January 2012

Multi-time Scales Stochastic Dynamic Processes: Modeling, Methods, Algorithms, Analysis, and Applications

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by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
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Date of Approval:
August 13, 2012

Keywords: Fractional integrals and derivatives, stochastic differential equations, Lyapunov function, numerical algorithms.

Dedication To Kristen, Chloe and Rose.

Acknowledgements

I would like to express my profound gratitude to Professor G. S. Ladde, my supervisor, for his many suggestions and constant support during this research.

I am very thankful to the examining committee members Professor Tsokos, Professor Ramachandran, Professor McWaters, and the chairperson of the defense Professor Dagne, for their careful reading, for the useful comments and helpful suggestions they made for improvement in this work.

I am grateful to my family, especially my brother Gilbert, my mother Marguerite, my wife Rose and daughters Kristen and Chloe, for their patience, support and love.

I wish to thank the following families and friends: the Tesi, the Kouokam, the Sukam for their friendship and invaluable support.

This research was supported by the Mathematical Science Division, US Army Office, Grants No. W911NF-07-1-0283 and W911NF-12-1-0090.

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Abstract

By introducing a concept of dynamic process operating under multi-time scales in sciences and engineering, a mathematical model is formulated and it leads to a system of multi-time scale stochastic differential equations. The classical Picard-Lindelöf successive approximations scheme is expended to the model validation problem, namely, existence and uniqueness of solution process. Naturally, this generates to a problem of finding closed form solutions of both linear and nonlinear multi-time scale stochastic differential equations. To illustrate the scope of ideas and presented results, multi-time scale stochastic models for ecological and epidemiological processes in population dynamic are exhibited. Without loss in generality, the modeling and analysis of three time-scale fractional stochastic differential equations is followed by the development of the numerical algorithm for multi-time scale dynamic equations. The development of numerical algorithm is based on the idea if numerical integration in the context of the notion of multi-time scale integration. The multi-time scale approach is applied to explore the study of higher order stochastic differential equations (HOSDE) is presented. This study utilizes the variation of constant parameter technique to develop a method for finding closed form solution processes of classes of HOSDE. Then then probability distribution of the solution processes in the context of the second order equations is investigated.

Introduction

Historically, mathematical models for dynamic processes in chemical, biological, engineering and physical sciences are described by systems of differential equations [67, 69, 71, 73]. The problems of existence and uniqueness of solutions of initial value problem provide the basis for the model validation and further undertaking a study of the corresponding dynamic processes. By utilizing either approximation schemes or fixed point theory, the initial value problem is reduced to its equivalent integral equation problem [6, 13, 25, 31, 44, 48, 57]. And hence, the problem of the notion of integration.

There is no phenomenon that is deterministic in character. Everything around us is highly random in nature. As a result, mathematical descriptions of a phenomenon invariably result in random or stochastic equations [72]. Randomness arises in a system due to different types of unforseen exogenous factors. The random nature of phenomenon makes their behaviors unknown. Therefore, the parameter describing the randomness of such phenomenon is governed by some probability distribution.

Around 1960, systems of ordinary stochastic differential equations of Itô-Doob type [4, 7, 14, 27, 47, 60], stochastic partial differential equations [26, 33], stochastic fractional differential equations [7, 60], and stochastic partial differential equations in abstract spaces [14] were defined in the framework of Itô-Doob type stochastic integral equations. The effects of random environmental fluctuations are characterized by normalized Wiener process [50]. This was further extended in the framework of local martingale integral equations [3, 23].

Based on the above summary of the development of dynamic modeling and to incorporate additional structural and/or environmental perturbation aspects of dynamic into the mathematical modeling described by dynamic equations, we observe that researchers constantly seek the modifications/extensions of integral concepts that utilize the idea of initial value problem to its equivalent integral equation problem. The by-product of this approach is that it forces one to lump the effects of the structural and environmental perturbations on to a notion of "single integral". Using this, the initial value problem (IVP) is represented in its equivalent integral equation. However, this is not the case of Itô-Doob type of stochastic differential equations and its extensions.

Very recently [55, 56], the classical mathematical modeling approach coupled with the stochastic methods were used to develop stochastic dynamic models for financial data (stock price). In order to extend this approach to more complex dynamic processes in sciences and engineering operating under internal structural and external environmental perturbations, we need to modify the existing mathematical models approach by incorporating certain significant attributable parameters or state variables, explicitly. This motivates us to initiate to partially characterize intra structural and external environmental perturbations by a set of linearly independent time-scales, for example $\{t,\ w(t),\ t^{\alpha},\ \eta(t)\}$, where w is the standard Wiener process, $\eta(.)$ is a function of bounded variation, and $\alpha \in (0,1)$.

The thesis is organized as follows. For easy reference, some preliminary results and definitions are presented in chapter 1. In fact, some of the preliminary results are presented in refined forms. In chapter 2, by introducing the concept of dynamic processes operating under linearly independent time-scales, a mathematical model of dynamic processes and it is described by a system of stochastic differential equations. For this purpose, it was necessary to introduce several concepts of calculus in multi-time scales setting. Section 2.2 deals with model validation issues, namely, the existence and uniqueness of solutions of the initial value problem defined in section 2.1. Knowing the existence of IVP, methods of finding close form solutions of both linear and nonlinear scalar stochastic differential equations under multi-time scales are developed in chapter 3. It is known that the close form solutions provide a mathematical tool to develop statistical models for dynamic processes in sciences engineering by using data-sets and statistical methods. In section 3.4, the developed concepts and results are applied to stochastic models in ecological and epidemiological processes of population dynamic under multi-time scales. In chapter 4, we presented a numerical scheme for approximating the solution process of SFDE developed in chapter 2. This is achieved by the usage of the multi-time scale integration concept. Furthermore, we prove that the global convergence result for the presented numerical scheme. Chapter 6 concerns mathematical models of several dynamic random processes are influenced by not only their state but also rates of change of states leading to higher order linear homogeneous stochastic differential equations. In this chapter, we are interested in finding exact or close form solution processes to such equations. After the formulation of the problem, we present a few basic preliminary results. In section 6.2, we develop a method of finding exact solutions of higher order (order $n \geq 2$) stochastic linear differential equations with constant coefficients. These solutions are classified in section 6.3 based on nature of the roots of the characteristic polynomial of associated with the higher order deterministic differential equation corresponding to the stochastic differential. We also illustrate the method developed in sections 6.2 and 6.3 with $n = 2$ and provide some useful examples. The last section of this chapter provides ideas about finding the probability distribution of the solution processes in the context of second order ($n = 2$) equation. We conclude the thesis with highlights of our future research work.

1 Preliminaries

In this chapter, we recall some important definitions and results that will be used in the subsequent chapters. For more details on these definitions and well known results, references are provided here for easy reading.

1.1 Some important definitions and results

This section covers a few well-known concepts and results in the fields of fractional and stochastic differential equations.

Definition 1.1.1 (Riemann-Liouville fractional integrals [40, 70]): Let $0 < \alpha < 1$ and $f \in L^1[a,b]$ ($L^1[a,b] = L^1[[a,b],\mathbb{R}^n] = \{y|y : [a,b] \to \mathbb{R}^n$ and y is Lebesgue integrable$\}$). The left-sided and right-sided Riemann-Liouville fractional integrals of order α are defined for almost all $t \in (a,b)$ by

$$(I^{\alpha}_{a+}f)(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \quad t > a, \tag{1.1.1}$$

and

$$(I^{\alpha}_{b-}f)(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1}f(s)ds, \quad t < b, \tag{1.1.2}$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} r^{\alpha-1}e^{-r}dr$ is the Euler function.

To present a modified version of Definition 1.1.1, we define the concept of absolutely continuous functions.

Definition 1.1.2 (Absolute Continuity [70]): A function f is said to be absolutely continuous on an interval J, if for $\varepsilon > 0$ there exists a $\delta > 0$ such that for any pairwise nonintersecting intervals $[a_k, b_k] \subset J$, $k = 1, ..., n$, such that $\sum_{k=1}^{n}(b_k - a_k) < \delta$, the inequality $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$ holds.

It is known (see Kolmogorov and Fomin [42] page 338) that the space $AC[a, b]$ of absolutely continuous functions on $[a, b]$ coincides with the Lebesgue summable functions:

$$f \in AC[a,b] \;\Leftrightarrow\; f(x) = c + \int_a^x \varphi(t)dt \quad (\varphi \in L^1[a,b]). \tag{1.1.3}$$

Definition 1.1.3 (Riemann-Liouville Fractional derivatives [40, 70]): Let f be defined and absolutely continuous on an interval $[a, b]$, and let $0 < \alpha < 1$,

$$(\mathcal{D}^{\alpha}_{a+} f)(t) \;=\; \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(t-a)^{\alpha}} + \int_a^t (t-s)^{-\alpha} f'(s)ds\right], \tag{1.1.4}$$

and

$$(\mathcal{D}^{\alpha}_{b-} f)(t) \;=\; \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(b)}{(b-t)^{\alpha}} - \int_t^b (s-t)^{-\alpha} f'(s)ds\right] \tag{1.1.5}$$

are called left-sided and right-sided Riemann-Liouville fractional derivatives, respectively.

In the sequel, we denote $(\mathcal{D}^{\alpha} f)$ or $f^{(\alpha)}$ for $\left(\mathcal{D}^{\alpha}_{a+} f\right)$, and 0 instead of a, unless otherwise specified. This is to simplify computations.

The following result characterizes interactions between Riemann-Liouville type fractional integral and fractional differential operators.

Proposition 1.1.4 ([40, 70]): *If $0 < \alpha < 1$ and $f \in L^p[a, b], 1 \leq p \leq \infty$, then we have*

$$\left(\mathcal{D}^{\alpha}_{a+} I^{\alpha}_{a+} f\right)(t) = f(t) \quad \textit{and} \quad \left(\mathcal{D}^{\alpha}_{b-} I^{\alpha}_{b-} f\right)(t) = f(t) \tag{1.1.6}$$

The space of summable functions plays an important role in defining the solution of FSDE.

Definition 1.1.5 (L^1_α Space, Kilbas et al. [40] p. 144:) Let $a, b \in \mathbb{R}$ and $0 < \alpha < 1$. The L^1_α space is defined as follows

$$L^1_\alpha[a,b] := \{y \in L^1[a,b] : \ \mathcal{D}^\alpha_{a+} y \in L^1[a,b]\} \tag{1.1.7}$$

where $L^1[a,b]$ is the space of summable or integrable functions in a finite interval $[a,b]$ of the real line $\mathbb{R}$.

Definition 1.1.6 (Caputo Fractional derivatives [40, 70]): For $0 < \alpha < 1$, the left-hand Caputo derivative of order α, denoted by ${}^C\mathcal{D}^\alpha_{a+} f$, is defined as the Riemann-Liouville type fractional derivatives (Definition 1.1.3):

$$\left({}^C\mathcal{D}^\alpha_{a+} f\right)(t) = \mathcal{D}^\alpha_{a+}\left[f(t) - f(a)\right] = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s) ds \tag{1.1.8}$$

and hence

$$\left({}^C\mathcal{D}^\alpha_{a+} f\right)(t) = \left(\mathcal{D}^\alpha_{a+} f\right)(t) - \left(\mathcal{D}^\alpha_{a+} f\right)(a) = \left(\mathcal{D}^\alpha_{a+} f\right)(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}. \tag{1.1.9}$$

Remark 1.1.7 *([40, 58]) (i) The main advantage of the Caputo derivative is that the initial conditions for fractional differential equations are the same form as that of ordinary differential equations with integer derivatives.*
(ii) The Caputo derivative of a constant c is zero, while the Riemann-Liouville fractional derivative of a constant c is not zero but equals to $\left(\mathcal{D}^\alpha_{a+} c\right)(t) = \frac{c(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$.

Definition 1.1.8 (The L^p-spaces [63]): For a given complete probability space $(\Omega, \mathcal{F}, P) \equiv \Omega$, if $X : \Omega \to \mathbb{R}^n$ is a random variable, and $p \in [1, \infty)$ is a constant, the L^p-norm of X, $\|X\|_p$, is defined by

$$\|X\|_p = \|X\|_{L^p(\Omega)} = \left(\int_\Omega |X(\omega)|^p dP(\omega)\right)^{\frac{1}{p}}. \tag{1.1.10}$$

If $p = \infty$, we set

$$\|X\|_\infty = \|X\|_{L^\infty(\Omega)} = \sup\{|X(\omega)|; \omega \in \Omega\}. \tag{1.1.11}$$

The corresponding L^p-spaces are defined by

$$L^p(P) = L^p(\Omega) = \sup\{X : \Omega \to \mathbb{R}^n; \|X\|_p < \infty\}. \tag{1.1.12}$$

With this norm the L^p-spaces are Banach spaces, i.e. complete normed linear spaces. If $p = 2$, the $L^2(P)$ is a Hilbert space, i.e. a complete inner product space, with the inner product

$$\langle X, Y\rangle_{L^2(P)} := E[X \cdot Y]; \quad X, Y \in L^2(P). \tag{1.1.13}$$

Definition 1.1.9 (Space of Hölder continuous functions [59]): Let γ be a real number, $0 < \gamma \leq 1$. A real value function f on finite interval $[t_0, T]$ is said to satisfy a Hölder condition of order γ, if there is a positive constant C such that for all $t, s \in [t_0, T]$, we have

$$|f(t) - f(s)| \leq C|t - s|^\gamma \tag{1.1.14}$$

For such a function, define

$$\|f\|_\gamma = \sup_t |f(t)| + \sup_{\substack{t,s \\ t \neq s}} \frac{|f(t) - f(s)|}{|t - s|^\gamma} \tag{1.1.15}$$

Proposition 1.1.10 (The Hölder inequality [9]): *Let S be a nonempty subset of $\mathbb{R}^n$, n positive integer, and $(S, \mathcal{M}, \mu)$ be a Lebesgue measure space. Suppose that $1 < p,\ q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(S, \mathcal{M}, \mu)$ and $g \in L^q(S, \mathcal{M}, \mu)$, and define $(f, g) : S \to \mathbb{R}$ by $(f, g)(x) = (f(x), g(x))$ for all $x \in S$ where $(\cdot, \cdot)$ denotes the usual inner product from $S \times S$ into $\mathbb{R}$. Then $(f, g) \in L^1(S, \mathcal{M}, \mu)$ and we have*

$$\|(f, g)\|_1 \leq \|f\|_p \|g\|_q \tag{1.1.16}$$

Proposition 1.1.11 (The Itô isometry [9, 63]): *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{V} = \mathcal{V}(t_0, T)$ be the class of functions*

$$f : [0, \infty) \times \Omega \to \mathbb{R}$$

such that

(i) *f is $\mathcal{B} \times \mathcal{F}$- measurable, where $\mathcal{B}$ denotes the Borel $\sigma-$algebra on $[0, \infty)$,*

(ii) *f is $\mathcal{F}_t-$adapted,*

(iii) $E\left[\int_{t_0}^{T} f(t,\omega)^2 dt\right] < \infty.$

The Itô isometry is given by

$$E\left[\left(\int_{t_0}^{T} f(t,\omega)dB_t\right)^2\right] = E\left[\int_{t_0}^{T} f(t,\omega)^2 dt\right] \text{ for all } f \in \mathcal{V}(t_0, T). \tag{1.1.17}$$

In the next section, we present result is the Taylor's series expansion of fractional order as developed by Jumarie (Proposition 3.1 in [35]). Here we provide a detailed proof of this result.

1.2 Fractional Taylor series approximations

Taylor's formula plays a very important role in the theory of fractional and stochastic differential equations. Various authors attempted to establish an analog to the Taylor series expansion in the fractional derivative sense. In this section, following the idea of Jumarie [35, 36, 37], we provide a detailed proof of the Taylor series approximations for non-differentiable functions which admit fractional derivatives of order $k\alpha$, $0 < \alpha < 1$ and $k \in \mathbb{N}$.

Definition 1.2.1 Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ be a continuous function; and let $h > 0$ denote a constant discretization span. Define the forward shift operator FW

$$[FW(h)f](x) := f(x+h) \tag{1.2.1}$$

then the fractional difference of order α, $0 < \alpha < 1$, is defined by the expression

$$\Delta^{\alpha} \cdot f(x) := (FW(h) - 1)^{\alpha} \cdot f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - h)k] \tag{1.2.2}$$

Theorem 1.2.2 (Taylor's series of fractional order [35, 37, 38]): *Assume that the function $f : \mathbb{R} \to \mathbb{R}$ has fractional derivatives of order $k\alpha$, $0 < \alpha < 1$, for any nonnegative integer k, then the following equality holds, which is*

$$f(t+h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha} f^{(k\alpha)}(t)}{\Gamma(1+k\alpha)} \tag{1.2.3}$$

where $f^{(k\alpha)}(t) = \mathcal{D}^{k\alpha} f(x)$ for $k = 0, 1, 2, \ldots$ and $0 < \alpha < 1$.

Proof. To prove Theorem 1.2.3 we proceed by the method of successive approximation and operator calculus.

Consider the following initial value problem

$$\mathcal{D}^{\alpha} FW(h) = \partial_t^{\alpha} FW(h), \quad FW_0(h) = \frac{h^{-\alpha}}{\Gamma(1-\alpha)} \tag{1.2.4}$$

Then

$$FW(h) = FW_0(h) + \frac{\partial_t^{\alpha}}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} FW(s) ds \tag{1.2.5}$$

$$FW_0(h) = \frac{h^{-\alpha}}{\Gamma(1-\alpha)}$$

$$\begin{aligned}
FW_1(h) &= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} FW_0(s) ds \\
&= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} \frac{s^{-\alpha}}{\Gamma(1-\alpha)} ds \\
&= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^h (h-s)^{\alpha-1} s^{-\alpha} ds \\
&= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 h^{\alpha-1}(1-u)^{\alpha-1} h^{-\alpha} u^{-\alpha} h \, du, \\
&\qquad \text{by setting } s = hu \\
&= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} h^{\alpha-1-\alpha+1} \beta(\alpha, 1-\alpha) \\
&= FW_0(h) + \partial_t^\alpha \ \ \partial_t^\alpha \cdot 1 + \partial_t^\alpha
\end{aligned}$$

$$\begin{aligned}
FW_2(h) &= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} FW_1(s) ds \\
&= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} FW_0(s) ds + \frac{\partial_t^{2\alpha}}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} ds \\
&= \partial_t^\alpha \cdot 1 + \partial_t^\alpha + \frac{h^\alpha \partial_t^{2\alpha}}{\Gamma(1+\alpha)} \\
&= FW_1(h) + \frac{h^\alpha \partial_t^{2\alpha}}{\Gamma(1+\alpha)}
\end{aligned}$$

$$\begin{aligned}
FW_3(h) &= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} FW_2(s) ds \\
&= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} FW_1(s) ds + \frac{\partial_t^{3\alpha}}{\Gamma(\alpha)} \int_0^h (h-s)^{\alpha-1} \frac{s^\alpha}{\Gamma(1+\alpha)} ds \\
&= \partial_t^\alpha \cdot 1 + \partial_t^\alpha + \frac{\partial_t^{3\alpha}}{\Gamma(\alpha)\Gamma(1+\alpha)} h^{\alpha-1+\alpha+1} \beta(\alpha, 1+\alpha) \\
&= \partial_t^\alpha \cdot 1 + \partial_t^\alpha + \frac{h^\alpha \partial_t^{2\alpha}}{\Gamma(1+\alpha)} + \frac{h^{2\alpha} \partial_t^{3\alpha}}{\Gamma(1+2\alpha)} \\
&= FW_2(h) + \frac{h^{2\alpha} \partial_t^{3\alpha}}{\Gamma(1+2\alpha)}
\end{aligned}$$

Now suppose that for each $k = 1, 2, \ldots, n$, $FW_k(h) = FW_{k-1}(h) + \frac{h^{(k-1)\alpha}\partial_t^{k\alpha}}{\Gamma(1+(k-1)\alpha)}$, $n \geq 1$. We want to show that $FW_{n+1}(h) = FW_n(h) + \frac{h^{n\alpha}\partial_t^{(n+1)\alpha}}{\Gamma(1+n\alpha)}$.

$$\begin{aligned}
FW_{n+1}(h) &= FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)}\int_0^h (h-s)^{\alpha-1} FW_n(s)ds \\
&= \underbrace{FW_0(h) + \frac{\partial_t^\alpha}{\Gamma(\alpha)}\int_0^h (h-s)^{\alpha-1} FW_{n-1}(s)ds}_{\overset{||}{FW_n(h)}} \quad + \\
&\quad \frac{\partial_t^\alpha}{\Gamma(\alpha)}\int_0^h (h-s)^{\alpha-1}\frac{s^{(n-1)\alpha}\partial_t^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}ds \\
&= FW_n(h) + \frac{\partial_t^{(n+1)\alpha}}{\Gamma(\alpha)\Gamma(1+(n-1)\alpha)}\int_0^h (h-s)^{\alpha-1}s^{(n-1)\alpha}ds \\
&= FW_n(h) + \frac{\partial_t^{(n+1)\alpha}}{\Gamma(\alpha)\Gamma(1+(n-1)\alpha)}h^{\alpha-1+(n-1)\alpha+1}\beta(\alpha, 1+(n-1)\alpha) \\
&= FW_n(h) + \frac{h^{n\alpha}\partial_t^{(n+1)\alpha}}{\Gamma(1+n\alpha)}
\end{aligned}$$

Therefore for all $n \geq 0$,

$$\begin{aligned}
FW_{n+1}(h) &= FW_n(h) + \frac{h^{n\alpha}\partial_t^{(n+1)\alpha}}{\Gamma(1+n\alpha)} \\
&= \partial_t^\alpha\left[1 + I + \frac{h^\alpha\partial_t^\alpha}{\Gamma(1+\alpha)} + \ldots + \frac{h^{n\alpha}\partial_t^{n\alpha}}{\Gamma(1+n\alpha)}\right] + \frac{h^{n\alpha}\partial_t^{(n+1)\alpha}}{\Gamma(1+n\alpha)}
\end{aligned}$$

What we just achieved is finding the the solution the following:

$$\mathcal{D}^\alpha FW_n(h) = \partial_t^\alpha FW_n(h), \quad n \geq 0, \quad FW_0(h) = \frac{h^{-\alpha}}{\Gamma(1-\alpha)} \tag{1.2.6}$$

What we just achieved is finding the the solution the following:

$$\mathcal{D}^\alpha FW_n(h) = \partial_t^\alpha FW_n(h), \quad n \geq 0, \quad FW_0(h) = \frac{h^{-1}}{\Gamma(1-\alpha)} \tag{1.2.7}$$

Taking limits of the first equality in (1.2.7) yield

$$\lim_{n\to\infty}\mathcal{D}^\alpha FW_n(h) = \partial_t^\alpha \lim_{n\to\infty} FW_n(h) + \lim_{n\to\infty}\frac{h^{(n-1)\alpha}\partial_t^{n\alpha}}{\Gamma(1+(n-1)\alpha)}, \tag{1.2.8}$$

which yields

$$\mathcal{D}^\alpha FW(h) = \partial_t^\alpha \left(1 + \sum_{k=0}^{\infty} \frac{[h^\alpha \partial_t^\alpha]^k}{\Gamma(1+k\alpha)}\right), \tag{1.2.9}$$

On the other hand

$$\mathcal{D}^\alpha E_\alpha(h^\alpha \partial_t^\alpha) = \partial_t^\alpha \left(1 + \sum_{k=0}^{\infty} \frac{[h^\alpha \partial_t^\alpha]^k}{\Gamma(1+k\alpha)}\right), \tag{1.2.10}$$

Thus we conclude that the solution to IVP (1.2.4) is

$$FW(h) = E_\alpha(h^\alpha \mathcal{D}^\alpha), \tag{1.2.11}$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function defined above.

