1/4}} = \gamma_z$$

and

$$x_2 = -\gamma_x$$
$$y_2 = -\gamma_y$$
$$z_2 = -\gamma_z$$

and

$$x_3 = \gamma_x$$
$$y_3 = \gamma_y$$
$$z_3 = -\gamma_z$$

and

$$x_4 = -\gamma_x$$
$$y_4 = -\gamma_y$$
$$z_4 = \gamma_z$$

or $c^4 = 1 - \omega_o \implies \left(a^4 + b^4\right) = \omega_o$ gives norming points

$$x_1 = a \cdot \frac{2^{1/4} \cdot 3^{1/4} \left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/12}}{\left(3 + \sqrt{3 + 2\sqrt{3}}\right)^{1/4} \left(\left(3 - \sqrt{3 - 2\sqrt{3}}\right)^{1/3} + \left(3 + \sqrt{3 + 2\sqrt{3}}\right)^{1/3}\right)^{1/4}} = \delta_x$$

$$y_1 = b \cdot \frac{2^{1/4} \cdot 3^{1/4} \left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/12}}{\left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/4} \left(\left(3 + \sqrt{3 - 2\sqrt{3}}\right)^{1/3} + \left(3 + \sqrt{3 + 2\sqrt{3}}\right)^{1/3}\right)^{1/4}} = \delta_y$$

$$z_1 = \frac{3^{5/12} \cdot 2^{5/12} \left(3 + \sqrt{3 + 2\sqrt{3}}\right)^{1/12}}{\left(6 \cdot 2^{2/3} \cdot 3^{2/3} \left(3 - \sqrt{3 + 2\sqrt{3}}\right)^{1/3} + 6 \left(108 - 36\sqrt{3 - 2\sqrt{3}}\right)^{1/3}\right)^{1/4}} = \delta_z$$

and

$$x_2 = -\delta_x$$
$$y_2 = -\delta_y$$
$$z_2 = -\delta_z$$

and

$$x_3 = \delta_x$$
$$y_3 = \delta_y$$
$$z_3 = -\delta_z$$

and

$$x_4 = -\delta_x$$
$$y_4 = -\delta_y$$
$$z_4 = \delta_z$$

So, for case 3, $max||P_r||_4$ has, at most, FOUR norming points. Therefore,

$$||P_r||_4^4 = max\left\{\Phi\left(a^4\right), \Phi\left(b^4\right), \Phi\left(c^4\right)\right\} = \Gamma$$

if and only if $||P_r||_4^4$ has, at most, four different norming points.

4.2 Analytic Results

Now we present the main result of this paper. In ℓ_4^3, is the hyperplane constant, with respect to radial projections, equal to the hyperplane constant with respect to all projections? This is a natural question arising from the fact that minimal projections in $\mathbb{R}^2$ are radial.

Before offering our theorem, we first we need a theorem by Shekhtman and Skrzypek. [**?**]

Theorem 4.1: [Theorem 2.2: **?**] A minimal projection from $L_p(\mu)$, $1 < p < \infty$, onto a two dimensional subspace has at least six different norming points $\pm x_1, \pm x_2, \pm x_3$.

Theorem 4.2 Given a projection $P = I - f \otimes z$, $P : \ell_4^3 \longrightarrow \ker f$ then

$$\max_f \{\lambda_r(\ell_4^3, \ker f)\} > \max_f \{\lambda(\ell_4^3, \ker f)\}$$

Proof:

Choose f_o, such that the minimal projection onto $\ker f_o$ equals max/min projection over all f, in other words

$$\lambda(\ell_4^3, \ker f_o) = \max_f \{\lambda(\ell_4^3, \ker f)\}$$

Now, with respect to f_o, we have the minimal projection

$$P_{m_{f_o}} : \ell_4^3 \longrightarrow \ker f_o$$

and the radial projection

$$P_{r_{fo}} : \ell_4^3 \longrightarrow \ker f_o$$

We have two possibilities with regard to the norms $||P_{m_{fo}}||$ and $||P_{r_{fo}}||$

Case 1: The radial projection is not minimal.

$$||P_{m_{fo}}|| < ||P_{r_{fo}}||$$

Then

$$\max_f \{\lambda(\ell_4^3, \ker f)\} = ||P_{m_{fo}}|| < ||P_{r_{fo}}|| \leq \max_f \{\lambda_r(\ell_4^3, \ker f)\}$$

and the proof is done.

Case 2: The radial projection is minimal.

$$||P_{m_{fo}}|| = ||P_{r_{fo}}||$$

Now Case 2 leaves us with two other possibilities.

Case 2(a):

$$||P_{r_{fo}}|| < \max_f\{\lambda_r(\ell_4^3, \ker f)\}$$

then

$$\max_f\{\lambda(\ell_4^3, \ker f)\} = ||P_{m_{fo}}|| = ||P_{r_{fo}}|| < \max_f\{\lambda_r(\ell_4^3, \ker f)\}$$

and we are done.

Case 2(b):

$$||P_{r_{fo}}|| = \max_f\{\lambda_r(\ell_4^3, \ker f)\}$$

so then

$$\max_f\{\lambda(\ell_4^3, \ker f)\} = ||P_{m_{fo}}|| = ||P_{r_{fo}}|| = \max_f\{\lambda_r(\ell_4^3, \ker f)\}$$

In this case, $P_{r_{fo}}$ is a max/minimal projection and is, therefore, also a minimal projection. By Theorem 4.1, $P_{r_{fo}}$ will have at least six norming points; however, in Section 4.1 we show that the max radial projection has, at most, four norming points. Therefore $||P_{r_{fo}}|| \neq \max_f\{\lambda_r(\ell_4^3, \ker f)\}$. ■

Chapter 5
Conclusion

Although we have shown that the maximum radial projection $P_r : \ell_4^3 \longrightarrow \ker f$ is not a minimal projection, it is crucial to note that we have not proven the converse. The maximum minimal projection could still, very well, be radial. In fact, if we look at the numerical results in Figure 5, the hyperplane constant is achieved for $f_1 = f_2 = f_3 = z_1 = z_2 = z_3 \approx .57735$. Now if we normalize these functionals, we see the projection for which the maximum relative projection constant is attained is both orthogonal and radial.

Conjecture:

The maximum relative projection constant

$$\max_f \{\lambda(\ell_p^n, \ker f)\}$$

is obtained at $\ker\{1, ...1\}$*, and, furthermore, this projection is not only minimal, but also orthogonal and radial.*

In support of the conjecture, it is important to note the confidence we have in our numerical analysis. First we note the maximum value obtained for $||P_r||$ numerically, in Figure 4, and compare it to the value computed by maximizing $||P_r||$ analytically (4.61). We see that the values are equivalent to 1×10^{-10}.

From Skrzypek [**?**] we have a method to calculate the relative projection constant of $\ker\{1, ..., 1\}$ in ℓ_n^p.

$$\lambda\left(\ker\{1, ..., 1\}, \ell_n^p\right) = \frac{\left((n-1)^{\frac{p}{q}} + 1\right)^{\frac{1}{p}} \left((n-1)^{\frac{q}{p}} + 1\right)^{\frac{1}{q}}}{n}$$

So for $p = 4$ and $n = 3$ we have $\lambda\left(\ker\{1, ..., 1\}, \ell_n^p\right) \approx 1.064165862858$

Again, this value is equivalent to the maximal relative projection constant obtained numerically, in Figure 5, to 1×10^{-10}; therefore we may infer that this is a minimal projection. These relatively accurate numerical calculations also serve to substantiate and verify the accuracy of our algorithm.