■

The implications of Theorem 1.2.2 are presented in the following remarks.

Remark 1.2.3 *([35, 37, 43]) Assume that f in Theorem 1.2.2 is $\alpha-$differentiable, then the following equalities hold:*

$$f^{(\alpha)}(t) \;=\; \lim_{h\downarrow 0} \frac{\Delta^\alpha f(t)}{h^\alpha} \;=\; \Gamma(\alpha+1) \lim_{h\downarrow 0} \frac{\Delta f(t)}{h^\alpha}, \;\; 0 < \alpha \leq 1, \tag{1.2.12}$$

where $\Delta f(t) = f(t+h) - f(t)$.

A useful relation in (1.2.12) is the relationship between the fractional differential and the classical differential of f as:

$$d^{\,\alpha} f \;=\; \Gamma(\alpha+1) df, \;\; 0 < \alpha \leq 1. \tag{1.2.13}$$

Here, "$d^{\,\alpha}$" and "d" are referred to as the fractional and classical (Cauchy-Riemann-Lebesgue) differentials, respectively.

The relationship between the fractional differential and the classical differential of f in

(1.2.13) plays a very significant role in our subsequent discussion. In fact, this idea is utilized to include fractional time scale as one of the time scales in our model formulation.

2 Modeling and analysis of dynamic processes

Mathematical models for dynamic processes in chemical, biological, engineering and physical sciences are described by systems of differential equations [67, 69, 71, 73]. The problems of existence and uniqueness of solutions of initial value problem provide the basis for the model validation and further undertaking a study of the corresponding dynamic processes. By utilizing either approximation schemes or fixed point theory, the initial value problem is reduced to its equivalent integral equation problem [6, 13, 25, 31, 44, 48, 57]. The theory of initial value problems centered around ordinary differential equations [13, 48, 57] and partial differential equations [25, 74] with their equivalent representations in Cauchy-Riemann-Darboux or Lebesgue single/multiple integrals, ordinary fractional differential equations [40, 58, 68, 70] under Riemann-Liouville integral, and ordinary differential equations in abstract spaces [44] in the sense of either Bochner or Pattis integrals. Furthermore, the study is extended to systems of time delay/functional differential equations [6, 19, 28] that incorporate the hereditary aspect of dynamic processes in sciences and engineering.

Although a deterministic model offers a good start in the mathematical analysis of a dynamic process, it is incomplete because such a process is subject to random perturbations. Randomness arises in a system due to different types of unforseen exogenous factors. The random nature of the process makes its behavior unknown. Therefore, the parameter describing the randomness is governed by some probability distribution. The effects of random environmental fluctuations are characterized by normalized Wiener process [50]. Aroud 1960, this motivates the development of systems of ordinary stochastic differential equations of Itô-Doob type [4, 27, 47], stochastic fractional differential equations [7, 60], were defined in the framework of Itô-Doob type stochastic integral equations.

The hybrid approach consisting of classical dynamic modeling approach, financial data and statistical methods [55, 56], is proved to be a very promising tool to investigate the stock price dynamic process. We propose to extend this approach for general complex mathematical models for dynamic processes in chemical, biological, engineering, medical, physical, and social sciences. Several dynamic processes in sciences and engineering are under the influence of intra structural and external environmental random perturbations. In general, intra and external structural perturbations of dynamic processes are represented by parameters.

2.1 Mathematical modeling of dynamic processes

Some of the attributable parameters or variables are influenced by the classical clock time "t". In view of this, certain internal and external perturbations can be described by linearly independent functions of t, for $t \in \mathbb{R}$. For example, t is the classical time scale measured by stop-clock in the laboratory, $T_1(t) := t$ signifies the ideal and controlled environmental condition; $T_2(t) := B(t)$ describes the environmental random perturbations (B is the standard Wiener process); $T_3(t) := t^\alpha,\ \ 0 < \alpha < 1$ indicates the time varying delay or lagged process; last but not least, $T_4(t) := \eta(t)$ exhibits the hereditary effects (η is a function of bounded variation, e.g. the incubation period in epidemiology). The effects of these intra-inter structural perturbations are characterized by the linearly independent time-scale functions. This leads to an introduction of the following type of dynamic process.

Definition 2.1.1 A dynamic process is said to be operating under multi-time scales if the effects of certain intra structural and external environmental perturbations are characterized by a set of linearly independent time scales monitored by the classical time.

Prior to the formulation of dynamic model of a complex system operating under the multi-time scales, we need to define some basic elementary calculus concepts in the context of multi-time scales that are linearly independent functions of the usual time t.

Definition 2.1.2 Multi-time scale Integral: For $p \in \mathbb{N},\ p > 1$, let $\{T_1, T_2, \ldots, T_p\}$ be a set of linearly independent time scales. Let $f : [a, b) \times \mathbb{R}^{p-1} \to \mathbb{R}^n$ be a continuous function defined by $f(t) := f(T_1(t), T_2(t), \ldots, T_p(t))$. The multi-time scale integral of the composite function f over an interval $[t_0, t] \subseteq (a, b)$ is defined as the sum of p integrals with respect to the time-scales $T_1, T_2, \ldots, T_p$. We denote it by If,

$$(If)(t) \ = \ \int_{t_0}^{t} f(s)ds \ = \ \sum_{j=1}^{p} (I_j f)(t) \tag{2.1.1}$$

where the sense of the integral

$$(I_j f)(t) \ = \ \int_{t_0}^{t} f(s)dT_j(s) \tag{2.1.2}$$

depends on the time scale T_j, for each $j = 1, 2, \ldots, p$.

Example 2.1.3 (i) For $p = 3$, consider the linearly independent set consisting of time scales $T_1(t) := t,\ T_2(t) := B(t)$ where B is the standard Wiener process, and $T_3(t) := t^{\alpha},\ \ 0 < \alpha < 1$ as defined before. In this case, $f(t) \equiv f(T_1(t), T_2(t), T_3(t))$ and

$$(If)(t) = (I_1 f)(t) + (I_2 f)(t) + (I_3 f)(t) \tag{2.1.3}$$

where the integrals

$$\begin{aligned}
(I_1 f)(t) \ &= \ \int_{t_0}^{t} f(s)ds, \\
(I_2 f)(t) \ &= \ \int_{t_0}^{t} f(s)dB(s), \\
(I_3 f)(t) \ &= \ \int_{t_0}^{t} \frac{(t-s)^{\alpha-1} f(s)}{\Gamma(\alpha)} ds
\end{aligned}$$

are Cauchy-Riemann/Lebesgue, Itô-Doob, and Riemann-Liouville type, respectively.

(ii) If $p = 2$ and the times scales are $T_1(t) := t,\ T_2(t) := B(t)$, where B is the standard

Wiener process, then

$$(If)(t) = (I_1 f)(t) + (I_2 f)(t) \tag{2.1.4}$$

where $I_1 f$ and $I_2 f$ are defined in (i). This is indeed Itô-Doob type of integral [4].

Definition 2.1.4 Multi-time scale Differential: Let f be a function defined in Definition 2.1.2. The multi-time scale differential of the composite function f is defined to be the sum of the partial differentials of f with respect to the times scales $T_1(t), T_2(t), \ldots,$ and $T_p(t)$. We denote it by df,

$$(df)(t) \; = \; \sum_{j=1}^{p} (d_j f)(t), \tag{2.1.5}$$

where for each $j = 1, 2, ..., p$,

$$\begin{aligned} (d_j f)(t) \; &= \; f(T_1(t), ..., T_{j-1}(t), T_j(t + \Delta t), T_{j+1}(t), ..., T_p(t)) \\ &- \; f(T_1(t), ..., T_{j-1}(t), T_j(t), T_{j+1}(t), ..., T_p(t)), \end{aligned} \tag{2.1.6}$$

$\Delta t \simeq dt$ for small Δt, and $(d_j f)(t)$ corresponds to the integral $(I_j f)(t)$ in (2.1.2).

Remark 2.1.5 *In addition to the assumptions in Definition 2.1.4, if the function f has continuous partial derivatives with respect to each time scale, then*

$$(df)(t) \; = \; \sum_{j=1}^{p} \frac{\partial f}{\partial T_j}(t) dT_j(t). \tag{2.1.7}$$

Example 2.1.6 (i) For $p = 3$, let f satisfy the condition of Definition 2.1.4, where the set of linearly independent time scales is defined in Example 2.1.3. Then

$$(df)(t) = (d_1 f)(t) + (d_2 f)(t) + (d_3 f)(t). \tag{2.1.8}$$

In addition, if f has continuous partial derivatives with respect to each time scale, then $d_1 f(t) = f_{T_1(t)}(t)dt = \frac{\partial f}{\partial t}(t)dt$, $d_2 f(t) = f_{T_2(t)}(t)dB(t) = \frac{\partial f}{\partial B(t)}(t)dB(t)$, and $d_3 f(t) = f_{T_3(t)}(t)(dt)^\alpha = \frac{\partial f}{\partial t^\alpha}(t)(dt)^\alpha$. The differentials dt, $dB(t)$ and $(dt)^\alpha$ are in the sense of Cauchy-Riemann or Lebesgue [59], Itô-Doob [27], and Jumarie [35, 37], respectively.

(ii) If $p = 2$ and the times scales are $T_1(t) := t$, $T_2(t) := B(t)$ as in Example 2.1.3 (ii), then

$$(df)(t) = (d_1 f)(t) + (d_2 f)(t), \tag{2.1.9}$$

where $d_1 f$ and $d_2 f$ are defined in (i). This is indeed Itô-Doob type of differential [4, 53].

Remark 2.1.7 *Under the assumptions of Definition 2.1.4, the differential* (2.1.5) *is equivalent to the following integral:*

$$f(t + \Delta t) \ = \ f(t) + \sum_{j=1}^{p} \int_{t}^{t+\Delta t} (d_j f)(s). \tag{2.1.10}$$

We are now ready to describe the development of the mathematical model of dynamic process influenced by intra-inter structural perturbations and corresponding p attributable parameters/variables that are characterized by a set of linearly independent time-scales $\{T_1, T_2, \ldots, T_p\}$.

Let an n-dimensional vector $x(t) := x(T_1(t), T_2(t), \ldots, T_p(t))$ be the state of the dynamic process operating under the influence of these linearly independent time-scales at time t (the classical time scale). We note that the state $x(t)$ of the system at a time t is the composite function of p time scales. For $\Delta t > 0$, the state of the system at time $t + \Delta t$ is $x(t + \Delta t)$. The overall change of the state of the system over an interval $[t, t + \Delta t]$ is

$$\Delta x(t) = x(t + \Delta t) - x(t). \tag{2.1.11}$$

In order to incorporate the effects of intra-inter structural perturbations characterized by the corresponding linearly independent time-scale $T_1, T_2, \ldots, T_p$, we need to find the partial changes of the state of the system over the interval $[t, t+\Delta t]$. These changes are determined by the corresponding changes in the multi-time scales over the interval $[t, t + \Delta t]$. For each $j = 1, 2, ..., p$, the change in a j-th time-scale and the corresponding partial change of state over the interval $[t, t + \Delta t]$ are as follows:

$$\Delta T_j(t) \;=\; T_j(t + \Delta t) - T_j(t), \quad \text{and} \tag{2.1.12}$$

$$\begin{aligned} \Delta_j x(t) \;&=\; x(T_1(t), ..., T_{j-1}(t), T_j(t + \Delta t), T_{j+1}(t), ..., T_p(t)) \\ &-\; x(T_1(t), ..., T_{j-1}(t), T_j(t), T_{j+1}(t), ..., T_p(t)), \end{aligned} \tag{2.1.13}$$

respectively. The aggregate partial change of state over the interval $[t, t + \Delta t]$ is

$$\Delta x_{ag}(t) = \sum_{j=1}^{p} \Delta_j x(t). \tag{2.1.14}$$

For small $\Delta t > 0$, the overall change of state of system (2.1.11) can be approximated by the aggregate partial change (2.1.14), i.e.

$$\Delta x(t) \;=\; \sum_{j=1}^{p} \Delta_j x(t). \tag{2.1.15}$$

For fixed $T_j(t)$, using the dynamic laws (if any), allowing the experimental or field based data collection on the time interval $[t, t + \Delta t]$, and by imitating the technique developed in [53], the partial change of state under the influence of the structural change corresponding to the change in time-scale T_j is described by

$$\Delta_j x(t) \;=\; \vartheta_j(t, x)\Delta T_j(t), \tag{2.1.16}$$

where $\vartheta_j(t, x)$ is the n-dimensional microscopic rate of change of x per unit change of ΔT_j due to the influence of the structural perturbation characterized by the time-scale $T_j(t)$, $j = 1, 2, ..., p$.

For small $\Delta t > 0$, $\Delta_j x(t) \simeq d_j x(t)$ and $\Delta T_j(t) \approx dT_j(t)$. From (2.1.16), (2.1.15) reduces to the following differential equation:

$$dx(t) \; = \; \sum_{j=1}^{p} \vartheta_j(t,x) dT_j(t). \tag{2.1.17}$$

This type of differential equation is referred to as a system of multi-time scale differential equations.

From Definition 2.1.4 and Remark 2.1.7, the differential equation in (2.1.17) is equivalent to the following multi-time scales integral equation

$$x(t+\Delta t) \; = \; x(t) + \sum_{j=1}^{p} \int_{t}^{t+\Delta t} \vartheta_j(u, x(u)) dT_j(u). \tag{2.1.18}$$

where $a \leq t \leq t + \Delta t \leq b$.

Example 2.1.8 Under the set of time scales in Example 2.1.3 (i), the system of multi-time scale differential equations in (2.1.17) is called stochastic fractional differential equation (SFDE) of Itô-Doob type. Moreover, the corresponding SFDE of Itô-Doob type initial value problem (with initial condition (t_0, x_0)) is as

$$dx = b(t,x)dt \; + \; \sigma_1(t,x)dB(t) \; + \; \sigma_2(t,x)(dt)^{\alpha}, \quad x(t_0) = x_0, \tag{2.1.19}$$

where $\alpha \in (0,1)$, $b, \sigma_2 \in C[[t_0, t_0+T] \times \mathbb{R}^n, \mathbb{R}^n]$, $\sigma_1 \in C[[t_0, t_0+T] \times \mathbb{R}^n, \mathbb{R}^{nm}]$, and $B = \{B(t), t \geq 0\}$ is a m-dimensional Brownian motion on a complete probability space $\Omega \equiv (\Omega, \mathcal{F}, \mathcal{P})$.

From Definition 2.1.7, we can rewrite equation (2.1.19) in its equivalent integral form as follows:

$$x(t) - x(t_0) = \int_{t_0}^{t} b(s, x(s))ds + \int_{t_0}^{t} \sigma_1(s, x(s))dB(s) + \alpha \int_{t_0}^{t} \frac{\sigma_2(s, x(s))}{(t-s)^{1-\alpha}} ds \tag{2.1.20}$$

The scope of the formulated mathematical model (2.1.17) is further exhibited by the following remark.

Remark 2.1.9 *We note that*

(i) *If* $\sigma_2(\cdot,\cdot) \equiv 0$ *in Example 2.1.8, then IVP* (2.1.19) *is reduced to known Itô-Doob type stochastic IVP*

$$dx = d_1x + d_2x = b(t,x)dt \, + \, \sigma_1(t,x)dB(t), \;\; x(t_0) = x_0. \tag{2.1.21}$$

whose fundamental properties and applications have been well studied for more than half-century [4, 47].

(ii) *For* $\sigma_1(\cdot,\cdot) \equiv 0$ *in* (2.1.19), *we have the following generalized version of the classical deterministic fractional differential equations [40, 68].*

$$dx = d_1x + d_3x = b(t,x)dt \, + \, \sigma_2(t,x)(dt)^{\alpha}, \;\; x(t_0) = x_0. \tag{2.1.22}$$

(iii) *If* $b(\cdot,\cdot) \equiv 0$ *and* $\sigma_1(\cdot,\cdot) \equiv 0$, *then* (2.1.19) *becomes the IVP with Caputo type fractional differential equation [40, 58, 68].*

$$^{C}\mathcal{D}^{\alpha}_{t_0}x \; = \; \sigma_2(t,x), \;\; x(t_0) = x_0. \tag{2.1.23}$$

(iv) *If* $\sigma_1(\cdot,\cdot) \equiv 0$ *and* $\sigma_2(\cdot,\cdot) \equiv 0$, *then* (2.1.19) *is to the deterministic IVP*

$$dx \; = \; d_1x \; = \; b(t,x)dt, \;\; x(t_0) = x_0. \tag{2.1.24}$$

In the following lemma we develop a result that paves the way for developing methods for determining explicit/implicit solution processes for linear and nonlinear stochastic fractional differential equations. The result is a straightforward extension of the classical Itô-Doob formula.

Lemma 2.1.10 *Let* $p = 3$ *and the set of time scale be the defined in Example 2.1.3* ($T_1(t) =$

t, $T_2(t) = B(t)$, $T_3(t) = t^\alpha$ and $1/2 < \alpha < 1$). Let $x(t)$ satisfy

$$dx = b(t,x)dt + \sigma_1(t,x)dB(t) + \sigma_2(t,x)(dt)^\alpha, \tag{2.1.25}$$

Furthermore, let $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^m]$, and assume that V_t, V_x, V_{xx} exist and continuous for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where V_x is an $m \times n$ Jacobian matrix of $V(t,x)$ and $V_{xx}(t,x)$ is an $m \times n$ Hessian matrix whose elements are m-dimensional vectors. Then we have

$$dV(t,x) = L_1V(t,x)dt + L_2V(t,x)dB(t) + L_3V(t,x)(dt)^\alpha, \tag{2.1.26}$$

$$\text{where} \quad L_1V(t,x) = V_t(t,x) + V_x(t,x)b(t,x) + \frac{1}{2}\sigma_1(t,x)^T V_{xx}(t,x)\sigma_1(t,x), \tag{2.1.27}$$

$$L_2V(t,x) = V_x(t,x)\sigma_1(t,x), \tag{2.1.28}$$

$$\text{and} \quad L_3V(t,x) = V_x(t,x)\sigma_2(t,x). \tag{2.1.29}$$

Proof. The idea of the proof of this result is exactly parallel to the proof of the existing result of Itô's formula. For the sake of simplicity, we briefly outline the proof.

First we define

$$F(\theta) = V(t + \theta\Delta t, x + \theta\Delta x), \quad 0 \le \theta \le 1 \tag{2.1.30}$$

F is differentiable on $(0, 1)$ and we have

$$\begin{aligned}
dV(t,x) &= V(t+\theta\Delta t, x+\theta\Delta x) - V(t,x) \\
&= F(1) - F(0) \\
&= \int_0^1 \left[V_t(t+\theta\Delta t, x+\theta\Delta x)\Delta t + V_x(t+\theta\Delta t, x+\theta\Delta x)(\Delta x)\right] d\theta \\
&= V_t(t,x)\Delta t + V_x(t,x)\Delta x + \int_0^1 \left[V_t(t+\theta\Delta t, x+\theta\Delta x) - V_t(t,x)\right] d\theta\Delta t \\
&\quad + \int_0^1 \left[V_x(t+\theta\Delta t, x+\theta\Delta x) - V_x(t,x)\right] d\theta(\Delta x) \\
&= V_t(t,x)\Delta t + V_x(t,x)\Delta x + o(\Delta t) \\
&\quad + \int_0^1 \left[V_x(t+\theta\Delta t, x+\theta\Delta x) - V_x(t,x)\right] d\theta(\Delta x). \qquad (2.1.31)
\end{aligned}$$

We observe that $V_x(t,x)\Delta x = \Delta x^T V_x^T(t,x)$.

Next we define

$$G(\eta) = V_x(t+\theta\Delta t, x+\eta\theta\Delta x), \quad 0 \le \eta \le 1. \qquad (2.1.32)$$

and $V_{xx}(t,x) = \frac{\partial}{\partial x}V_x^T(t,x)$. G is differentiable on $(0, 1)$ because V is twice differentiable in x, and we have

$$\begin{aligned}
&V_x(t+\theta\Delta t, x+\theta\Delta x)(\Delta x) - V_x(t,x)(\Delta x) \\
&= G(1) - G(0) \\
&= \int_0^1 G'(\eta)d\eta \\
&= \int_0^1 (\Delta x)^T V_{xx}(t+\theta\Delta t, x+\eta\theta\Delta x)\theta\Delta x d\eta \\
&= (\Delta x)^T V_{xx}(t,x)\Delta x\theta + (\Delta x)^T \int_0^1 \left[V_{xx}(t+\theta\Delta t, x+\eta\theta\Delta x) - V_{xx}(t,x)\right]\Delta x d\eta\theta \\
&= (\Delta x)^T V_{xx}(t,x)\Delta t\theta + (\Delta x)^T \int_0^1 \left[V_{xx}(t+\theta\Delta t, x+\eta\theta\Delta x) - V_{xx}(t+\theta\Delta t, x)\right]\Delta x d\eta\theta \\
&\quad + (\Delta x)^T \int_0^1 \left[V_{xx}(t+\theta\Delta t, x) - V_{xx}(t,x)\right]\Delta x d\eta\theta. \qquad (2.1.33)
\end{aligned}$$

Now, substituting the quantity on the right hand side of (2.1.33) into the integral in (2.1.31), we obtain

$$
\begin{aligned}
&\int_0^1 [V_x(t+\theta\Delta t, x+\theta\Delta x) - V_x(t,x)]\, d\theta(\Delta x) \\
&\quad = \Delta x^T \int_0^1 [V_{xx}(t,x)\Delta x\theta d\theta + \int_0^1\int_0^1 \Delta x^T\, [(V_{xx}(t+\theta\Delta t, x) - V_{xx}(t,x))\Delta x]\, d\eta d\theta \\
&\qquad + \ (\Delta x)^T \int_0^1\int_0^1 [(V_{xx}(t+\theta\Delta t, x+\eta\theta\Delta x) - V_{xx}(t+\theta\Delta t, x))(\Delta x)]\, \theta d\eta d\theta \\
&\quad = \frac{1}{2}\Delta x^T V_{xx}(t,x)\Delta x + o(\Delta t)\ \text{(By the continuity of } V_{xx} \text{ and } \Delta t \ll 1). \qquad (2.1.34)
\end{aligned}
$$

Therefore, since $\Delta t \simeq dt$, $\Delta x \simeq dx$; by combining (2.1.31) and (2.1.34), we get

$$
\Delta V(t,x) = V_t(t,x)\Delta t + V_x(t,x)\Delta x + \frac{1}{2}(\Delta x)^T V_{xx}(t,x)(\Delta x). \qquad (2.1.35)
$$

Since x satisfies (2.1.25), under the assumption $1/2 < \alpha < 1$, (2.1.35) becomes

$$
\begin{aligned}
&(\Delta x)^T(\Delta x) = b(t,x)^T b(t,x)(\Delta t)^2 + b(t,x)^T\sigma_1(t,x)(dt)\Delta B(t) + b(t,x)^T\sigma_2(t,x)(dt)(dt)^\alpha \\
&+ \Delta B(t)^T\sigma_1(t,x)^T b(t,x)\Delta t + \Delta B(t)^T\sigma_1(t,x)^T\sigma_1(t,x)\Delta B(t) + \Delta B(t)^T\sigma_1(t,x)^T\sigma_2(\Delta t)^\alpha \\
&+ \sigma_2(t,x)^T b(t,x)(\Delta t)^{\alpha+1} + \sigma_2(t,x)^T\sigma_1(t,x)\Delta B(t)(\Delta t)^\alpha + \sigma_2(t,x)^T\sigma_2(t,x)(\Delta t)^{2\alpha}.
\end{aligned}
$$

We have

$(\Delta t)(\Delta t) \simeq o(\Delta t) \simeq (\Delta t)(\Delta t)^\alpha$, $(\Delta t)(\Delta B(t)) \simeq o(\Delta t) \simeq (\Delta B(t))(\Delta t)^\alpha$, $(\Delta t)^\alpha(\Delta t)^\alpha \simeq o(\Delta t)$, and $(\Delta B(t))^2 \simeq \Delta t$. Therefore, $(dx)(dx)^T = \sigma_1(t,x)\sigma_1(t,x)^T dt$, and (2.1.35) becomes

$$
\begin{aligned}
dV(t,x) &= V_t(t,x)dt + V_x(t,x)[b(t,x)dt + \sigma_1(t,x)dB(t) + \sigma_2(t,x)(dt)^\alpha] \\
&\quad + \frac{1}{2}\sigma_1(t,x)^T V_{xx}(t,x)\sigma_1(t,x)dt \\
&= L_1V(t,x)dt\ +\ L_2V(t,x)dB(t)\ +\ L_3V(t,x)(dt)^\alpha, \qquad (2.1.36)
\end{aligned}
$$

as desired.

■

Remark 2.1.11 (i) *If* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *(i.e.* $m = 1$*), then* $\sigma_1(t,x)^T V_{xx}(t,x)\sigma_1(t,x)$ *can be written as trace*$\big[V_{xx}(t,x)\sigma_1(t,x)\sigma_1(t,x)^T\big]$.

(ii) *Lemma 2.1.10 can be considered as Itô-Doob type of formula for system of time-scale stochastic differential equation* (2.1.25).

(iii) *The presented lemma can easily be extended to a more general set of linearly independent time scales including, but not limited to,* $0 < \alpha < 1/2$.

(iv) *The Development of Lemma 2.1.10 is motivated by our current research project that depends on long range dependent dynamic processes.*

2.2 Model validation

In this section, without loss of generality, we focus on the problem of model validation in the context of (2.1.19). The most fundamental problem for the validity of the presented mathematical model is the problem of existence and uniqueness of the solution processes (2.1.19). This is achieved by utilizing the classical Picard-Lindelöf successive approximations scheme in a natural way [58, 63].

Definition 2.2.1 (Space of solutions): Let $\{T_1(t) = t, T_2(t) = B(t), T_3(t) = t^\alpha\}$ be the set of linearly independent time scales defined in Example 2.1.3 (i). A random process $x = \{x(t), t_0 < t < t_0 + T\}$ is called solution of the IVP (2.1.19), if the composite function x $(x(t) \equiv x(t, w(t), t^\alpha))$ is sample path continuous with respect to each of the time scales T_j, $j = 1, 2, 3$.

Theorem 2.2.2 ***Existence and uniqueness:*** *Assume that, for* $(t,x) \in [t_0, t_0 + T] \times \mathbb{R}^n$, $\alpha \in (1/2, 1)$, $b, \sigma_2 \in C[[t_0, t_0+T]\times\mathbb{R}^n, \mathbb{R}^n]$, $\sigma_1 \in C[[t_0, t_0+T]\times\mathbb{R}^n, \mathbb{R}^{nm}]$, *and* $B = \{B(t), t \geq 0\}$ *is a m-dimensional Brownian motion on a complete probability space* $\Omega \equiv (\Omega, \mathcal{F}, \mathcal{P})$*, the following inequalities hold:*

$$|b(t,x)|^2 + |\sigma_1(t,x)|^2 + |\sigma_2(t,x)|^2 \leq K^2(1 + |x|^2); \text{ (Linear growth bound)} \tag{2.2.1}$$

for some constant $K > 0$, the Lipschitz condition

$$|b(t,x) - b(t,y)| + |\sigma_1(t,x) - \sigma_1(t,y)| + |\sigma_2(t,x) - \sigma_2(t,y)| \leq L|x - y| \tag{2.2.2}$$

for some constant $L > 0$.

Let x_0 be a random variable defined on $(\Omega, \mathcal{F}, \mathcal{P})$ and it is independent of the σ-algebra $\mathcal{F}_s^t \subseteq \mathcal{F}$ generated by $\{B(s), t \geq s \geq 0\}$ and such that $E|x_0|^2 < \infty$.

Then the initial value problem (2.1.19) *has a unique solution which is t-continuous with the property that $x(t,\omega)$ is adapted to the filtration $\mathcal{F}_t^{x_0}$ generated by x_0 and $\{B(s)(\cdot), s \leq t\}$, and*

$$\sup_{t_0 \leq t \leq T} E\left(|x(t)|^2\right) < \infty \tag{2.2.3}$$

Proof. **Existence:** First, we imitate the classical method of successive approximations [4, 41] to establish the existence of solution of IVP (2.1.19).

Let $x^{(0)}(t) = x_0$ and $x^{(k)}(t) = x^{(k)}(t,\omega)$ inductively as follows:

$$\begin{aligned} x^{(k+1)}(t) &= x_0 + \int_{t_0}^t b\left(s, x^{(k)}(s)\right) ds + \int_{t_0}^t \sigma_1\left(s, x^{(k)}(s)\right) dB(s) \\ &\quad + \alpha \int_{t_0}^t (t-s)^{\alpha-1} \sigma_2\left(s, x^{(k)}(s)\right) ds. \end{aligned} \tag{2.2.4}$$

From the assumption $E|x_0|^2 < \infty$ and the definition of $x^{(0)}(t)$, it is clear that

$$E\left(\left|x^{(0)}(t)\right|^2\right) < \infty$$

Applying the Schwarz inequality $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, the Itô's isometry [27] and the linear growth condition (2.2.1) to (2.2.4) we obtain

$$
\begin{aligned}
&E\left(\left|x^{(k+1)}(t)\right|^2\right)\\
&\quad\leq 4E\left|x^{(0)}(t)\right|^2+4E\left|\int_{t_0}^{t} b\left(s,x^{(k)}(s)\right)ds\right|^2+4E\left|\int_{t_0}^{t}\sigma_1\left(s,x^{(k)}(s)\right)dB(s)\right|^2\\
&\quad+\ 4E\left|\alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\sigma_2\left(s,x^{(k)}(s)\right)ds\right|^2\\
&\quad\leq\ \ 4E\left|x^{(0)}(t)\right|^2+4(T-t_0)E\left(\int_{t_0}^{t}\left|b\left(s,x^{(k)}(s)\right)\right|^2ds\right)\\
&\quad+4E\left(\int_{t_0}^{t}\left|\sigma_1\left(s,x^{(k)}(s)\right)\right|^2ds\right)+\ \frac{4\alpha^2(t-t_0)^{2\alpha-1}}{2\alpha-1}E\left(\int_{t_0}^{t}\left|\sigma_2\left(s,x^{(k)}(s)\right)\right|^2ds\right)\\
&\quad\leq\ \ 4E\left|x^{(0)}(t)\right|^2+4\left(1+(t-t_0)+\frac{\alpha^2(t-t_0)^{2\alpha-1}}{2\alpha-1}\right)K^2E\left(\int_{t_0}^{t}\left(1+\left|x^{(k)}(s)\right|^2\right)ds\right)
\end{aligned}
$$

for $k=0,1,2,....$ Thus, by induction we have

$$
\sup_{t_0\leq t\leq T}E\left(\left|x^{(n)}(t)\right|^2\right)\leq C_0<\infty \tag{2.2.5}
$$

for $n=1,2,....$

Similarly, applying the Schwarz inequality, the Itô's isometry [27] and the Lipschitz condition (2.2.2), we obtain

$$
\begin{aligned}
E\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2 \;\le\; & 3E\left[\left|\int_{t_0}^{t}\left(b\left(s,x^{(k)}(s)\right)-b\left(s,x^{(k-1)}(s)\right)\right)ds\right|^2\right] \\
& +\; 3E\left[\left|\int_{t_0}^{t}\left(\sigma_1\left(s,x^{(k)}(s)\right)-\sigma_1\left(s,x^{(k-1)}(s)\right)\right)dB(s)\right|^2\right] \\
& +\; 3E\left[\left|\alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\left[\sigma_2\left(s,x^{(k)}(s)\right)-\sigma_2\left(s,x^{(k-1)}(s)\right)\right]ds\right|^2\right] \\
\le\; & 3(t-t_0)\int_{t_0}^{t}E\left|b\left(s,x^{(k)}(s)\right)-b\left(s,x^{(k-1)}(s)\right)\right|^2 ds \\
& +\; 3\int_{t_0}^{t}E\left|\sigma_1\left(s,x^{(k)}(s)\right)-\sigma_1\left(s,x^{(k-1)}(s)\right)\right|^2 ds \\
& +\; 3\frac{\alpha^2(t-t_0)^{2\alpha-1}}{2\alpha-1}\int_{t_0}^{t}E\left|\sigma_2\left(s,x^{(k)}(s)\right)-\sigma_2\left(s,x^{(k-1)}(s)\right)\right|^2 ds \\
\le\; & 3L^2\left[1+t-t_0+\frac{\alpha^2(t-t_0)^{2\alpha-1}}{2\alpha-1}\right]\int_{t_0}^{t}E\left|x^{(k)}(t)-x^{(k-1)}(t)\right|^2 ds \\
& \text{(By using assumptions (2.2.2) and } \alpha>1/2\text{)} \\
\le\; & 3L^2\left[1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right]\int_{t_0}^{t}E\left|x^{(k)}(t)-x^{(k-1)}(t)\right|^2 ds, \qquad (2.2.6)
\end{aligned}
$$

where $T\in\mathbb{R}_+$ such that $t_0\le t\le t_0+T$.

From equation (2.2.4), by applying again the Schwarz inequality, the Itô's isometry together with the growth condition (2.2.1) for $k=1$, we have

$$
\begin{aligned}
E\left[\left|x^{(1)}(t)-x^{(0)}(t)\right|^2\right] \;\le\; & 3(t-t_0)\int_{t_0}^{t}E|b(s,x_0)|^2ds+3\int_{t_0}^{t}E|\sigma_1(s,x_0)|^2ds \\
& +3\frac{\alpha^2(t-t_0)^{2\alpha-1}}{2\alpha-1}\int_{t_0}^{t}E|\sigma_2(s,x_0)|^2ds \\
\le\; & 3L^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\int_{t_0}^{t}3K^2\left(1+E|x_0|^2\right)ds \\
\le\; & K^2 3^2 L^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\left(1+E|x_0|^2\right)(t-t_0). \qquad (2.2.7)
\end{aligned}
$$

Now, for $k=1$, replacing $E\left[\left|x^{(1)}(t)-x^{(0)}(t)\right|^2\right]$ in inequality (2.2.6) with the value on the right hand side of inequality (2.2.7) and integrating, we obtain

$$
\begin{aligned}
E\left|x^{(2)}(t)-x^{(1)}(t)\right|^2 &\leq 3L^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\int_{t_0}^{t}E\left|x^{(1)}(s)-x^{(0)}(s)\right|^2 ds \\
&\leq K^2\left(1+E|x_0|^2\right)\left[3^2L^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\right]^2\int_{t_0}^{t}(s-t_0)ds \\
&\leq K^2\left(1+E|x_0|^2\right)\left[3^2L^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\right]^2\frac{(s-t_0)^2}{2!}. \quad (2.2.8)
\end{aligned}
$$

For $k=2$, proceeding as before, we have

$$
E\left|x^{(3)}(t)-x^{(2)}(t)\right|^2 \leq K^2\left(1+E|x_0|^2\right)\left[3^2L^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)\right]^3\frac{(s-t_0)^3}{3!}. \quad (2.2.9)
$$

Thus, by principle of mathematical induction, this procedure yields

$$
E\left(\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2\right) \leq \frac{BM^{k+1}(t-t_0)^{k+1}}{(k+1)!},\ k=0,1,2,\ldots,\ t_0\leq t\leq t_0+T \quad (2.2.10)
$$

where $B=K^2\left(1+E|x_0|^2\right)$, and $M=3^2L^2\left(1+T+\frac{\alpha^2T^{2\alpha-1}}{2\alpha-1}\right)$ is a constant depending only on α, T, L^2 and $E|x_0|^2$.

Thus

$$
\sup_{t_0\leq t\leq t_0+T}E\left(\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2\right) \leq \frac{BM^{k+1}(t-t_0)^{k+1}}{(k+1)!}, \quad k=0,1,2,\ldots. \quad (2.2.11)
$$

This implies the mean-square convergence of the successive approximations uniformly on $[t_0, t_0+T]$.

$$\begin{aligned}
\left\|x^{(m)}(t)-x^{(n)}(t)\right\|_{L^2(P)}^2 &= \left\|\sum_{k=n}^{m-1} x^{(k+1)}(t)-x^{(k)}(t)\right\|_{L^2(P)}^2 \\
&\leq \sum_{k=n}^{m-1}\left\|x^{(k+1)}(t)-x^{(k)}(t)\right\|_{L^2(P)}^2 \\
&= \sum_{k=n}^{m-1}\int_{t_0}^{t_0+T} E\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2 dt \\
&\leq \sum_{k=n}^{m-1}\int_{t_0}^{t_0+T} \frac{BM^{k+1}(t-t_0)^{k+1}}{(k+1)!}dt \\
&= \sum_{k=n}^{m-1}\frac{BM^{k+1}T^{k+2}}{(k+2)!} \to 0 \text{ as } m,n\to\infty
\end{aligned}$$

By the Borel-Cantelli lemma and Weierstrass's convergence criterion [42], the convergence of the following series is guaranteed:

$$\sum_{k=1}^{\infty} P\left[\sup_{t_0\leq t\leq t_0+T}\left(\left|x^{(k+1)}(t)-x^{(k)}(t)\right|^2\right) > \frac{1}{k^2}\right] \leq \sum_{k=1}^{\infty}\frac{BM^{k+1}T^{k+2}k^4}{(k+2)!}. \tag{2.2.12}$$

From this, we conclude that there exists a random variable $x(t)$ satisfying

$$\lim_{n\to\infty}\left(x^{(0)}(t)+\sum_{k=1}^{n}\left(x^{(k)}(t)-x^{(k-1)}(t)\right)\right) = \lim_{n\to\infty} x^{(n)} = x(t) \quad \text{a.s} \tag{2.2.13}$$

uniformly on $[t_0, t_0+T]$. Since $x(t)$ is the limit of a nonanticipating functions and the uniform limit of a sequence of continuous functions, it is itself nonanticipating and continuous. From (2.2.4) we have

$$x(t) = x_0(t) + \int_0^t b(s,x(s))ds + \int_0^t \sigma_1(s,x(s))dB(s) + \alpha\int_0^t (t-s)^{\alpha-1}\sigma_2(s,x(s))ds$$

for all $t\in[t_0, t_0+T]$. This completes the proof of existence.

Uniqueness: The uniqueness follows from the Itô isometry [27] and the Lipschitz condition (2.2.2). Let $x(t,\omega)=x_t(\omega)$ and $y(t,\omega)=y_t(\omega)$ be solution processes through the initial data (t_0, x_0) and (t_0, y_0) , respectively, i.e. $x(0,\omega) = x_0(\omega)$ and $y(0,\omega) = y_0(\omega)$, $\omega \in \Omega$.

Let $b^{*k}(t,\omega) = b(t,x_t) - b(t,y_t)$ and $\sigma_i^{*k}(t,\omega) = \sigma_i(t,x_t) - \sigma_i(t,y_t)$, $i = 1,2$. Then, by virtue of the Schwarz inequality and the Itô's isometry, we have

$$\begin{aligned} E\left[|x_t - y_t|^2\right] &\leq 4E\left[|x_0 - y_0|^2\right] + 4(t-t_0)\int_{t_0}^{t}\left|b^{*k}(s)\right|^2 ds + 4E\left[\int_{t_0}^{t}\left|\sigma_1^{*k}(s)\right|^2 ds\right] \\ &\quad + 4\left(\frac{\alpha^2(t-t_0)^{2\alpha-1}}{2\alpha-1}\right)\int_{t_0}^{t}\left|\sigma_2^{*k}(s)\right|^2 ds \\ &\leq 4E\left|x_0 - y_0\right|^2 + 4L^2\left(1 + T + \frac{\alpha^2 T^{2\alpha-1}}{2\alpha-1}\right)\int_{t_0}^{t} E\left|x_s - y_s\right|^2 ds \end{aligned}$$

We define $u(t) = E\left[|x_t - y_t|^2\right]$. Then the function u satisfies

$$u(t) \leq u_0 + K\int_{t_0}^{t} u(s)ds, \tag{2.2.14}$$

where $u_0 = 4E\left[|x_0 - y_0|^2\right]$ and $K = 4L^2\left(1 + T + \frac{\alpha^2 T^{2\alpha-1}}{2\alpha-1}\right)$. By the application of the Grönwall's inequality [42, 59], we conclude that

$$u(t) \leq u_0 \exp\{Kt\}$$

Now, assume that $x_0 = y_0$. Then $u_0 = 0$ and so $u(t) = 0$ for all $t \geq t_0$. Hence

$$P\left\{|x_s - y_s| = 0 \text{ for all } t \in \mathbb{Q} \cap [t_0, t_0 + T]\right\} = 1,$$

where $\mathbb{Q}$ denotes the rational numbers. By the continuity of $t \mapsto |x_t - y_t|$, it follows that

$$P\left\{|x(t,\omega) - y(t,\omega)| = 0 \text{ for all } t \in [0,T]\right\} = 1,$$

and the uniqueness is proved.

■

Remark 2.2.3 *We note that Theorem 2.2.2 is proved for* $1/2 < \alpha < 1$. *This was moti-*

vated by the long range delay dependent dynamic process that is part of our current project. However, we propose to investigate the proposed problem for not only $0 < \alpha < 1$, *but also more general set of linearly independent multi time-scales.*

3 Method of finding solutions of SFDE

The validity of the mathematical model (2.1.19) in section 2.1 naturally generates problem of finding a close form solution (either explicit or implicit form). Again, without loss of generality, we are developing the methods of solving SFDE (2.1.19). The method can be easily extended for solving linear and nonlinear multi-time scale stochastic differential equations (2.1.17). In this chapter, we develop the methods of finding a general solution process of a class of linear and nonlinear SFDEs. The close form solution representation in terms of attributable variables characterized by multi-time scales provides a very important promising tool in statistical analysis, in particular time series and regression analysis [29]. In Section 3.4, the concepts and results developed in the preceding chapter are applied to stochastic models in ecological and epidemiological processes of population dynamic under multi-time scales.

3.1 Method of finding solutions of linear homogeneous SFDE

In this section, we utilize the eigenvalue-eigenvector type of method to compute a close form solution of a linear homogeneous stochastic fractional differential equations of the following type:

$$dx = b(t)xdt + \sigma_1(t)xdB(t) + \sigma_2(t)x(dt)^\alpha, \tag{3.1.1}$$

where $\alpha \in (1/2, 1)$, $b, \sigma_2 \in C[[t_0, t_0 + T] \times \mathbb{R}^n, \mathbb{R}^n]$, $\sigma_1 \in C[[t_0, t_0 + T] \times \mathbb{R}^n, \mathbb{R}^{nm}]$, and $B = \{B(t), t \geq 0\}$ is a m-dimensional Brownian motion process on a complete probability space $\Omega \equiv (\Omega, \mathcal{F}, \mathcal{P})$. Moreover, rate coefficient functions satisfy linear growth (2.2.1) and

Lipschitz (2.2.2) conditions that ensure the existence of solution of (3.1.1).

Procedure of finding a general solution of (3.1.1)

The method presented here is an alternative approach to the integrating factor approach of deterministic homogeneous stochastic fractional differential equations. To achieve our goal using Definition 2.1.4, we begin by decomposing equation (3.1.1) into its linearly independent time scales (deterministic, stochastic and fractional parts) as:

$$d_1x = b(t)xdt, \tag{3.1.2}$$

$$d_2x = \sigma_1(t)xdB(t), \quad \text{and} \tag{3.1.3}$$

$$d_3x = \sigma_2(t)x(dt)^\alpha, \tag{3.1.4}$$

respectively. Our objective of finding solution of (3.1.1) is decomposed into the following three sub-goals consist of:

(a) finding solutions x^d, x^s and x^f of (3.1.2), (3.1.3) and (3.1.4), respectively;

(b) creating a candidate solution:

$$x = x^d x^s x^f, \tag{3.1.5}$$

(c) testing the validity of the candidate solution in (3.1.5).

Note: By taking the product in (b), we assume independence (here and in the sequel) with respect to entities which constitute each system. In what follows, we outline a procedure to fulfill the stated sub goals in (a), (b) and (c). Let us seek solutions of (3.1.2), (3.1.3) and (3.1.4) of the following form:

$$x^d(t) = \exp\left(\int_{t_0}^t \lambda^d(s)ds\right) c^d \ \text{(Riemann-Cauchy/Lebesgue integral)}, \tag{3.1.6}$$

$$x^s(t) = \exp\left(\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right) c^s \ \text{(Itô-Doob integral)}, \tag{3.1.7}$$

and

$$x^f(t) = E_\alpha \left[\alpha \int_{t_0}^{t} (t-s)^{\alpha-1} \lambda^f(s) ds \right] c^f \quad \text{(Riemann-Liouville integral)}, \tag{3.1.8}$$

respectively, where c^d, c^s, and c^f are arbitrary unknown constants that are either real numbers or real-valued random variables independent of the Brownian motion B. λ^d, λ^s, and λ^f) are unknown functions defined on J, into $\mathbb{R}^n$.

Case 1: If either $c^d = 0$, $c^s = 0$, and $c^f = 0$ (for any λ^d, λ_1^s, λ_2^s, λ^f), then either x^d in (3.1.2) or x^s in (3.1.3) or x^f in (3.1.4) is trivial solution process (zero solution, that is, zero random process or function on J) of either (3.1.2) or (3.1.3) or (3.1.4). In this case, the candidate for the solution defined in (3.1.5) is always the trivial solution of (3.1.1).

Case 2: It follows from case 1 that our major goal is reduced to seeking a nontrivial solution process (non-zero solutions) of (3.1.1). For this purpose, we need to find unknown functions λ^d, λ_1^s, λ_2^s, λ^f, and unknown non-zero constants c^d, c^s, and c^f in (3.1.2), (3.1.3), and (3.1.4). This is achieved in the following.

By applying the stochastic calculus to (3.1.5) and imitating the procedure described in [53], the problem of finding solution $x^d(t)$ of (3.1.2) is summarized as follows:

$$dx^d(t) = \lambda^d(t) \exp\left(\int_{t_0}^{t} \lambda^d(s) \right) c^d ds, \tag{3.1.9}$$

$$\lambda^d(t) \exp\left(\int_{t_0}^{t} \lambda^d(s) \right) c^d ds = b(t) \exp\left(\int_{t_0}^{t} \lambda^d(s) \right) c^d ds, \tag{3.1.10}$$

$$\lambda^d(t) = b(t), \tag{3.1.11}$$

and hence

$$x^d(t) = \exp\left(\int_{t_0}^{t} b(s) ds \right) c^d. \tag{3.1.12}$$

This completes the determination of solution process of (3.1.2) under the influence of the classical time scale $T_1(t) = t$.

To find solution of (3.1.3), again we imitate the procedure described in [53] in the context of Itô-Doob stochastic calculus. The Itô-Doob differential of $x^s(t)$ yields

$$\begin{aligned} dx^s(t) &= \exp\left[\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right]\left(\lambda_1^s(t) + \frac{1}{2}(\lambda_1^s)^2(t)\right)c^s dt \\ &+ \exp\left[\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right]\lambda_2^s(t)c^s dB(t). \end{aligned} \tag{3.1.13}$$

From the fact that $x^s(t)$ is a solution of (3.1.3), we also have

$$dx^s(t) = \sigma_1(t)\exp\left[\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right]c^s dB(t). \tag{3.1.14}$$

Subtracting the side by side members of (3.1.13) from those of (3.1.14), the resulting equation yields

$$\begin{aligned} &\exp\left[\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right]\left(\lambda_1^s(t) + \frac{1}{2}(\lambda_1^s)^2(t)\right)c^s dt \\ &+ \exp\left[\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right](\lambda_2^s(t) - \sigma_1(t))c^s dB(t) = 0. \end{aligned} \tag{3.1.15}$$

Since the time scales $T_1(t) = t$ and $T_2(t) = B(t)$ are linearly independent, so are their differentials $dT_1(t) = dt$ and $dT_2(t) = dB(t)$. Therefore, considering the fact that $c^s \neq 0$, equation (3.1.15) yields

$$\exp\left[\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right]\left(\lambda_1^s(t) + \frac{1}{2}(\lambda_1^s)^2(t)\right) = 0 \tag{3.1.16}$$

$$\exp\left[\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right](\lambda_2^s(t) - \sigma_1(t)) = 0. \tag{3.1.17}$$

For any integrable λ_1^s and λ_2^s, we note that $\exp\left[\int_{t_0}^t \lambda_1^s(u)du + \int_{t_0}^t \lambda_2^s(u)dB(u)\right] \neq 0$ on $\mathbb{R}^n$. Therefore, from (3.1.16) and (3.1.17), we have the system of equations

$$\lambda_1^s(t) + \frac{1}{2}(\lambda_2^s)^2(t) = 0, \tag{3.1.18}$$

$$\lambda_2^s(t) = \sigma_1(t). \tag{3.1.19}$$

Solving this systems of equations for $\lambda_1^s(t)$ and $\lambda_2^s(t)$ yields

$$\lambda_2^s(t) = \sigma_1(t) \quad \text{and} \quad \lambda_1^s(t) = -\frac{1}{2}\sigma_1^2(t). \tag{3.1.20}$$

Thus, it follows that

$$x^s(t) \;=\; \exp\left[-\frac{1}{2}\int_{t_0}^{t}\sigma_1^2(u)du + \int_{t_0}^{t}\sigma_1(u)dB(u)\right]c^s. \tag{3.1.21}$$

This completes the determination of solution process of (3.1.3) characterized by the effect of environmental random perturbations described by Wiener process B.

By following procedure for finding solution of (3.1.4) detailed in [37], we obtained

$$dx^f(t) \;=\; E_\alpha\left[\alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\lambda^f(s)ds\right]\lambda^f(t)c^f(dt)^\alpha \tag{3.1.22}$$

$$= E_\alpha\left[\alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\sigma_2(s)ds\right]\sigma_2(t)c^f(dt)^\alpha. \tag{3.1.23}$$

For any function λ^s, we note that $E_\alpha\left[\alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\lambda^f(s)ds\right] \neq 0$. Hence, solving for λ^f yields $\lambda^f = \sigma_2$ so that

$$x^f(t) = E_\alpha\left[\alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\sigma_2(s)ds\right]c^f. \tag{3.1.24}$$

Therefore, a candidate solution of (3.1.1) is:

$$\begin{aligned} x(t) &= x^d(t)x^s(t)x^f(t) \\ &= \exp\left[\int_{t_0}^{t}\left[b(s)-\frac{1}{2}\sigma_1^2(s)\right]ds + \int_{t_0}^{t}\sigma_1(s)dB(s)\right]\times E_\alpha\left[\Gamma(\alpha+1)\left(I^\alpha\sigma_2\right)(t)\right]c, \end{aligned} \tag{3.1.25}$$

where $c = c^d c^s c^f \neq 0$ and $\Gamma(\alpha+1)\left(I^\alpha\sigma_2\right)(t) = \alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\sigma_2(s)ds$.

Finally, we need to justify the validity of $x(t)$ (sub-goal (c)) defined in (3.1.25) as a nontrivial solution of (3.1.1). We note that $x(t)$ is a well-defined random process/function defined on J as long as b, σ_1 and σ_2 are defined and continuous on J. It is nontrivial because $\exp\left[\int_{t_0}^t \lambda_1(s)ds + \int_{t_0}^t \lambda_2(s)dB(s)\right] \neq 0$, $E_\alpha\left[\alpha\int_{t_0}^t (t-s)^{\alpha-1}\lambda_3(s)ds\right] \neq 0$ and $c \neq 0$. It remains to show that $x(t)$ defined in (3.1.25) satisfies (3.1.1). From Definition 2.1.4 ($d = d_1 + d_2 + d_3$) and differential equations (3.1.2), (3.1.3), and (3.1.4), and the product rule for differentials appropriately, we obtain the following:

$$
\begin{aligned}
dx(t) &= d_1x(t) + d_2x(t) + d_3x(t) \\
&= [b(t)x^d dt]x^s(t)x^f(t) + x^d(t)[\sigma_1(t)x^s(t)dB(t)]x^f(t) \\
&\quad + x^d(t)x^s(t)[\sigma_2(t)x^f(t)(dt)^\alpha] \text{ (by using (3.1.2), (3.1.3), and (3.1.4), respectively)} \\
&= b(t)x(t)dt + \sigma_1(t)x(t)dB(t) + \sigma_2(t)x(t)(dt)^\alpha. \qquad (3.1.26)
\end{aligned}
$$

This exhibits the fact that the non-zero process defined in (3.1.25) is indeed a solution process of (3.1.1).

Remark 3.1.1 *In the following, we make a few observations based on the above discussions.* (i) *We can rewrite each of the solution processes of* (3.1.2)-(3.1.4) *as follows:*

$$x^d(t) = \exp\left(\int_{t_0}^t b(s)ds\right)c^d \;=\; \Phi^d(t)c^d, \qquad (3.1.27)$$

$$x^s(t) = \exp\left[-\frac{1}{2}\int_{t_0}^t \sigma_1^2(u)du + \int_{t_0}^t \sigma_1(u)dB(u)\right]c^s \;=\; \Phi^s(t)c^s, \qquad (3.1.28)$$

$$x^f(t) = E_\alpha\left[\alpha\int_{t_0}^t (t-s)^{\alpha-1}\sigma_2(s)ds\right]c^f \;=\; \Phi^f(t)c^f, \qquad (3.1.29)$$

where c^d, c^s *and* c^f *are non-zero arbitrary constants defined in* (3.1.12), (3.1.21) *and* (3.1.24), *respectively;*

$$\Phi^d(t) \;=\; \exp\left(\int_{t_0}^t b(s)ds\right), \qquad (3.1.30)$$

$$\Phi^s(t,\omega(t)) \;\equiv\; \Phi^s(t) \;=\; \exp\left[-\frac{1}{2}\int_{t_0}^t \sigma_1^2(u)du + \int_{t_0}^t \sigma_1(u)dB(u)\right], \qquad (3.1.31)$$

$$\Phi^f(t) \;=\; E_\alpha\left[\alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\sigma_2(s)ds\right]. \tag{3.1.32}$$

(ii) We note that $\Phi^d(t)$, $\Phi^s(t)$ and $\Phi^f(t)$ are all non-zero functions for any $t_0, t \in J$ and $t \geq t_0$. They all have algebraic inverses. In fact,

$$(\Phi^d)^{-1}(t) \;=\; \exp\left(-\int_{t_0}^{t} b(s)ds\right), \tag{3.1.33}$$

$$(\Phi^s(t,\omega(t)))^{-1} \;\equiv\; (\Phi^s)^{-1}(t) \;=\; \exp\left[\frac{1}{2}\int_{t_0}^{t}\sigma_1^2(u)du - \int_{t_0}^{t}\sigma_1(u)dB(u)\right]. \tag{3.1.34}$$

Φ^f invertibility comes from the fact that the Mittag-Leffler function E_α is monotone increasing when $0 < \alpha < 1$, and therefore invertible. Let $(\Phi^f)^{-1}(t)$ denote its algebraic inverse. Also, we note that if $c^d = c^s = c^f = 1$, form (3.1.27), (3.1.28), (3.1.29), *we further conclude that $\Phi^d(t)$, $\Phi^s(t)$ and $\Phi^f(t)$ general fundamental solution processes of* (3.1.2), (3.1.3) *and* (3.1.4), *respectively. In fact,*

$$d_1\Phi^d(t) \;=\; b(t)\Phi^d(t)dt, \tag{3.1.35}$$

$$d_2\Phi^s(t) \;=\; \sigma_1(t)\Phi^s(t)dB(t), \tag{3.1.36}$$

$$\text{and} \qquad d_3\Phi^f(t) \;=\; \sigma_2(t)\Phi^f(t)(dt)^\alpha. \tag{3.1.37}$$

(iii) Let us define a process

$$\Phi(t,\omega(t)) = \Phi^d(t)\Phi^s(t,\omega(t))\Phi^f(t) \;=\; \Phi(t) = \Phi^d(t)\Phi^s(t)\Phi^f(t). \tag{3.1.38}$$

$\Phi(t)$ is the general fundamental solution process of (3.1.1), *and because $\Phi^d(t)$, $\Phi^s(t)$ and $\Phi^f(t)$ are algebraically invertible, so is $\Phi(t)$. Moreover, we have*

$$d\Phi(t) = b(t)\Phi(t)dt + \sigma_1(t)\Phi(t)dB(t) + \sigma_2(t)\Phi(t)(dt)^\alpha. \tag{3.1.39}$$

A nontrivial solution of (3.1.1) *can be represented by*

$$
\begin{aligned}
x(t) &= \exp\left[\int_{t_0}^{t}\left[b(s)-\frac{1}{2}\sigma_1^2(s)\right]ds+\int_{t_0}^{t}\sigma_1(s)dB(s)\right]\times E_\alpha\left[\Gamma(\alpha+1)\left(I^\alpha\sigma_2\right)(t)\right]c, \\
&= \Phi(t)c, \qquad (3.1.40)
\end{aligned}
$$

where $t_0, t \in J$, $\Phi(t)$ *is defined in* (3.1.38) *on* $J = [t_0, t_0+T]$, *and* c *is arbitrary constant.* $x(t)$ *is called as the general solution process of* (3.1.1).

In the following, we present an analytic test for the invertibility of the fundamental solution Φ of (3.1.1).

Theorem 3.1.2 *: Let* Φ *be the fundamental solution process of* (3.1.1). *Then,* Φ *is invertible in the algebraic sense, and it satisfies the following first order linear stochastic fractional differential equation of Itô-Doob type:*

$$
d\Phi^{-1}(t) \;=\; [-b(t)+\sigma_1^2(t)]\Phi^{-1}(t)dt-\sigma_1(t)\Phi^{-1}(t)dB(t)-\sigma_2(t)\Phi^{-1}(t)(dt)^\alpha. \qquad (3.1.41)
$$

Proof. The fundamental solution process $\Phi(t) = \Phi^d(t)\Phi^s(t)\Phi^f(t)$ of (3.1.1) is invertible due to the fact that both Φ_1, Φ_2, and Φ_3 are. The case of Φ_3 is justifies by the fact that the Mittag-Leffler function E_α, $1/2 \le \alpha \le 1$ is monotone increasing for each α. We have the following relations.

$$
\begin{aligned}
d_1(\Phi(t)) &= d_1[\Phi^d(t)\Phi^s(t)\Phi^f(t)] \;=\; d_1[\Phi^d(t)]\Phi^s(t)\Phi^f(t) \\
&= b(t)\Phi^d(t)\Phi^s(t)\Phi^f(t)dt \;=\; b(t)\Phi(t)dt. \qquad (3.1.42)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
d_2(\Phi(t)) &= \sigma_1(t)\Phi(t)dB(t), \qquad \text{and} \qquad (3.1.43) \\
d_3(\Phi(t)) &= \sigma_2(t)\Phi(t)(dt)^\alpha. \qquad (3.1.44)
\end{aligned}
$$

The algebraic inverse of $\Phi(t)$ is $\Phi^{-1}(t) = \Phi^f(t))^{-1}(\Phi^s(t))^{-1}(\Phi^d(t))^{-1}$. From Definition 2.1.4 and the product rule, we get the following differential:

$$d(\Phi\Phi^{-1}) \;=\; d\Phi\Phi^{-1} + \Phi d\Phi^{-1} + d\Phi d\Phi^{-1} \;=\; d(1) \;=\; 0. \tag{3.1.45}$$

This yields

$$\begin{aligned}
d\Phi^{-1}(t) \;&=\; -\Phi^{-1}(t)[d\Phi(t)\Phi^{-1}(t) + d\Phi(t)d\Phi^{-1}(t)], \;\; \text{(by solving (3.1.45) for } d\Phi^{-1}(t))\\
&=\; -\Phi^{-1}(t)d\Phi(t)\Phi^{-1}(t) - \Phi^{-1}(t)d\Phi(t)d\Phi^{-1}(t)\\
&\quad -\Phi^{-1}(t)[b(t)\Phi(t)dt + \sigma_1(t)\Phi(t)dB(t) + \sigma_2(t)\Phi(t)(dt)^{\alpha}]d\Phi^{-1}(t)\\
&\quad \text{(by using relations (3.1.42), (3.1.43) and (3.1.44))}\\
&=\; -[b(t)dt + \sigma_1(t)dB(t) + \sigma_2(t)(dt)^{\alpha}]\Phi^{-1}(t) - \sigma_1^2(t)\Phi^{-1}(t)(dB(t))^2\\
&\quad \text{(by using the random part of } d\Phi^{-1})\\
&=\; [-b(t) + \sigma_1^2(t)]\Phi^{-1}(t)dt + \sigma_1(t)\Phi^{-1}(t)dB(t) + \sigma_2(t)\Phi^{-1}(t)(dt)^{\alpha}
\end{aligned}$$

This completes the proof.

■

Initial Value Problem (IVP) associated with (3.1.1)

Consider the initial value problem for linear homogeneous SFDE of the form

$$dx = \; b(t)xdt + \sigma_1(t)xdB(t) + \sigma_2(t)x(dt)^{\alpha}, \;\; x(t_0) = x_0, \tag{3.1.46}$$

b, σ_1, σ_2, $B(t)$ and x are as defined in (3.1.1); $t_0 \in J$; x_0 is real-valued random variable on a complete probability space $(\Omega, \mathcal{F}, P)$, and it is independent of $B(t)$ for all $t \in J$. By application of Theorem 2.2.2, the IVP (3.1.46) has a unique solution.

Remark 3.1.3 *We recall that a solution process of the IVP* (3.1.46) *is a solution of* (3.1.1) *that also satisfies the initial condition* $x(t_0) = x_0$.

Since $x(t_0) = x_0$, from (3.1.40), we have $\Phi(t_0)c = x_0$. Moreover, from the invertibility of Φ, we obtain

$$c = \Phi^{-1}(t_0)x_0 \tag{3.1.47}$$

which depends on b, σ_1, σ_2, $B(t)$ and x_0.

The solution to the IVP (3.1.46) is determined by substituting the expression in (3.1.47) into (3.1.40). Hence, we have

$$\begin{aligned} x(t) &= \Phi(t)c \quad \text{ from (3.1.40)} \\ &= \Phi(t)\Phi^{-1}(t_0)x_0 \quad \text{ (from (3.1.47))} \\ &= \Phi(t, t_0)x_0 \quad \text{(by notation)} \end{aligned} \tag{3.1.48}$$

where $\Phi(t, t_0) \equiv \Phi(t)\Phi^{-1}(t_0)$. More explicitly,

$$\begin{aligned} \Phi(t, t_0) &= [\Phi^d(t)\Phi^s(t)\Phi^f(t)][(\Phi^f(t_0))^{-1}(\Phi^s(t_0))^{-1}(\Phi^d(t_0))^{-1}] \\ &= [\Phi^d(t)(\Phi^d(t_0))^{-1}][\Phi^s(t)(\Phi^s(t_0))^{-1}][\Phi^f(t)(\Phi^f s(t_0))^{-1}] \\ &= \Phi^s(t, t_0)\Phi^s(t, t_0)\Phi^f(t, t_0) \end{aligned} \tag{3.1.49}$$

where $\Phi^d(t, t_0) = \Phi^d(t)(\Phi^d)^{-1}(t_0)$ $\Phi^s(t, t_0) = \Phi^s(t)(\Phi^s)^{-1}(t_0)$, $\Phi^f(t, t_0) = \Phi^f(t)(\Phi^f)^{-1}(t_0)$. $x(t) = x(t, t_0, x_0) = \Phi(t, t_0)x_0$ is referred to as a particular solution of (3.1.1). $\Phi(t, t_0)$ defined in (3.1.48) is called normalized fundamental solution process of (3.1.1); because $\Phi(t, t_0)$ has an algebraic inverse, $\Phi(t_0, t_0) = 1$.

3.2 Method of finding solutions of linear nonhomogeneous SFDE

In this section, we discuss a procedure for finding solution for the linear nonhomogeneous SFDE of the following type:

$$dx = [b(t)x + p(t)]dt + [\sigma_1(t)x + q(t)]dB(t) + [\sigma_2(t)x + v(t)](dt)^\alpha, \tag{3.2.1}$$

where b, σ_1, σ_2 are as in (3.1.1), and p, q, and v are continuous functions defined on an interval $J = [t_0, t_0 + T]$. The concept of solution process defined in Remark 3.1.3 is directly applicable to (3.2.1).

Procedure for finding general solution of (3.2.1)

To find a general solution of (3.2.1), we utilize the following basic steps:

(i) find a general solution process x_c of the linear homogeneous scalar SFDE corresponding to (3.2.1).

(ii) find a particular solution x_p of (3.2.1), and

(iii) setup a candidate $x = x_c + x_p$ for testing the general solution of (3.2.1).

The answer to (i), recall that the linear homogeneous scalar SFDE corresponding to (3.2.1) is exactly the equation (3.1.1) whose general fundamental solution is provided in (3.1.40) i.e.

$$\begin{aligned}\Phi(t) &= \Phi^d(t)\Phi^s(t)\Phi^f(t)\\ &= \exp\left[\int_{t_0}^{t}\left[b(s) - \frac{1}{2}\sigma_1^2(s)\right]ds + \int_{t_0}^{t}\sigma_1(s)dB(s)\right] \times E_\alpha\left[\alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\sigma_2(s)ds\right],\end{aligned}$$

To verify that x_p is solution process of (3.2.1), we employ the method of variation of constants parameters.

Variation of constants parameters

A particular solution process x_p of (3.2.1) is a random function that satisfies the stochastic fractional differential equation (3.2.1). For this purpose, we define a random function a

$$x_p(t) \;=\; \Phi(t)c(t), \quad c(a) = c_a, \tag{3.2.2}$$

where $c(t)$ is an unknown random function with $c(a) = c_a,\ a \in J$. Let us assume that $x_p(t)$ is a solution process of (3.2.1). Applying Definition 2.1.4 and the product rule to x_p and the fact that $\Phi(t)$ is solution to (3.1.1) to get

$$dx_p(t) = d\Phi(t)c(t) + \Phi(t)dc(t) + d\Phi(t)dc(t) \tag{3.2.3}$$

Now, using the algebraic inverse $\Phi^{-1}(t)$ of $\Phi(t)$, we solve (3.2.3) for $dc(t)$, and obtain

$$\begin{aligned} dc(t) &= \Phi^{-1}(t)\left[dx_p(t) - d\Phi(t)c(t) - d\Phi(t)dc(t)\right] \quad \text{(from (3.2.3))} \\ &= \Phi^{-1}(t)\left[(b(t)x_p(t)+p(t))dt + (\sigma_1(t)x_p(t)+q(t))dB(t) + (\sigma_2(t)x_p(t)+v(t))(dt)^\alpha\right] \\ &\quad -\Phi^{-1}(t)\left[b(t)\Phi(t)dt + \sigma_1(t)\Phi(t)dB(t) + \sigma_2(t)\Phi(t)(dt)^\alpha\right]c(t) - \Phi^{-1}(t)d\Phi(t)dc(t) \\ &\quad \text{(since } x_p(t) \text{ satisfies (3.2.2))} \\ &= \Phi^{-1}(t)p(t)dt + \Phi^{-1}(t)q(t)dB(t) + \Phi^{-1}(t)v(t)(dt)^\alpha - \Phi^{-1}(t)\Phi(t)dc(t). \end{aligned}$$

On the other hand

$$\begin{aligned} d\Phi(t)dc(t) &= [\sigma_1(t)\Phi(t)dB(t)]\Phi^{-1}q(t)dB(t) \text{ (using the random part of } dc(t)) \\ &= \sigma_1(t)\Phi(t)\Phi^{-1}q(t)(dB(t))^2 \\ &= \sigma_1(t)q(t)dt. \end{aligned}$$

Therefore we have

$$dc(t) = \Phi^{-1}(t)[p(t) - \sigma_1(t)q(t)]dt + \Phi^{-1}(t)q(t)dB(t) + \Phi^{-1}(t)v(t)(dt)^\alpha, \; c(a) = c_a$$

and
$$\begin{aligned} c(t) &= c_0 + \int_{t_0}^{t}\Phi^{-1}(s)[p(s) - \sigma_1(s)q(s)]ds + \int_{t_0}^{t}\Phi^{-1}(s)q(s)dB(s) \\ &\quad + \alpha\int_{t_0}^{t}(t-s)^{\alpha-1}\Phi^{-1}(s)v(s)ds. \end{aligned}$$

Thus

$$
\begin{aligned}
x_p(t) &= \Phi(t)c(t) \\
&= \Phi(t)c_0 + \int_{t_0}^{t} \Phi(t,s)[p(s) - \sigma_1(s)q(s)]ds + \int_{t_0}^{t} \Phi(t,s)q(s)dB(s) \\
&\quad + \alpha \int_{t_0}^{t} \frac{\Phi(t,s)}{(t-s)^{1-\alpha}} v(s)ds,
\end{aligned}
$$

where $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$.

For (iii), it easy to see that from the above argument, if $x_c(t) = \Phi(t)c$, where c is any arbitrary constant, then $x_c(t)$ is solution of

$$dx = b(t)xdt + \sigma_1(t)xdB(t) + \sigma_2(t)x(dt)^{\alpha}$$

and $x(t) = x_p(t) + x_c(t)$ is the general solution of (3.2.1).

This completes the procedure for finding general solution of (3.2.1).

Initial Value Problem (IVP) associated with (3.2.1)

The procedure elaborated for determining solution for the IVP corresponding to the linear homogeneous SFDE is applicable to the nonhomogeneous case

$$dx = [b(t)x+p(t)]dt+[\sigma_1(t)x+q(t)]dB(t)+[\sigma_2(t)x+v(t)](dt)^{\alpha}, \;\; x(t_0) = x_0. \tag{3.2.4}$$

By following the procedure of finding a general solution of (3.2.4), we have

$$
\begin{aligned}
x(t) = \Phi(t)c &+ \int_{t_0}^{t} \Phi(t,s)[p(s) - \sigma_1(s)q(s)]ds + \int_{t_0}^{t} \Phi(t,s)q(s)dB(s) \\
&+ \; \alpha \int_{t_0}^{t} (t-s)^{\alpha-1}\Phi(t,s)v(s)ds
\end{aligned} \tag{3.2.5}
$$

where $a, t \in J$ and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$.

To solve the IVP (3.2.4), we use $x(t_0) = x_0$ to obtain

$$c = \Phi^{-1}(t_0)x_0 - \int_{t_0}^{t} \Phi^{-1}(s)[p(s) - \sigma_1(s)q(s)]ds + \int_{t_0}^{t} \Phi^{-1}(s)q(s)dB(s) + \alpha \int_{t_0}^{t} (t-s)^{\alpha-1}\Phi^{-1}(s)v(s)ds \quad (3.2.6)$$

And substituting the right hand side of (3.2.6) for c in the right hand side of (3.2.5) yields the solution of IVP (3.2.4).

Next, we explore the particular type of nonlinear equation known as Bernouilli type equation. We conclude this section two applications on the of this type of equation.

3.3 Example of Nonlinear Reducible Equations: the Bernouilli type equation

In this subsection, for the sake of simplicity, we attempt to illustrate a particular nonlinear stochastic fractional differential equations. In fact, these ideas can be successfully utilized to solve many types of SFDE. For more on this topic, refer to [51, 53]. A Bernouilli type stochastic fractional differential equation is defined by

$$dx = \left[P(t)x + Q(t)x^n + \frac{n}{2}Y^{2n-1}\right] dt \; + \; [S(t)x + Y(t)x^n]\, dB(t) + \; [U(t)x + Z(t)x^n]\,(dt)^{\alpha}, \;\; x(0) = x_0 \quad (3.3.1)$$

We need the following assumptions:

(H1): growth and Lipschitz conditions on the coefficients of (2.1.19), and

(H2): In addition to **(H1)**, we assume that b, σ_1 and σ_2 are continuously differentiable with respect to both t and x. Furthermore $\sigma_1(t,x) \neq 0$ and $G(t,x) = \int_c^x \frac{ds}{\sigma_1(t,s)}$ is invertible in x for each t.

(H3): It is assumed that functions P, Q, S, Y, and U are continuous functions. Moreover (3.3.1) has solution process. In addition, we assume that either $S(t)$ or $Y(t)$ or $U(t)$ are not identically 0.

(H1) guarantees the existence of a unique solution to the IVP (2.1.19), $x(t_0) = x_0$.

To solve (3.3.1), we need to reduce it to the form (3.2.4), and to do so we execute the following steps:

Seek the energy function

We seek an unknown function $V(t,x)$ (Energy/Lyapunov function) defined on $J \times \mathbb{R}$ into $\mathbb{R}$ with the following properties

(a) $V(t,x)$ is continuous on $J \times \mathbb{R}$.

(b) for $(t,x) \in J \times \mathbb{R}$, $V(t,x)$ is monotonic in x for each t

(c) V is continuously differentiable with respect to t for each x, and twice continuously differentiable with respect to x for each t

(d) for all $t \in J$, $V(t,x)$ has an inverse function $E(\cdot,\cdot)$ defined on $J \times \mathbb{R}$ into $\mathbb{R}$, that is, $V(t, E(t,x)) = x = E(t, V(t,x))$.

Differential of the energy function along differential field

By applying Lemma 2.1.10 to the energy function V, we obtain

$$dV(t,x) = LV(t,x)dt + \sigma_1(t,x)\frac{\partial}{\partial x}V(t,x)dB(t) + \sigma_2(t,x)\frac{\partial}{\partial x}V(t,x)(dt)^{\alpha} \tag{3.3.2}$$

where L is a linear differential associated with (3.3.2), and it is defined by

$$LV(t,x) = \frac{\partial}{\partial x}V(t,x) + b(t,x)\frac{\partial}{\partial x}V(t,x) + \frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2}{\partial x^2}V(t,x) \tag{3.3.3}$$

Now set $m(t) = V(t,x)$.

Using of conceptual choice of functions

We set seek the reduced form of equation (3.3.1). So we set

$$\mu(t)V(t,x)+p(t)=\frac{\partial}{\partial x}V(t,x)+\left[P(t)x+Q(t)x^n+\frac{n}{2}Y^{2n-1}\right]\frac{\partial}{\partial x}V(t,x)$$
$$+\frac{1}{2}\left[S(t)x+Y(t)x^n\right]\frac{\partial^2}{\partial x^2}V(t,x), \tag{3.3.4}$$

$$\nu(t)V(t,x)+q(t)=\left[S(t)x+Y(t)x^n\right]\frac{\partial}{\partial x}V(t,x), \tag{3.3.5}$$

$$\eta(t)V(t,x)+v(t)=\left[U(t)x+Z(t)x^n\right]\frac{\partial}{\partial x}V(t,x). \tag{3.3.6}$$

Finding candidate for $V(t,x)$

We use (3.3.5) or (3.3.6) to find candidate for $V(t,x)$. From these two equations and in view of the structure of rate functions in the Bernouilli differential equation, we have

$$S(t)x\frac{\partial}{\partial x}V(t,x)=\nu V(t,x), \tag{3.3.7}$$

$$Y(t)x^n\frac{\partial}{\partial x}V(t,x)=q(t), \tag{3.3.8}$$

$$U(t)x\frac{\partial}{\partial x}V(t,x)=\eta V(t,x), \tag{3.3.9}$$

$$Z(t)x^n\frac{\partial}{\partial x}V(t,x)=v(t). \tag{3.3.10}$$

From (H3), assuming that $S(t)\neq 0$, we have $S(t)x\neq 0$, and chose $V(t,x)$ that satisfies (3.3.7) with a choice of $\nu(t)=\delta S(t)$ for an unknown non-zero constant δ ($\delta\neq 0$). the goal is to find $V(t,x)$, rate $\mu(t)$, $p(t)$, $\nu(t)$, $q(t)$, $\eta(t)$, and $v(t)$ of the differential equation in its reduced form

$$dm=[\mu(t)m+p(t)]dt+[\nu(t)m+q(t)]dB(t)+[\eta(t)m+v(t)](dt)^{\alpha}, \tag{3.3.11}$$

where $m=V(t,x)$.

To find $\nu(t)$ it is enough to obtain δ; we rewrite (3.3.7) as follows:

$$\begin{aligned}\frac{\partial}{\partial x}V(t,x) &= \frac{\nu}{S(t)x}V(t,x) \text{ (assuming } S(t)x \neq 0)\\ &= \frac{\delta}{x}V(t,x) \text{ (from } \mathbf{(H3)}). \end{aligned} \tag{3.3.12}$$

Thus $\frac{\frac{\partial}{\partial x}V(t,x)}{V(t,x)} = \frac{\delta}{x}$ is independent of t so that $V(t,x) = x^{\delta}C$, C nonzero constant of integration.

$$\frac{\partial}{\partial x}V(t,x) = \delta x^{\delta-1}C, \tag{3.3.13}$$

$$\frac{\partial^2}{\partial x^2}V(t,x) = \delta(\delta-1)x^{\delta-2}C, \tag{3.3.14}$$

$$\begin{aligned} q(t) &= Y(t)x^n \frac{\partial}{\partial x}V(t,x)\\ &= \delta x^{n+\delta-1}Y(t)C. \end{aligned} \tag{3.3.15}$$

By using **(H3)** and separating the functions of t and x from (3.3.15), we have

$$x^{n+\delta-1} = \frac{q(t)}{\delta Y(t)C} \text{ is a function of } t \text{ only.}$$

Therefore choose $q(t)$ and δ so that $x^{n+\delta-1} = x^0 = 1$; that is $\delta = 1-n$.

$$q(t) = \delta Y(t)x^{n+\delta-1}C = (1-n)Y(t)C. \tag{3.3.16}$$

From (3.3.4) and the fact that $V(t,x) = x^{\delta}C$, we obtain

$$\mu(t)x^{\delta}C + p(t) = (1-n)\left[x^{\delta}\left(P(t) - \frac{n}{2}S^2(t)\right)C + \left(Q(t) - nS(t)Y(t)\right)C\right]. \tag{3.3.17}$$

The method of undetermined coefficients leads to

$$\mu(t) = (1-n)\left[P(t) - \frac{n}{2}S^2(t)\right], \tag{3.3.18}$$

$$p(t) = (1-n)\left[Q(t) - nS(t)Y(t)\right]C. \tag{3.3.19}$$

The procedure used to find $q(t)$ and ν can be used to obtain

$$\eta(t) = (1-n)U(t), \tag{3.3.20}$$

$$v(t) = (1-n)Z(t)C. \tag{3.3.21}$$

Taking $C = 1$ sice it is arbitrary, the reduced equation becomes

$$\begin{aligned} dm =& (1-n)\left[\left(P(t) - \frac{n}{2}S^2(t)\right)m + (Q(t) - nS(t)Y(t))\right]dt \\ &+ \ (1-n)\left[S(t)m + Y(t)\right]dB(t) + (1-n)\left[U(t)m + Z(t)\right](dt)^{\alpha}. \end{aligned} \tag{3.3.22}$$

A particular solution to (3.3.22) is given by

$$\begin{aligned} m_p(t) = \Phi(t)C_0 &+ \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)\left[p(s) - \nu(s)q(s)\right]ds \\ &+ \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)q(s)dB(s) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)v(s)ds. \end{aligned} \tag{3.3.23}$$

or in our case of Bernouilli type equation we have

$$\begin{aligned} m_p(t) \ &= \Phi(t)C_0 + (1-n)\int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)\left[Q(s) - (n+1)S(s)Y(s)\right]ds \\ &+ (1-n)\int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)Y(s)dB(s) + \frac{(1-n)}{\Gamma(\alpha)}\int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)Z(s)ds. \end{aligned} \tag{3.3.24}$$

A general solution to (3.3.11) is of the form $m(t) = m_p(t) + m_c(t)$ where $m_c(t) = \Phi(t)\cdot c$, c an arbitrary constant.

Since $V(t,x) = x^{\delta}$ and $\delta = 1-n$,the inverse of V is $E(t,x) = x^{1/(1-n)}$.

Therefore $x(t) = E(t, V(t,x)) = E(t, m(t))$.

$$
\begin{aligned}
x(t) &= \Bigg[\Phi(t)C_0 + (1-n)\int_{t_0}^{t}\Phi(t,s)\left[Q(s)-(n+1)S(s)Y(s)\right]ds + \\
&\quad +(1-n)\int_{t_0}^{t}\Phi(t,s)Y(s)dB(s) + \frac{(1-n)}{\Gamma(\alpha)}\int_{t_0}^{t}\Phi(t,s)Z(s)ds\Bigg]^{\frac{1}{1-n}},
\end{aligned}
\tag{3.3.25}
$$

where $\Phi(t,s) = \Phi(t)\Phi^{-1}(s) = \Phi_1(t)\Phi_1^{-1}(s)\Phi_2(t)\Phi_2^{-1}\Phi_3(t)(s)\Phi_3^{-1}(s)$.

3.4 Dynamical modeling of ecological and epidemiological processes

In this section, we develop stochastic models of dynamical processes in ecology and epidemiology. The dynamic of these processes is affected by both random and hereditary perturbations. Stochastic fractional differential equations are suitable mathematical framework that can be utilized to study these kinds of dynamic processes in a systematic and coherent manner. The application of fractals and scaling to finance [61] and other fields has grown significantly in the past two decades. Mathematical and stochastic models in ecology and epidemiology have of great interest to scientists [1, 2, 5, 67, 69].

Example 3.4.1 (Ecological Process) The biological processes have a natural tendency to maintain its steady (equilibrium) states, or the gradual adjustment to its states under environmental changes. The birth and death processes in the biological systems play a significant role. For the development of a mathematical model of single-species ecological processes (SEP) in biological sciences, we make the following assumptions (for more detail see e.g. [53]):

In the open communities/systems, the size of species are integral values. The successive generations may not overlap in time. The breeding as well as environmental parameters may vary in discrete time. As a result of this, it is difficult to control the population growth by only birth and death processes. For example, the migration is a natural process to be taken into account of the growth rate. The environmental random fluctuations can also cause to modify the resources and also to cause the migration process. The natural mathematical model of these types of ecosystems can be described by the stationary Gaussian model with independent increments. Because of these considerations, the modified stochastic model we propose is directly appropriate for these types of species in the communities. Furthermore, the hereditary perturbations affecting such systems may delay the growth rate or the expansion of such communities. Let $N(t)$ be the size of the species/organism in the community at a time t. A mathematical model suitable for the dynamic of these systems is represented by

$$dN = \theta(\overline{\kappa} - N)Ndt + \sigma_1 NdB(t) + \sigma_2 N(dt)^{\alpha}, \quad N(t_0) = N_0, \tag{3.4.1}$$

where $0 < \alpha < 1$, $\theta = \beta - \delta$ is the intrinsic rate of natural increase, β and δ being the birth and death rates per individual member of the community, respectively. The deterministic rate functions $b(t, N) = \theta(\kappa - N)N$, (with $\kappa = \overline{\kappa} + \xi(t)$), $\sigma_1(t, N) = \sigma_1 N$, and $\sigma_2(t, N) = \sigma_2 N$ are smooth enough to guarantee existence of solution of (3.4.1). $B(t) = \xi dt$, $\xi(t)$ is the stationary Gaussian process with independent increments that characterizes the effects of the environmental random fluctuations with $E[\xi(t)] = 0$; $\sigma_1 \neq 0$ describes the random growth rate due to the migration of the population, and ($B(t) = \int_{t_0}^{t} \xi(s)ds$ or $\xi(t) = \frac{d}{dt}B(t)$). $\sigma_2(t, N)(dt)^{\alpha}$ characterizes the delay effects due to response time delay as well as the variable breeding time and developmental time delay, $\sigma_2 > 0$ is the rate describing magnitude of the hereditary perturbations [45, 49].

Note that this equation is a Bernouilli type stochastic fractional differential equation in which $n = 2$, $P(t) = \theta\overline{\kappa}$, $Q(t) = -\theta$, $S(t) = \sigma_1$, $Y(t) = 0$, $U(t) = \sigma_2$, and $Z(t) = 0$. The candidate for Energy function (with $C = 1$) is

$$V(x) = x^{\delta}$$

Following the procedure above for finding general solution of stochastic fractional differential equation, we obtain the reduced linear nonhomogeneous stochastic fractional differential equation:

$$dm = -[(\theta\overline{\kappa} - \sigma_1^2)m - \theta]dt - \sigma_1 m dB(t) - \sigma_2 m(dt)^{\alpha}. \tag{3.4.2}$$

A particular solution to (3.4.2) is

$$m_p(t) = \Phi(t)c, \tag{3.4.3}$$

where

$$\begin{aligned}\Phi(t) &= \exp\left[\int_{t_0}^{t}\left((\theta\overline{\kappa} - \sigma_1^2) - \frac{1}{2}\sigma_1^2\right)ds + \int_{t_0}^{t}\sigma_1 dB(s)\right] E_\alpha\left[\Gamma(1+\alpha)^2(I^\alpha\sigma_2)(t)\right] \\ &= \exp\left[(\theta\overline{\kappa} - \frac{3}{2}\sigma_1^2)t + \sigma_1 B(t)\right] E_\alpha\left[\sigma_2\Gamma(1+\alpha)t^\alpha\right].\end{aligned} \tag{3.4.4}$$

Since $V(t, N) = N^\delta$ and $\delta = -1$,the inverse of V is $E(t, N) = N^{-1}$.

And the general solution to (3.4.1) is $(N(t) = E(t, V(t, N)) = E(t, m(t)))$ given by

$$\begin{aligned}N(t) &= \left[\Phi(t)C_0 + (1-n)\int_{t_0}^{t}\Phi(t,s)\left[Q(s) - (n+1)S(s)Y(s)\right]ds + \right. \\ &\left. +(1-n)\int_{t_0}^{t}\Phi(t,s)Y(s)dB(s) + \frac{(1-n)}{\Gamma(\alpha)}\int_{t_0}^{t}\Phi(t,s)Z(s)ds\right]^{\frac{1}{1-n}} \\ &= \left[\Phi(t)C_0 + \theta\int_{t_0}^{t}\Phi(t,s)ds\right]^{-1}.\end{aligned} \tag{3.4.5}$$

where $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$.

Example 3.4.2 (Epidemic model): Most epidemiological phenomena are very complex. Mathematical models provide an information about epidemic of communicable diseases to the international, national, state and local health departments for their planning and decision making processes. For simplicity, we assume that the population is divided into two disjoint groups of individuals: the infective and the susceptible (for more detail [53]). The environmental random fluctuations can also cause to modify the resources as well as the migration process. The natural mathematical model of these types of ecosystems can be described by the stationary Gaussian model with independent increments. Because of these considerations, the stochastic fractional studied above model is applicable to these types of species in the communities. We admit to have open communities/systems (in which it is difficult to control "well-mixed/homogeneous" condition) with the possibility of a nonlocal interaction beside the local interaction in the epidemic spreading. For instance, both types of interaction were widely observed in the recent SARS and birds flu outbreaks. The non-local interacting epidemic has been modeled by a fractional differential equation [1]. The growth rate of number of infective $I = I(t)$ is proportional to I as well as $S = N - I$ with a positive constant of proportionality θ also known as the specific rate of infection. N is the size of the species/organism and S is the susceptible population size in the community at a time t. N can be treated as a random parameter: $N = \widetilde{N} + \xi(t)$, where ξ is stationary Gaussian process with independent increments satisfying $E[\xi(t)] = 0$. Moreover, in view of modified stochastic model, $dB(t) = \xi(t)dt$. From the above mention assumptions, the model is well described by the following stochastic fractional differential equation:

$$dI = \theta(\widetilde{N} - I)Idt + \sigma_1 IdB(t) + \sigma_2 I(dt)^{\alpha}, \quad I(0) = 1, \tag{3.4.6}$$

where $\theta > 0$, $\sigma_1 \neq 0$, $\sigma_2 \neq 0$. The deterministic rate functions $b(t, I) = \theta(\widetilde{N} - I)I$, $\sigma_1(t, I) = \sigma_1 I$ for some $\sigma_1 \neq 0$, and $\sigma_2(t, I) = \sigma_2 I$ are smooth enough to guarantee existence of solution of (5.0.1). $(dt)^{\alpha}$ represents the time lag/delay characterizing the time varying incubation process and the individual immune response with the magnitude $\sigma_2 > 0$. The incorporation of these types of hereditary effects extends the existing non-hereditary [2, 46, 49] as well as finite time delay processes.

We observe that (5.0.1) is a Bernouilli type equation in which $n = 2$, $P(t) = \theta\widetilde{N}$, $Q(t) = -\theta$, $S(t) = \sigma_1$, $Y(t) = 0$, $U(t) = \sigma_2$, and $Z(t) = 0$. The candidate for Energy function (with $C = 1$) is

$$V(x) = x^{\delta}.$$

Following the procedure above for finding general solution of stochastic fractional differential equation, we obtain the reduced linear nonhomogeneous stochastic fractional differential equation:

$$dm = -[(\theta\widetilde{N} - \sigma_1^2)m - \theta]dt - \sigma_1 mdB(t) - \sigma_2 m(dt)^{\alpha}. \tag{3.4.7}$$

A particular solution to (5.1.1) is

$$m_p(t) = \Phi(t)c, \tag{3.4.8}$$

where

$$\begin{aligned}\Phi(t) &= \exp\left[\int_{t_0}^{t}\left((\theta\widetilde{N} - \sigma_1^2) - \frac{1}{2}\sigma_1^2\right)ds + \int_{t_0}^{t}\sigma_1 dB(s)\right] E_\alpha\left[\Gamma(1+\alpha)^2(I^\alpha\sigma_2)(t)\right] \\ &= \exp\left[\left(\theta\widetilde{N} - \frac{3}{2}\sigma_1^2\right)t + \sigma_1 B(t)\right] E_\alpha\left[\sigma_2\Gamma(1+\alpha)t^\alpha\right]. \end{aligned} \tag{3.4.9}$$

Since $V(t, I) = I^\delta$ and $\delta = -1$, the inverse of V is $E(t, I) = I^{-1}$.

And the general solution to (3.4.1) is $I(t) = E(t, V(t, I)) = E(t, m(t))$ given by

$$\begin{aligned} I(t) &= \left[\Phi(t)C_0 + (1-n)\int_{t_0}^{t}\Phi(t,s)\left[Q(s) - (n+1)S(s)Y(s)\right]ds + \right. \\ &\quad \left. + (1-n)\int_{t_0}^{t}\Phi(t,s)Y(s)dB(s) + \frac{(1-n)}{\Gamma(\alpha)}\int_{t_0}^{t}\Phi(t,s)Z(s)ds\right]^{\frac{1}{1-n}} \\ &= \left[\Phi(t)C_0 + \theta\int_{t_0}^{t}\Phi(t,s)ds\right]^{-1}, \end{aligned} \tag{3.4.10}$$

where $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$.

4 Numerical Methods for SFDE

In this chapter, we derive a numerical scheme for the stochastic fractional differential equation (2.1.19) studied in chapters 2 and 3. We realize that a numerical method for (2.1.19) is actually a numerical integration for the equivalent integral equation (2.1.20). We conclude this chapter with the proof of convergence of the presented numerical scheme.

4.1 Derivation of a numerical scheme for SFDE

Due to the equivalency between the differential equation (2.1.19) and the integral equation (2.1.20), we recognize that a numerical scheme for (2.1.19) is indeed a numerical integration of (2.1.20). From this observation, a numerical scheme for (2.1.19) consists of numerical integration schemes depending on the time scales defined in Remark 3.1 in [64], $T_1(t) = t$, $T_2(t) = B(t)$, and $T_3(t) = t^{\alpha}$. In view of this idea, the development of numerical scheme for (2.1.19) depends on the numerical integration technique for (2.1.20). In the following we utilize this idea to develop a numerical scheme for (2.1.19).

Throughout this section, for each time interval $[t_0, T]$ and integer $N > 1$, we assume that the partition $t_0 < t_1 < \cdots < t_N = T$ is equally spaced, that is, letting $h = (T - t_0)/N$, the times at the grid points are given as $t_k = t_0 + hk,\ k = 0, 1, \ldots, N$. For the sake of simplicity, we shall use $t_0 = 0$ so that $t_k = hk$, and define $x_k = x(t_k) = x(kh)$. We also use the notations $\Delta t_k = t_{k+1} - t_k$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, and $\sigma_{i,k} = \sigma_1(t_k, x_k),\ i = 1, 2$.

For this purpose, we employ the classical Euler scheme [11], the Maruyama scheme [41], a numerical approximation of the Abel-Volterra type integral equation [15, 16, 39] to approximate $I_1x(t)$, $I_2x(t)$, and $I_3x(t)$, respectively. In fact by the application of the Euler scheme

[11], we have

$$I_1(t) \approx \sum_{k=0}^{N} b(t_k, x_k)\Delta t_k. \tag{4.1.1}$$

By applying the Maruyama scheme [41], we have

$$I_2(t) \approx \sum_{k=0}^{N} \sigma_1(t_k, x_k)\Delta B_k. \tag{4.1.2}$$

For $I_3x(t)$, we need to provide more details about the development of a numerical integration technique. For this purpose, we need to recall the two-point Lagrange interpolation formula [24].For $k = 0, 1, 2, \ldots, N$, and for any $j = 0, 1, 2, \ldots, k$, the two-point Lagrange linear interpolation formula for $\sigma_2(t, x(t))$ on the interval $[t_j, t_{j+1}]$ is defined by

$$\begin{aligned} \tilde{\sigma}_{2,k+1}(t, x(t)) &= \frac{t_{j+1} - t}{t_{j+1} - t_j}\sigma_2(t_j, x(t_j)) + \frac{t - t_j}{t_{j+1} - t_j}\sigma_2(t_{j+1}, x(t_{j+1})) + e_j(t_k, t) \\ &= \frac{(t_{j+1} - t)}{h}\sigma_2(t_j, x(t_j)) + \frac{(t - t_j)}{h}\sigma_2(t_{j+1}, x(t_{j+1})) + e_j(t_k, t). \end{aligned} \tag{4.1.3}$$

For each $j = 0, 1, 2, \ldots, k$, setting $e_j(t) \equiv 0$, and using(4.1.3), we obtain

$$\begin{aligned} &\int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\sigma_2(t, x(t))dt \approx \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\tilde{\sigma}_{2,k+1}(t, x(t))dt \\ &\quad = \sum_{j=0}^{k} \int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\tilde{\sigma}_{2,k+1}(t, x(t))dt \\ &\quad = \sum_{j=0}^{k} \int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \left[\frac{(t_{j+1} - t)}{h}\sigma_2(t_j, x(t_j)) + \frac{(t - t_j)}{h}\sigma_2(t_{j+1}, x(t_{j+1}))\right] dt \\ &\quad = \sum_{j=0}^{k} \frac{\sigma_2(t_j, x(t_j))}{h} \int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}(t_{j+1} - t)dt \\ &\qquad + \sum_{j=0}^{k} \frac{\sigma_2(t_{j+1}, x(t_{j+1}))}{h} \int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}(t - t_j)dt \end{aligned} \tag{4.1.4}$$

Setting $t = t_j + ph$ with $0 \leq p \leq 1$, $dt = hdp$ and recalling that $t_n = nh$ for each $n = 0, 1, 2, \ldots N$, we evaluate the integrals in the first and second terms in the right hand side of (4.1.4)

$$\int_{t_j}^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}(t-t_j)dt = \int_0^1 ((k+1)h - jh - ph)^{\alpha-1}(h-ph)hdp$$
$$= h^{\alpha+1}\int_0^1 (k+1-j-p)^{\alpha-1}(1-p)dp$$
$$= \frac{h^{\alpha+1}}{\alpha(\alpha+1)}\left[(k-j)^{\alpha+1} + (\alpha+j-k)(k+1-j)^{\alpha}\right] \quad (4.1.5)$$

and

$$\int_{t_j}^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}(t_{j+1}-t)dt$$
$$= \int_0^1 ((k+1)h - jh - ph)^{\alpha-1} phhdp$$
$$= h^{\alpha+1}\int_0^1 (k+1-j-p)^{\alpha-1} pdp$$
$$= \frac{h^{\alpha+1}}{\alpha(\alpha+1)}\left[(k+1-j)^{\alpha+1} - (\alpha+1+k-j)(k-j)^{\alpha}\right] \quad (4.1.6)$$

From (4.1.5) and (4.1.6), (4.1.4) reduces to

$$\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\sigma_2(t,x(t))dt \approx \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\tilde{\sigma}_{2,k+1}(t,x(t))dt$$
$$= \sum_{j=0}^{k} \frac{\sigma_2(t_j,x(t_j))}{h}\int_{t_j}^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}(t_{j+1}-t)dt$$
$$+ \sum_{j=0}^{k} \frac{\sigma_2(t_{j+1},x(t_{j+1}))}{h}\int_{t_j}^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}(t-t_j)dt$$
$$= \sum_{j=0}^{k} \frac{\sigma_2(t_j,x(t_j))}{h}\frac{h^{\alpha+1}}{\alpha(\alpha+1)}\left[(k-j)^{\alpha+1} + (\alpha+j-k)(k+1-j)^{\alpha}\right]$$
$$+ \sum_{j=0}^{k} \frac{\sigma_2(t_{j+1},x(t_{j+1}))}{h}\frac{h^{\alpha+1}}{\alpha(\alpha+1)}\left[(k+1-j)^{\alpha+1} - (\alpha+1+k-j)(k-j)^{\alpha}\right]$$
$$= \frac{h^{\alpha}}{\alpha(\alpha+1)}\left\{\sigma_2(t_0,x(t_0))[k^{\alpha+1} + (\alpha-k)(k+1)^{\alpha}] + \sigma_2(t_{j+1},x(t_{j+1}))\right\}$$
$$+ \frac{h^{\alpha}}{\alpha(\alpha+1)}\sum_{j=1}^{k}\sigma_2(t_j,x(t_j))\left[(k+1-j+1)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k+1-j)^{\alpha}\right]$$
$$= \frac{1}{\alpha}\times\sum_{j=0}^{k} a_{j,k+1}\sigma_2(t_j,x(t_j)) + \frac{h^{\alpha}}{\alpha(\alpha+1)}a_{k+1,k+1}\sigma_2(t_{k+1},x(t_{k+1}) \quad (4.1.7)$$

where

$$a_{j,k+1} = \frac{h^\alpha}{\alpha+1} \times \begin{cases} k^{\alpha+1} + (\alpha - k)(k+1)^\alpha, & \text{if } j = 0 \\ \big[(k+1-j+1)^{\alpha+1} + (k-j)^{\alpha+1} & \\ \quad -2(k+1-j)^\alpha\big], & \text{if } 1 \le j \le k, \\ 1, & \text{if } j = k+1. \end{cases} \tag{4.1.8}$$

To reduce the error in the integration procedure for fractional integral discussed above, using a modified numerical technique, namely, a predictor-corrector method (fractional Adams-Bashforth-Moulton method) [15, 18], the term $\sigma_2(t_{k+1}, x_{k+1})$ in (4.1.7) is replaced with $\sigma_2(t_{k+1}, x^P_{k+1})$, where x^P_{k+1} is determined by the fractional Adams-Bashforth method

$$x^P_{k+1} = x_0 + \alpha \sum_{j=0}^{k} b_{j,k+1}\sigma_2(t_j, x_j), \tag{4.1.9}$$

where

$$b_{j,k+1} = \frac{h^\alpha}{\alpha}\left((k+1-j)^\alpha - (k-j)^\alpha\right). \tag{4.1.10}$$

Thus, we have

$$\alpha \int_0^{t_{k+1}} \frac{\sigma_2(t, x(t))}{(t_{k+1}-t)^{1-\alpha}} dt \approx \frac{h^\alpha}{\alpha+1}\sigma_2(t_{k+1}, x^P_{k+1}) \; + \; \sum_{j=0}^{k} a_{j,k+1}\sigma_2(t_j, x_j). \tag{4.1.11}$$

The importance of the predictor-corrector method [16] is that, for $0 < \alpha < 1$, the error behaves as

$$\max_{k=0,1,2,\ldots,N} |x(t_k) - x_k| = O(h^{1+\alpha}). \tag{4.1.12}$$

Now we are ready to formulate a numerical integration scheme for (2.1.20). From the Euler numerical integration (4.1.1) (for Cauchy-Riemann/Lebesgue integral), the Maruyama scheme (4.1.2) (for Itô-Doob integral), and the numerical approximation of the fractional integral by the predictor-corrector method (4.1.11), a numerical integration of the fractional stochastic integral equation (2.1.20) is defined by

$$\begin{aligned} x_{n+1} &= x_0 + \sum_{k=0}^{n} b(t_k, x_k)\Delta t_k + \sum_{k=0}^{n} \sigma_1(t_k, x_k)\Delta B_k + \sum_{k=0}^{n} a_{k,n+1}\sigma_2(t_k, x_k) \\ &\quad + \frac{h^\alpha}{\alpha+1}\sigma_2(t_{n+1}, x_{n+1}^P), \quad n = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{4.1.13}$$

where $\Delta t_k = t_{k+1} - t_k$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, and $a_{k,n+1}$ is defined in (4.1.8). From this, we conclude that the numerical integration scheme (4.1.13) is also a numerical scheme for the multi-time scales differential equation (2.1.19).

4.2 Convergence of the Numerical scheme for SFDE

Definition 4.2.1 (Strong Order of Convergence): A time-discredited approximation X^δ of a continuous-time process X, with δ the maximum time increment of the discretization, is said to be of general strong order of convergence γ to X if for any fixed time horizon T it holds true that

$$E\left|X^\delta(T) - X(T)\right| < C\delta^\gamma, \quad \forall\delta < \delta_0 \tag{4.2.1}$$

with $\delta_0 > 0$ and C a constant not depending on δ.

Definition 4.2.2 (Weak Order of Convergence): A time-discredited approximation X^δ is said to converge weakly of order β to X if for any fixed horizon T and any $2(\beta+1)$ continuous differentiable function g of polynomial growth, it holds true that

$$\left|E_g\left(X^\delta(T)\right) - E_g\left(X(T)\right)\right| < C\delta^\beta, \quad \forall\delta < \delta_0 \tag{4.2.2}$$

with $\delta_0 > 0$ and C a constant not depending on δ.

Remark 4.2.3 *Schemes of approximation of some order that strongly converge usually have a higher order of weak convergence. This is the case with the Euler scheme for two time scales dt and $dB(t)$, which is strongly convergent of order $\delta = \frac{1}{2}$ and weakly convergent of order $\beta = 1$ (under some smoothness conditions on the coefficients of the stochastic* *differential equation). While the schemes have their own order of convergence, it is usually the case that, for some actual specifications of the stochastic differential equations, they behave better.*

Theorem 4.2.4 *Under the assumptions of Theorem 2.2.2, a strongly consistent equidistant time discrete approximation $x^h(t)$ (in (4.1.13)) of the solution process $x(t)$ of a 1-dimensional stochastic fractional differential equation (2.1.19) converges strongly to $x(t)$.*

Proof. The proof of this theorem follows from those of 1-dimensional stochastic differential equations (Ref. Kloeden and Platen [41] pp 324-326) and the predictor-corrector method [15, 18] described above. For $0 \leq t \leq T$, we set

$$z(t) = \sup_{0 \leq s \leq t} E\left(\left|x^h_{n_s} - x(s)\right|^2\right) \tag{4.2.3}$$

we obtain

$$
\begin{aligned}
z(t) = \sup_{0\le s\le t} E \Bigg\{ \Bigg| & \sum_{k=0}^{n_s-1} b(t_k, x_k)\Delta t_k + \sum_{k=0}^{n_s-1} \sigma_1(t_k, x_k)\Delta B_k \\
& + \frac{h^\alpha \sigma_2(t_{n_s}, x_{n_s}^P)}{\alpha+1} + \sum_{k=0}^{n_s-1} a_{k,n_s}\sigma_2(t_k, x_k) - \int_0^s b(r, x(r))dr \\
& - \int_0^s \sigma_1(r, x(r))dB(r) - \alpha \int_0^s (s-r)^{\alpha-1}\sigma_2(r, x(r))dr \Bigg|^2 \Bigg\} \\
\le\ & C_0 \sup_{0\le s\le t} E \left(\left| \sum_{k=0}^{n_s-1} b(t_k, x_k)\Delta t_k - \int_0^{t_{n_s}} b(r, x(r))dr \right|^2 \right) \\
& + C_0 \sup_{0\le s\le t} E \left(\left| \sum_{k=0}^{n_s-1} \sigma_1(t_k, x_k)\Delta B_k - \int_0^{t_{n_s}} \sigma_1(r, x(r))dB(r) \right|^2 \right) \\
& + C_0 \sup_{0\le s\le t} E \left(\left| \frac{h^\alpha \sigma_2(t_{n_s}, x_{n_s}^P)}{\alpha+1} + \sum_{k=0}^{n_s-1} a_{k,n_s}\sigma_2(t_k, x_k) - \alpha \int_0^{t_{n_s}} \frac{\sigma_2(r, x(r))}{(t_{n_s}-r)^{1-\alpha}} dr \right|^2 \right) \\
& + C_0 \sup_{0\le s\le t} E \left(\left| \int_{t_{n_s}}^s b(r, x(r))dr + \int_{t_{n_s}}^s \sigma_1(r, x(r))dB(r) \right|^2 \right) \\
& + \alpha^2 C_0 \sup_{0\le s\le t} E \left(\left| \int_0^{t_{n_s}} \frac{\sigma_2(r, x(r))}{(t_{n_s}-r)^{1-\alpha}} dr - \int_0^{t_{n_s}} \frac{\sigma_2(r, x(r))}{(s-r)^{1-\alpha}} dr \right|^2 \right) \\
& + \alpha^2 C_0 \sup_{0\le s\le t} E \left(\left| \int_{t_{n_s}}^s (s-r)^{\alpha-1}\sigma_2(r, x(r))dr \right|^2 \right)
\end{aligned} \tag{4.2.4}
$$

We need to find estimate for each term in the right hand side of (4.2.4).

Using the Euler-Maruyama approximation scheme, the combined estimates for the 1st and 2nd terms in (4.2.4) is

$$
\begin{aligned}
C_0 \sup_{0\le s\le t} E & \left(\left| \sum_{k=0}^{n_s-1} b(t_k, x_k)\Delta t_k - \int_0^{t_{n_s}} b(r, x(r))dr \right|^2 \right) \\
& + C_0 \sup_{0\le s\le t} E \left(\left| \sum_{k=0}^{n_s-1} \sigma_1(t_k, x_k)\Delta B_k - \int_0^{t_{n_s}} \sigma_1(r, x(r))dB(r) \right|^2 \right) \le C_1 h,
\end{aligned} \tag{4.2.5}
$$

for some positive constant C_1 depending on b, σ_1, and T.

By applying the fact that the predictor-corrector method converges with order $1+\alpha$, the

3rd term in (4.2.4) has estimate

$$C_0 \sup_{0\le s\le t} E\left(\left|\frac{h^\alpha \sigma_2(t_{n_s}, x^P_{n_s})}{\alpha+1} + \sum_{k=0}^{n_s-1} a_{k,n_s}\sigma_2(t_k, x_k) - \alpha\int_0^{t_{n_s}} \frac{\sigma_2(r, x(r))}{(t_{n_s}-r)^{1-\alpha}}dr\right|^2\right)$$
$$\le\ C_2 h^{2(1+\alpha)}, \tag{4.2.6}$$

for some positive constant C_2 depending on σ_2 and T.

For the fourth term in (4.2.4), by using the Schwartz inequality, Itô isometry [4], the growth condition and the fact that $\sup_{0\le s\le T} E|x(s)|^2 < \infty$ (Theorem 4.1 in [64]), we have

$$\begin{aligned}
&\sup_{0\le s\le t} E\left(\left|\int_{t_{n_s}}^s b(r, x(r))dr\ +\ \int_{t_{n_s}}^s \sigma_1(r, x(r))dB(r)\right|^2\right)\\
&\le 2(s-t_{n_s})\int_{t_{n_s}}^s K^2(1+\sup_{0\le u\le T} E|x(u)|^2)dr + 2\int_{t_{n_s}}^s K^2(1+\sup_{0\le u\le T} E|x(u)|^2)dr\\
&\le 2(h+1)hK^2(1+\sup_{0\le u\le T} E|x(u)|^2)\\
&\le C_3 h, \qquad (4.2.7)
\end{aligned}$$

for some positive constant C_3 depending on b, σ_1, and T.

Now, let's consider the fifth term in (4.2.4):

$$\begin{aligned}
&\int_0^{t_{n_s}} (t_{n_s}-r)^{\alpha-1}\sigma_2(r, x(r))dr - \int_0^{t_{n_s}} (s-r)^{\alpha-1}\sigma_2(r, x(r))dr\\
&\quad = \int_0^{t_{n_s}} \frac{(s-r)^{1-\alpha} - (t_{n_s}-r)^{1-\alpha}}{(t_{n_s}-r)^{1-\alpha}(s-r)^{1-\alpha}}\sigma_2(r, x(r))dr\\
&\quad \le \int_0^{t_{n_s}} \frac{(s-r)^{1-\alpha} - (t_{n_s}-r)^{1-\alpha}}{(t_{n_s}-r)^{2(1-\alpha)}}\sigma_2(r, x(r))dr, \ \text{ since } t_{n_s} < s \text{ and } \alpha < 1. \qquad (4.2.8)
\end{aligned}$$

The function $f(r) = (s-r)^{1-\alpha} - (t_{n_s}-r)^{1-\alpha}$ on the interval $[0, t_{n_s}]$, is differentiable on $(0, t_{n_s})$ and $f'(r) = (1-\alpha)[(t_{n_s}-r)^{-\alpha} - (s-r)^{-\alpha} < 0$. Thus, f is a nonnegative and decreasing function on $[0, t_{n_s}]$ with maximum value occurring at 0,

$$f(0)\ =\ s^{1-\alpha} - t_{n_s}^{1-\alpha}\ \le\ t_{n_s+1}^{1-\alpha} - t_{n_s}^{1-\alpha}\ =\ h^{1-\alpha}[(n_s+1)^{1-\alpha} - (n_s)^{1-\alpha}]$$

Since $1-\alpha > 0$ and the function $g(x) = (x+1)^{1-\alpha} - x^{1-\alpha}$ is a nonnegative and decreasing function on $[0, \infty)$, we have

$$f(0) \; \leq \; h^{1-\alpha}[(n_s+1)^{1-\alpha} - (n_s)^{1-\alpha}] \; \leq \; h^{1-\alpha} g(0) = h^{1-\alpha} \tag{4.2.9}$$

On the other hand we have

$$\int_0^{t_{n_s}} \frac{1}{(t_{n_s}-r)^{2(1-\alpha)}} dr = \frac{-1}{2\alpha-1}(t_{n_s}-r)^{2\alpha-1}\Big|_0^{t_{n_s}} = \frac{t_{n_s}^{2\alpha-1}}{2\alpha-1} \leq \frac{T^{2\alpha-1}}{2\alpha-1}. \tag{4.2.10}$$

Now by using the linear growth condition and the fact that $\sup\limits_{0\leq s\leq T} E|x(s)|^2 < \infty$, (see Theorem 4.1 [64]), (4.2.8) and (4.2.10), the fifth term in (4.2.4) reduces to

$$\alpha^2 \sup_{0\leq s\leq t} E\left(\left|\int_0^{t_{n_s}} \frac{\sigma_2(r,x(r))}{(t_{n_s}-r)^{1-\alpha}} dr - \int_0^{t_{n_s}} \frac{\sigma_2(r,x(r))}{(s-r)^{1-\alpha}} dr\right|^2\right) \leq C_4 h^{1-\alpha}, \tag{4.2.11}$$

where C_4 is a positive constant depending on σ_2 and T.

Finally, for the last term in (4.2.4), an estimate is obtained by applying the Schwartz inequality, the linear growth condition, and the fact that $\sup\limits_{0\leq s\leq T} E|x(s)|^2 < \infty,\ t_0 \leq t \leq T$. Thus, we have

$$\begin{aligned}
&\alpha^2 C_0 \sup_{0\leq s\leq t} E\left(\left|\int_{t_{n_s}}^{s} (s-r)^{\alpha-1}\sigma_2(r,x(r))dr\right|^2\right)\\
&\qquad \leq C_5 \int_{t_{n_s}}^{s} (s-r)^{2(\alpha-1)} dr (s-t_{n_s})(1+\sup_{0\leq s\leq T} E|x(s)|^2))\\
&\qquad \leq C_5 \frac{-1}{2\alpha-1}(s-r)^{2\alpha-1}\Big|_{t_{n_s}}^{s} (s-t_{n_s})(1+\sup_{0\leq s\leq T} E(|x(s)|^2))\\
&\qquad \leq C_5 \frac{(s-t_{n_s})^{2\alpha-1}}{2\alpha-1}(s-t_{n_s})(1+\sup_{0\leq s\leq T} E(|x(s)|^2))\\
&\qquad \leq C_5 h^{2\alpha},
\end{aligned} \tag{4.2.12}$$

where C_5 is a some positive constant depending on σ_2 and T.

Therefore, from (4.2.5), (4.2.6), (4.2.7), (4.2.11), and (4.2.12), an estimate for $z(t)$ in (4.2.4) is given by

$$z(t) \leq C_1 h + C_2 h^{2(1+\alpha)} + C_3 h + C_4 h^{1-\alpha} + C_5 h^{2\alpha} \leq C h^{1-\alpha}. \tag{4.2.13}$$

Finally, using the Lyapunov inequality we can then conclude that

$$E\left|x^h(t) - x(t)\right| \leq \sqrt{z(t)} \leq C h^{(1-\alpha)/2}. \tag{4.2.14}$$

where C is a positive constant depending on b, σ_1, σ_2, and T.

■

Remark 4.2.5 *In the absence of one or two attributes in a dynamical systems modeled by the three-time scales fractional stochastic differential equations* (2.1.19)*, we have six possible scenarios as follows:*

1. *Under the ordinary time scale "t" alone, i.e. in the absence of external environmental perturbations ($\sigma_1(\cdot,\cdot) \equiv 0$) and hereditary effects ($\sigma_2(\cdot,\cdot) \equiv 0$), the FSDE IVP* (2.1.19) *reduces to the deterministic IVP* (2.1.24) *and the scheme described in* (4.1.13) *is just the Euler scheme*

$$x_{n+1} = x_n + b(t_n, x_n)(t_{n+1} - t_n), \quad n = 0, 1, 2, 3, \ldots, N-1. \tag{4.2.15}$$

2. *In the absence of time scale "t" (i.e. $b(\cdot,\cdot) \equiv 0$) and hereditary effects ($\sigma_2(\cdot,\cdot) \equiv 0$),* (2.1.19) *is to the Itô-Doob IVP $dx = \sigma_1(t,x)dB(t)$ and* (4.1.13) *is Maruyama scheme*

$$x_{n+1} = x_n + \sigma_1(t_k, x_k)(B(t_{n+1}) - B(t_n)), \quad n = 0, 1, 2, 3, \ldots, N-1. \tag{4.2.16}$$

3. *When both $b(\cdot,\cdot) \equiv 0$ and $\sigma_1(\cdot,\cdot) \equiv 0$ (no external environmental perturbations), the FSDE IVP* (2.1.19) *is the deterministic fractional differential equation $dx = \sigma_2(t,x)(dt)^\alpha$. In this case,* (4.1.13) *is one-step predictor corrector algorithm for frac-*

tional differential equations,

$$x_{n+1} \;=\; x_0 \;+\; \sum_{k=0}^{n} a_{k,n+1}\sigma_2(t_k, x_k) \;+\; \frac{h^\alpha}{\alpha+1}\sigma_2(t_{n+1}, x_{n+1}^P), \qquad (4.2.17)$$

where $n = 0, 1, 2, \ldots, N-1$, $a_{k,n+1}$ *is defined in* (4.1.8) *and* x_{n+1}^P *in* (4.1.9).

4. *In the absence of time-scale* dt^α *alone, the FSDE IVP* (2.1.19) *reduces to the Itô-Doob IVP* (2.1.21) *and the scheme described in* (4.1.13) *is the Euler-Maruyama scheme*

$$x_{n+1} = x_n + b(t_n, x_n)(t_{n+1} - t_n) + \sigma_1(t_k, x_k)(B(t_{n+1}) - B(t_n)), \qquad (4.2.18)$$

where $n = 0, 1, 2, 3, \ldots, N-1$.

5. *If only* $b(\cdot,\cdot) \equiv 0$, *the FSDE IVP* (2.1.19) *becomes*

$$dx \;=\; \sigma_1(t,x)dB_t \;+\; \sigma_2(t,x)(dt)^\alpha, \quad x(t_0) = x_0, \qquad (4.2.19)$$

and the numerical scheme (4.1.13) *reduces to*

$$\begin{aligned} x_{n+1} \;&=\; x_0 + \sum_{k=0}^{n}\sigma_1(t_k, x_k)\Delta B_k \;+\; \sum_{k=0}^{n} a_{k,n+1}\sigma_2(t_k, x_k) \\ &\quad + \frac{h^\alpha}{\alpha+1}\sigma_2(t_{n+1}, x_{n+1}^P), \quad n = 0, 1, 2, \ldots, N-1, \end{aligned} \qquad (4.2.20)$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$, *and* $a_{k,n+1}$ *is defined in* (4.1.8).

6. *Finally, for a system under the influence of two the time-scales* t *and* t^α, *the FSDE IVP* (2.1.19) *becomes*

$$dx \;=\; b(t,x)dt \;+\; \sigma_2(t,x)(dt)^\alpha, \quad x(t_0) = x_0, \qquad (4.2.21)$$

and from (4.1.13)*, a numerical scheme for this IVP is given by*

$$\begin{aligned} x_{n+1} &= x_0 + \sum_{k=0}^{n} b(t_k, x_k)\Delta t_k \; + \; \sum_{k=0}^{n} a_{k,n+1}\sigma_2(t_k, x_k) \\ &\quad + \frac{h^\alpha}{\alpha+1}\sigma_2(t_{n+1}, x^P_{n+1}), \quad n = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{4.2.22}$$

where $\Delta t_k = t_{k+1} - t_k,$, *and* $a_{k,n+1}$ *is defined in* (4.1.8)*.*

in the next chapter, we utilize the scheme (4.1.13) to simulate sample paths of IVP (2.1.19) in the presence of external environmental perturbations (i.e. $\sigma_1(\cdot,\cdot) \neq 0$) and graphs of the solution of (2.1.19) when $\sigma_1(\cdot,\cdot) \equiv 0$).

5 Sample Paths Simulations Results

In this chapter, we apply the numerical scheme (4.1.13) to approximate the solution of the mathematical model for epidemiological processes studied in [64] to simulate sample paths of the the associated solution processes. We recall that the mathematical model for the epidemic process of communicable diseases studied in section3.4 is given by

$$dI = \theta(\widetilde{N} - I)Idt + \sigma_1 IdB(t) + \sigma_2 I(dt)^\alpha, \quad I(0) = I_0 \tag{5.0.1}$$

where $\theta > 0$, $\sigma_1 \neq 0$, $\sigma_2 \neq 0$. The rate functions $b(t, I) = \theta(\widetilde{N} - I)I$ (deterministic), $\sigma_1(t, I) = \sigma_1 I$ (stochastic) for some $\sigma_1 \neq 0$, and $\sigma_2(t, I) = \sigma_2 I$ (fractional) are smooth functions which guarantees the existence of solution of (5.0.1).

The general solution of (5.0.1) [64] is given by

$$I(t) = \left[\Phi(t)C_0 + \theta \int_{t_0}^{t} \Phi(t, s)ds\right]^{-1}, \tag{5.0.2}$$

where

$$\begin{aligned}\Phi(t) &= \exp\left[\int_{t_0}^{t}\left((\theta\widetilde{N} - \sigma_1^2) - \frac{1}{2}\sigma_1^2\right)ds + \int_{t_0}^{t}\sigma_1 dB(s)\right] E_\alpha\left[\Gamma(1+\alpha)^2(I^\alpha\sigma_2)(t)\right] \\ &= \exp\left[\left(\theta\widetilde{N} - \frac{3}{2}\sigma_1^2\right)t + \sigma_1 B(t)\right] E_\alpha\left[\sigma_2\Gamma(1+\alpha)t^\alpha\right];\end{aligned}$$

and $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$ is the fundamental solution.

5.1 Simulations study in the absence of environmental perturbations and hereditary effects

Let us consider the case where $\sigma_i \equiv 0$, $i = 1, 2$, that is, there is no environmental perturbations and hereditary effects influencing the system. Then equation (5.0.1) reduces to

$$dI = \theta(\widetilde{N} - I)Idt, \quad I(0) = I_0 \tag{5.1.1}$$

The exact solution of this equation is given by

$$I(t) = \frac{\widetilde{N} I_0}{I_0 + (\widetilde{N} - I_0)e^{-\theta \widetilde{N}(t-t_0)}} \tag{5.1.2}$$

The numerical scheme for (5.1.1) is given in (4.2.15). In fact it is as:

$$I_{n+1} = I_0 + \sum_{k=0}^{n} b(t_k, I_k)\Delta t_k, \quad n = 0, 1, 2, \ldots, N-1. \tag{5.1.3}$$

Moreover,

$$I_{n+1} = I_n + b(t_n, I_n)(t_{n+1} - t_n), \quad n = 0, 1, 2, \ldots, N-1. \tag{5.1.4}$$

All our subsequent simulation results are based on 1 unit=1000 or 100,000 or 1000,000.

If $I_0 < \widetilde{N}$, the graph of $I(t)$ in (5.1.2) has an S-shape with an inflection point occurring at $t = t_0 + \frac{\ln(\widetilde{N}-I_0)-\ln(I_0)}{\theta\widetilde{N}}$ (see Figure 5.1).

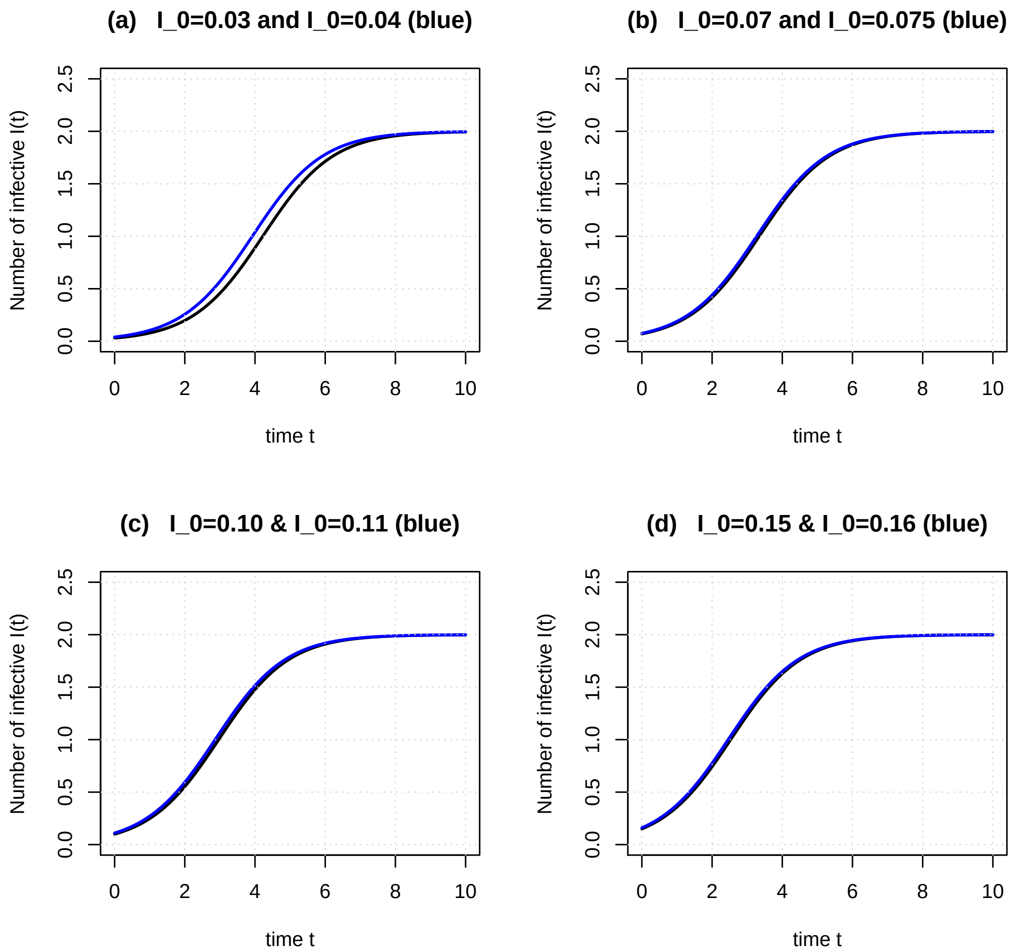

Figure 5.1: Graphs of solution $I(t)$ in (5.1.2) when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0$, $\sigma_2 = 0$ varying conditions: $I_0 = 0.03$ and $I_0 = 0.04$ (a), $I_0 = 0.07$ and $I_0 = 0.075$ (b), $I_0 = 0.10$ and $I_0 = 0.11$ (c), and $I_0 = 0.15$ and $I_0 = 0.16$ (d).

Observations: (1) The number of infective $I(t)$ grows faster (on an interval $[t_0, t_1]$ for some $t_1 < 10$) from the initial value I_0 and then reaches the saturation level of (population size of the species) $\widetilde{N} = 2$. This shows that the spread of the epidemic reaches the entire community rapidly (chaotic) if not contained early.
(2) The solution $I(t)$ of (5.1.2) is asymptotically stable as all these graphs show no distinction between two graphs with different initial conditions as the time increases. This indicates the stable and tight community in that what happens to one individual easily affects the entire ecosystem and so is the spread of communicable diseases.

If $I_0 > \widetilde{N}$, the graph of $I(t)$ falls sharply from left to right and approaches the saturation level $\widetilde{N}$, (see Figure 5.2).

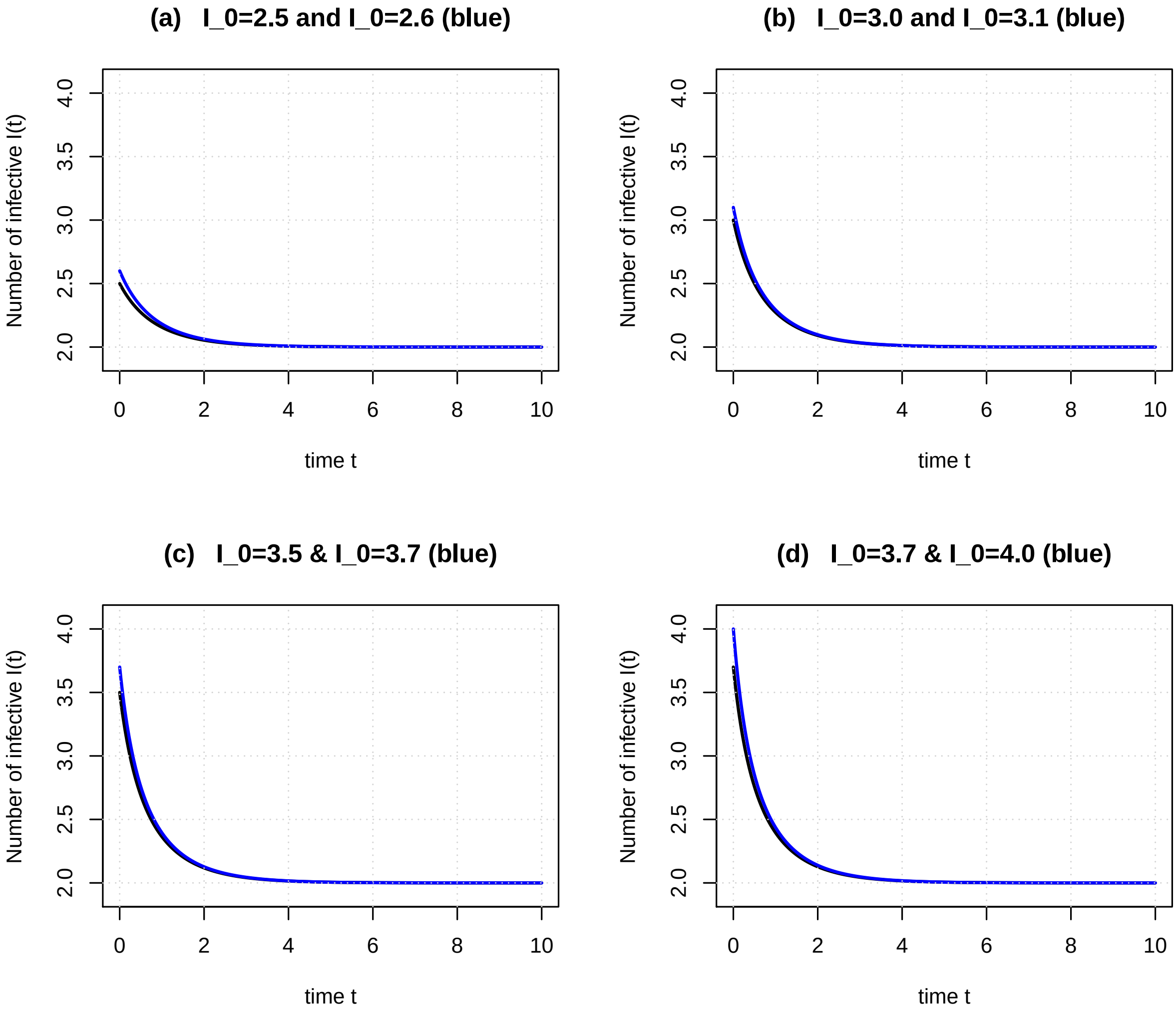

Figure 5.2: Graphs of solution $I(t)$ in (5.1.2) when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0$, $\sigma_2 = 0$ with varying conditions: $I_0 = 2.5$ and $I_0 = 2.6$ (a), $I_0 = 3.0$ and $I_0 = 3.1$ (b), $I_0 = 3.5$ and $I_0 = 3.7$ (c), and $I_0 = 4.1$ and $I_0 = 4.2$ (d).

Observations: When the initial value is larger than the critical level ($I_0 = 2$ in this case), we observe that it is stationary which is as expected.

5.2 Simulations study in the absence of hereditary effects (only)

Let us consider the case where $\sigma_2 \equiv 0$, that is, there is no hereditary effects influencing the system. Then equation (5.0.1) reduces to

$$dI = \theta(\widetilde{N} - I)Idt + \sigma_1 IdB(t), \quad I(0) = I_0 \tag{5.2.1}$$

As shown in [53, 64], the general solution of (5.0.1) is given by

$$I(t) = \left[\Phi(t)C_0 + \theta \int_{t_0}^{t} \Phi(t, s)ds\right]^{-1}, \tag{5.2.2}$$

where

$$\begin{aligned}
\Phi(t) &= \exp\left[\int_{t_0}^{t} \left((\theta\widetilde{N} - \sigma_1^2) - \frac{1}{2}\sigma_1^2\right) ds + \int_{t_0}^{t} \sigma_1 dB(s)\right] \\
&= \exp\left[\left(\theta\widetilde{N} - \frac{3}{2}\sigma_1^2\right) t + \sigma_1 B(t)\right],
\end{aligned}$$

and $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$ is the fundamental solution of (5.2.1).

To simulate the sample paths of $I(t)$, we utilize the numerical scheme described in (4.2.18), which is just the Euler-Maruyama scheme for stochastic differential equations. Thus we have, for $n = 0, 1, 2, \ldots, N-1$

$$I_{n+1} = I_0 + \sum_{k=0}^{n} b(t_k, I_k)\Delta t_k + \sum_{k=0}^{n} \sigma_1(t_k, I_k)\Delta B_k, \tag{5.2.3}$$

or simply

$$I_{n+1} = I_n + b(t_n, I_n)(t_{n+1} - t_n) + \sigma_1(t_n, I_n)(B(t_{n+1}) - B(t_n)). \tag{5.2.4}$$

In the remaining part of this section we'll only discuss the case $I_0 < \widetilde{N}$.

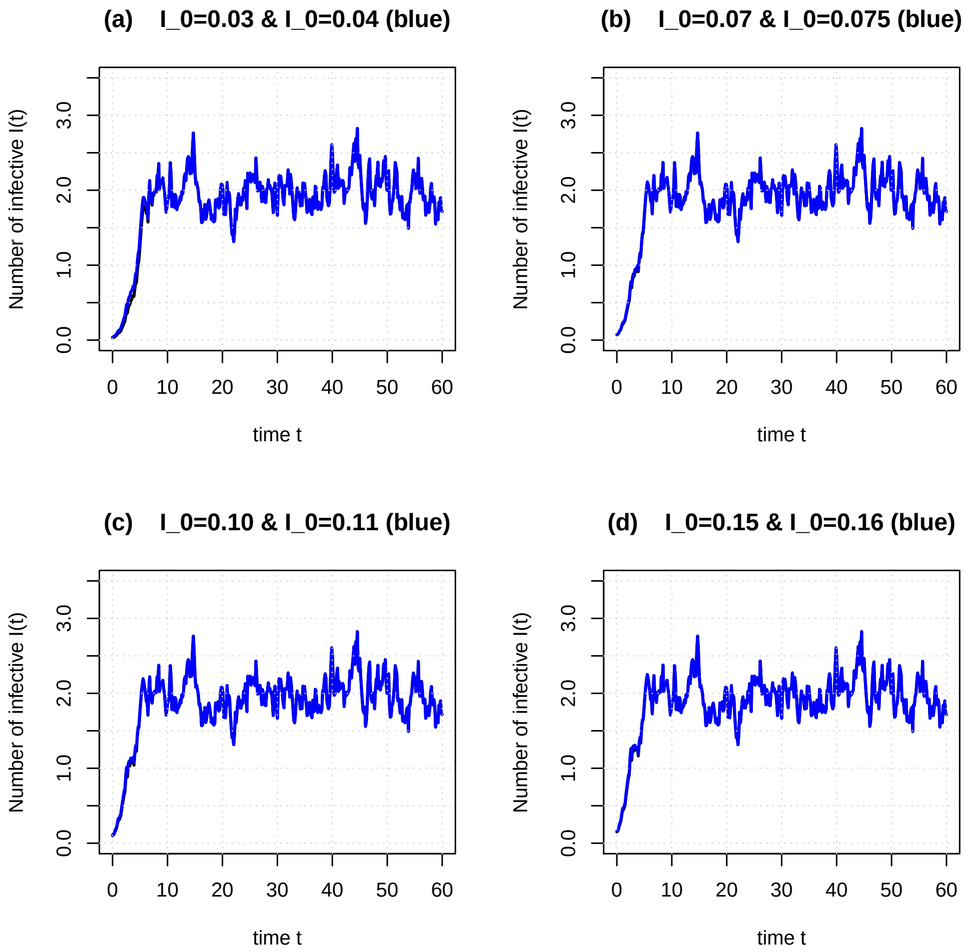

Figure 5.3: Sample paths of solution $I(t)$ of (5.2.1) when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0.2$ with varying conditions: $I_0 = 0.03$ and $I_0 = 0.04$ (a), $I_0 = 0.07$ and $I_0 = 0.075$ (b), $I_0 = 0.1$ and $I_0 = 0.11$ (c), and $I_0 = 0.15$ and $I_0 = 0.16$ (d).

Observations: (1) We note that small values of σ_1 (between 0 and around 0.3), $I(t)$ fluctuates about the curve of the deterministic case discussed above in Figure 5.1 (the expected steady state $\widetilde{N} = 2$). This means that, the environmental perturbations can be attributed to the effects of the influx of infective individuals from others neighboring communities.

(2) Again, we see that the solution $I(t)$ of (5.2.2) is asymptotically stable as all these graphs show no distinction between two graphs with different initial conditions as the time increases.

Here we vary the diffusion coefficient σ_1 in (5.2.1)

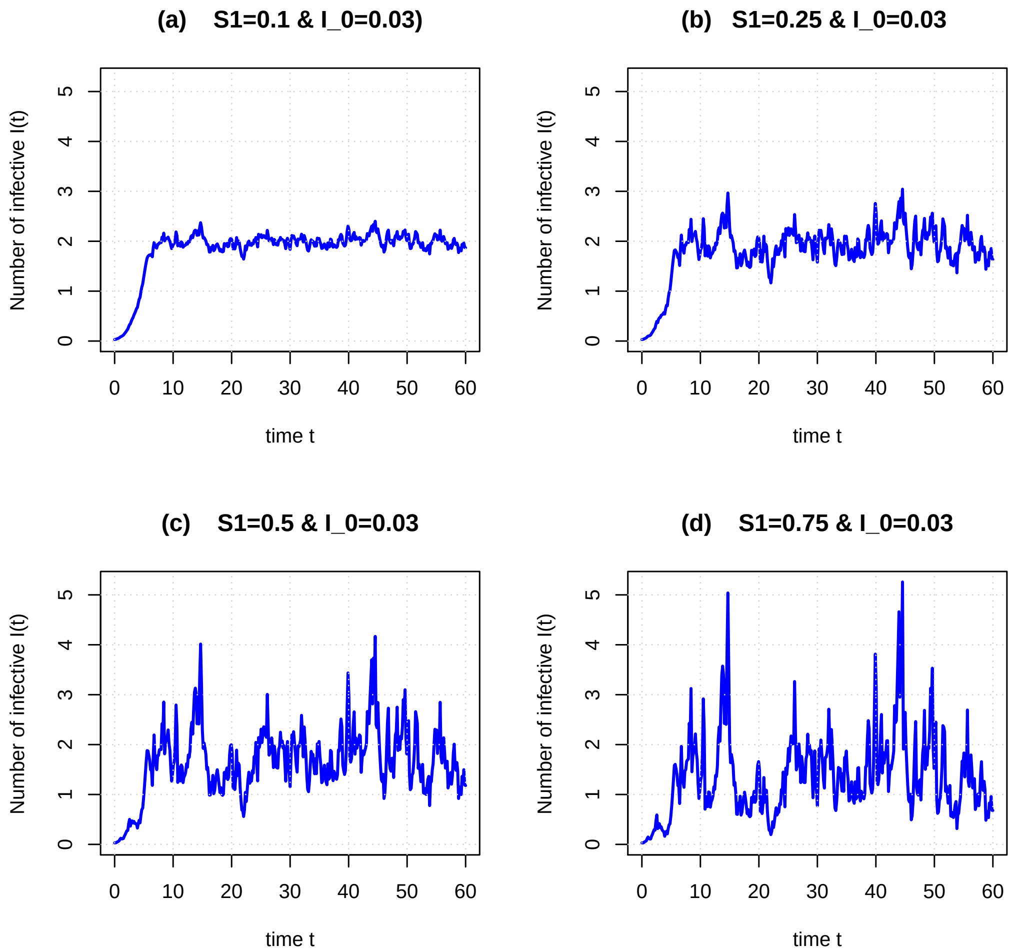

Figure 5.4: Sample paths for the solution $I(t)$ when $\theta = 0.5$, $\widetilde{N} = 2$ with varying conditions: $I_0 = 0.03$ and $I_0 = 0.04$. $\sigma_1 = 0.1$ (a), $\sigma_1 = 0.25$ (b), $\sigma_1 = 0.5$ (c), $\sigma_1 = 0.75$ (d)

Observations: We remak that as σ_1 increases, the fluctuations of $I(t)$ increases as well and still tend to average about the curve of the deterministic case discussed above. This means that the higher the rate of the environmental perturbations, the lesser the change to epidemic. The chance of controlling the epidemic diminishes.

(2) Again, we see that the solution $I(t)$ of (5.2.2) is asymptotically stable as all these graphs show no distinction between two graphs with different initial conditions as the time increases.

5.3 Simulations study in the absence of environmental perturbations (only)

By setting $\sigma_1 \equiv 0$, that is there is no environmental perturbations affecting the ecosystem. Then equation (5.0.1) reduces to

$$dI = \theta(\widetilde{N} - I)Idt + \sigma_2 I(dt)^\alpha, \quad I(0) = I_0 \tag{5.3.1}$$

The general solution of (5.3.1) is just simplified version of the solution of the three time-scale presented in [64],

$$I(t) = \left[\Phi(t)C_0 + \theta \int_{t_0}^{t} \Phi(t,s)ds\right]^{-1}, \tag{5.3.2}$$

where

$$\Phi(t) = \exp\left(\theta \widetilde{N} t\right) E_\alpha \left[\sigma_2 \Gamma(1+\alpha) t^\alpha\right]$$

and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ is the fundamental solution of (5.3.1).

To graph $I(t)$, we utilize the numerical scheme described in (**??**) as follows: for $n = 0, 1, 2, \ldots, N-1$

$$x_{n+1} = x_0 + \sum_{k=0}^{n} b(t_k, x_k)\Delta t_k + \sum_{k=0}^{n} a_{k,n+1}\sigma_2(t_k, x_k) + \frac{h^\alpha}{\alpha+1}\sigma_2(t_{n+1}, x_{n+1}^P), \tag{5.3.3}$$

where $\Delta t_k = t_{k+1} - t_k$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, and $a_{k,n+1}$ and x_{n+1}^P are defined in (4.1.8) and (4.1.9), respectively.

Let's let α varies.

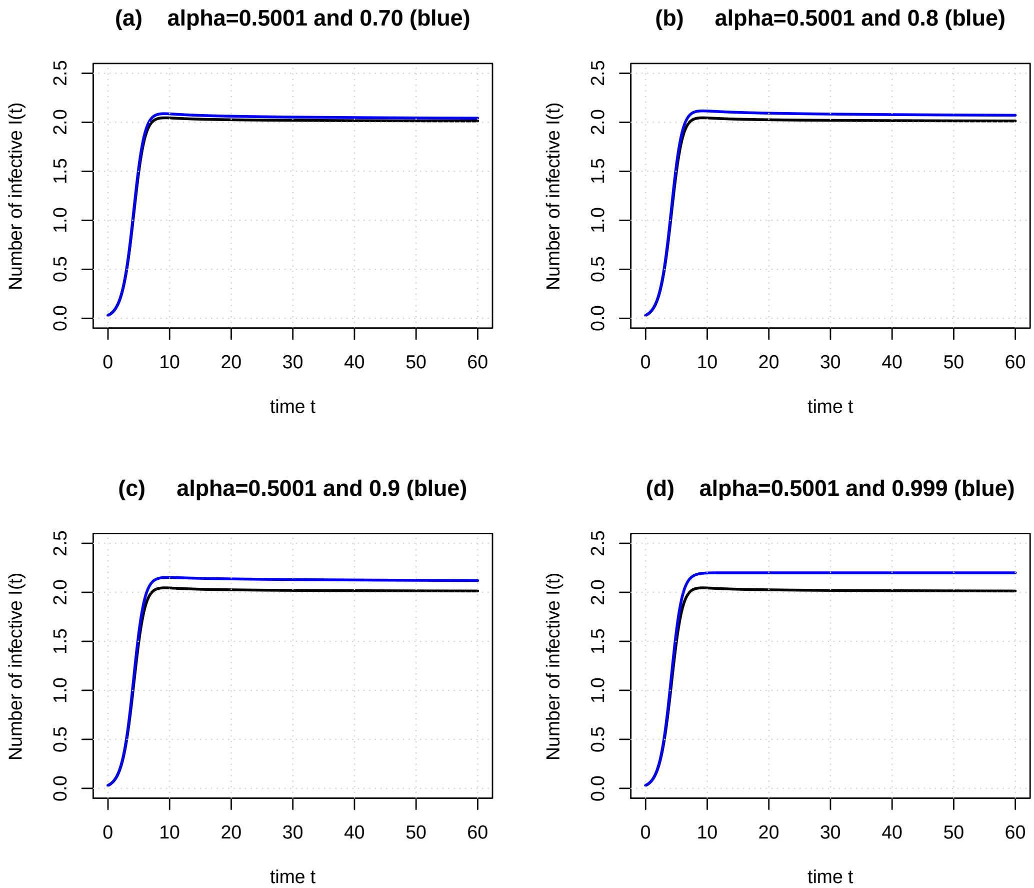

Figure 5.5: Graphs for the solution $I(t)$ when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_2 = 0.1$ with initial value $I_0 = 0.03$. $\alpha = 0.5001$ and $\alpha = 0.7$ (a), $\alpha = 0.5001$ and $\alpha = 0.8$ (b), $\alpha = 0.5001$ and $\alpha = 0.9$ (c), and $\alpha = 0.5001$ and $\alpha = 0.999$ (d).

Observations: We observe that as α increases $(0.5 < \alpha < 1)$, the graph of $I(t)$ maintains an S-shape as in the deterministic $dI = \theta(\widetilde{N} - I)Idt$ discussed in subsection 5.1 above. The only difference is that here the graph of $I(t)$ grows rapidly with time and crosses the line $I = 2$ (saturation level), then reaches its maximum before starts decreasing slowly toward the saturation level. The larger the value of α, the higher the maximum value attained by $I(t)$ and the slower it decreases toward the saturation level. This might be explained as follows: at the time the epidemic turns chaotic, it happens that there is also a influx of infected individuals from neighboring communities, and then, part of the infected population leave the overwhelmed community of die.

5.4 Simulations study for solution of (5.0.1)

From the three time scales in equation (5.0.1), using the numerical scheme described in (4.1.13), some of the sample paths of the solution process $I(t)$ are presented here.

Observations: We note that the sample paths of $I(t)$ fluctuates about the curve of

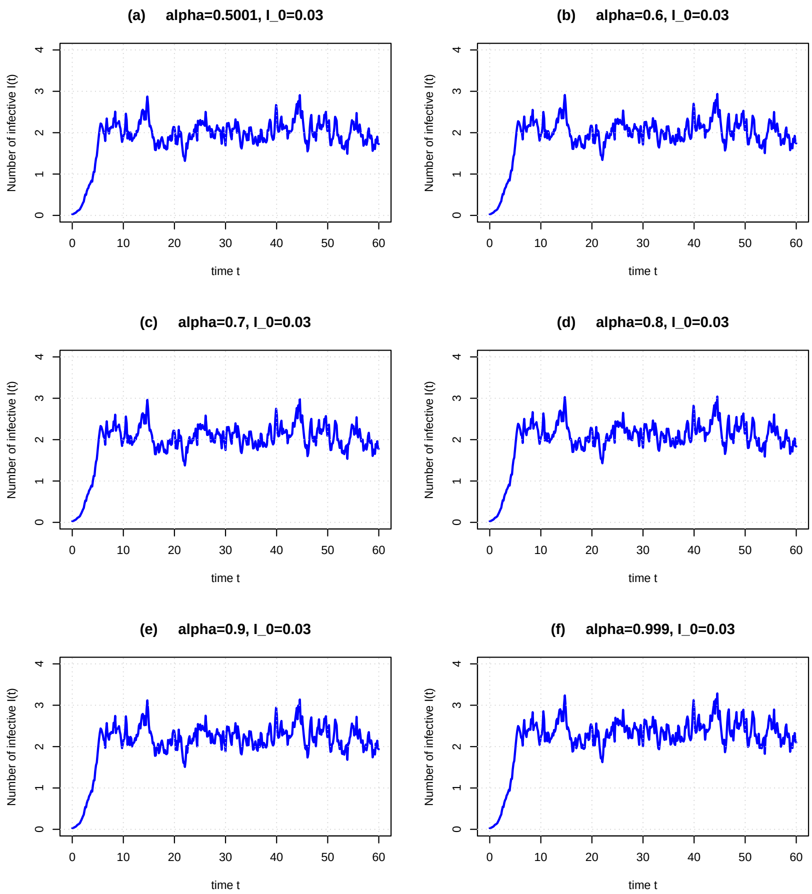

Figure 5.6: Sample paths of the solution $I(t)$ with $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$ with initial value $I_0 = 0.03$. $\alpha = 0.5001$ (a), $\alpha = 0.6$ (b), $\alpha = 0.7$ (c), $\alpha = 0.8$ (d), $\alpha = 0.9$ (e), and $\alpha = 0.999$ (f).

the two-time scale deterministic fractional differential $dI = \theta(\widetilde{N} - I)Idt + \sigma_2(t, I)(dt)^{\alpha}$ discussed in subsection 5.3 above. This means that the expected impact of the epidemic on the community is as explained in the subsection 5.3.

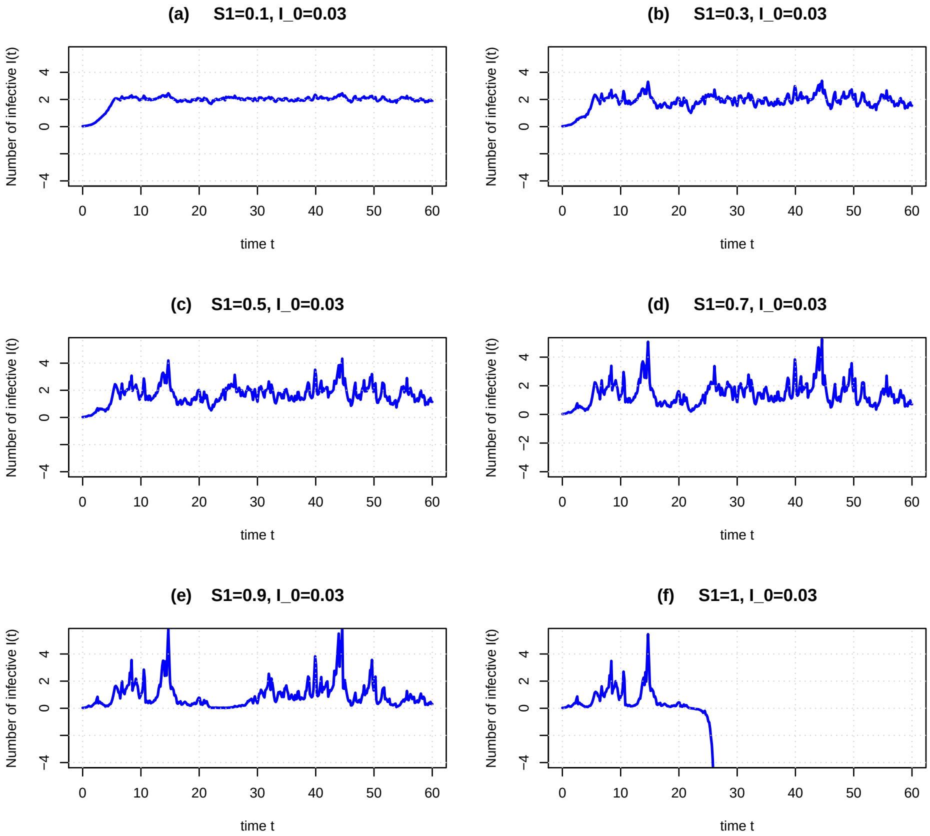

Figure 5.7: Sample paths of the solution $I(t)$ with $\theta = 0.5$, $\widetilde{N} = 2$, $\alpha = 0.5001$, $\sigma_2 = 0.1$ with initial value $I_0 = 0.03$. $\sigma_1 = 0.1$ (a), $\sigma_1 = 0.1$ (b), $\alpha = 0.7$ (c), $\alpha = 0.8$ (d), $\alpha = 0.9$ (e), and $\alpha = 0.999$ (f).

Observations: The sample paths for $I(t)$ show that the larger the value of σ_1, the highly unstable the system becomes despite the low value of σ_2.

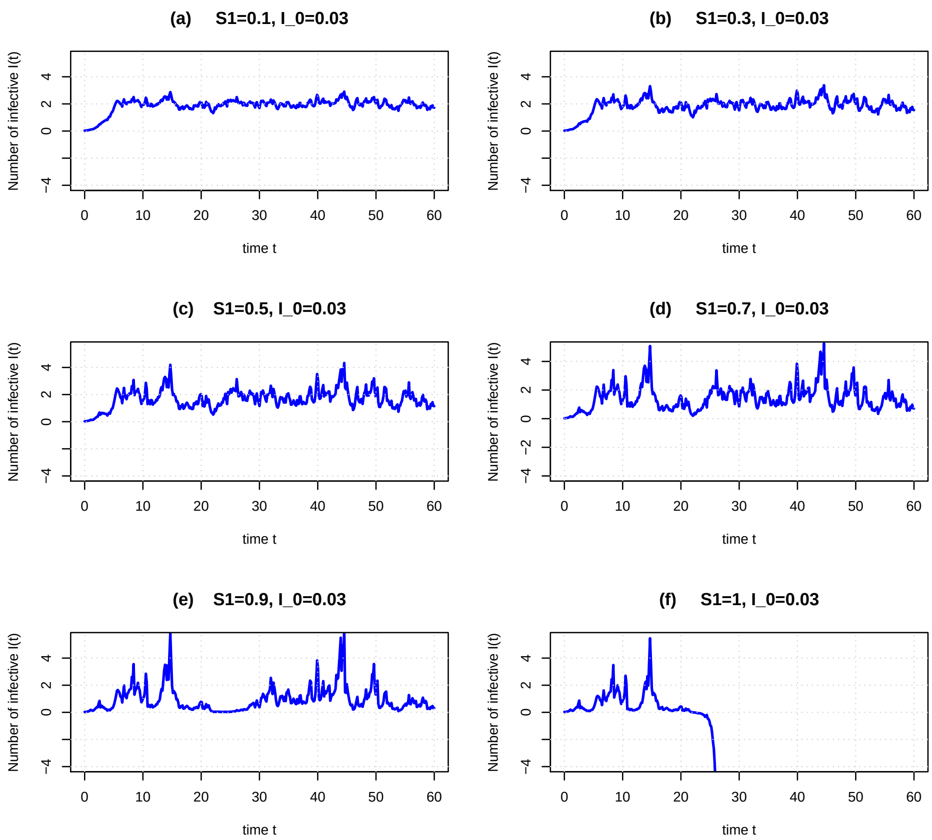

Figure 5.8: Sample paths of the solution $I(t)$ with $\theta = 0.5$, $\widetilde{N} = 2$, $\alpha = 0.5001$, $\sigma_2 = 0.2$ with initial value $I_0 = 0.03$. $\sigma_1 = 0.1$ (a), $\sigma_1 = 0.3$ (b), $\sigma_1 = 0.5$ (c), $\sigma_1 = 0.7$ (d), $\sigma_1 = 0.9$ (e), and $\sigma_1 = 1$ (f).

Observations: We note that the epidemic goes out of control with a 0.1 increase in σ_2 when the value of α is close to 1.

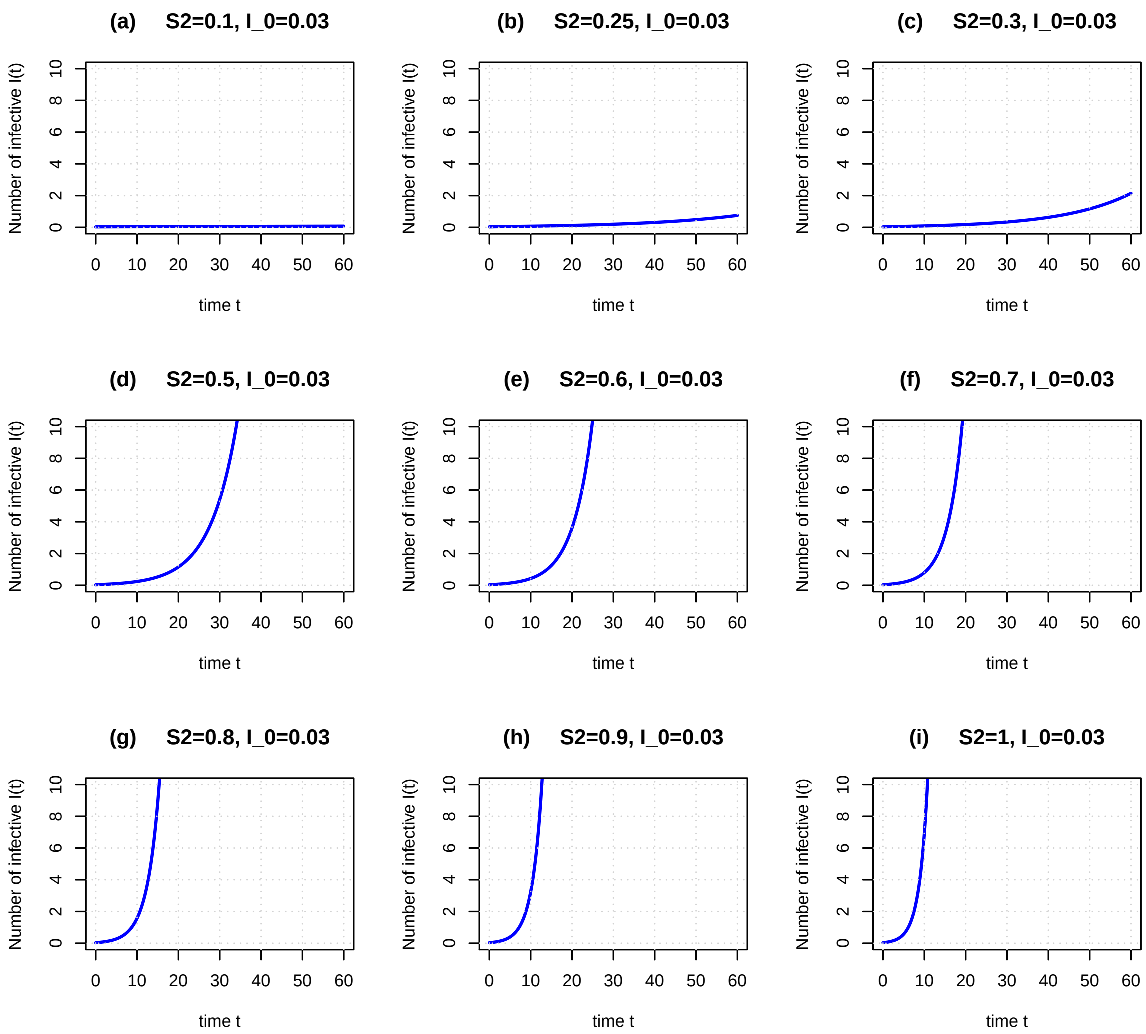

Figure 5.9: Graphs of the solution $I(t)$ with $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0$, the initial conditions is $I_0 = 0.03$ and $\alpha = 0.5001$. $\sigma_2 = 0.1$ (a), $\sigma_2 = 0.2$ (b), $\sigma_2 = 0.3$ (c), $\sigma_2 = 0.5$ (d), $\sigma_2 = 0.6$ (e), and $\sigma_2 = 0.7$ (f), $\sigma_2 = 0.8$ (g), $\sigma_2 = 0.9$ (h), and $\sigma_2 = 1$ (i).

Observations: From these graphs of $I(t)$, we see that for fixed value of $\sigma_2 = 0.2$, the larger the value of α, faster the average growth of the $I(t)$.

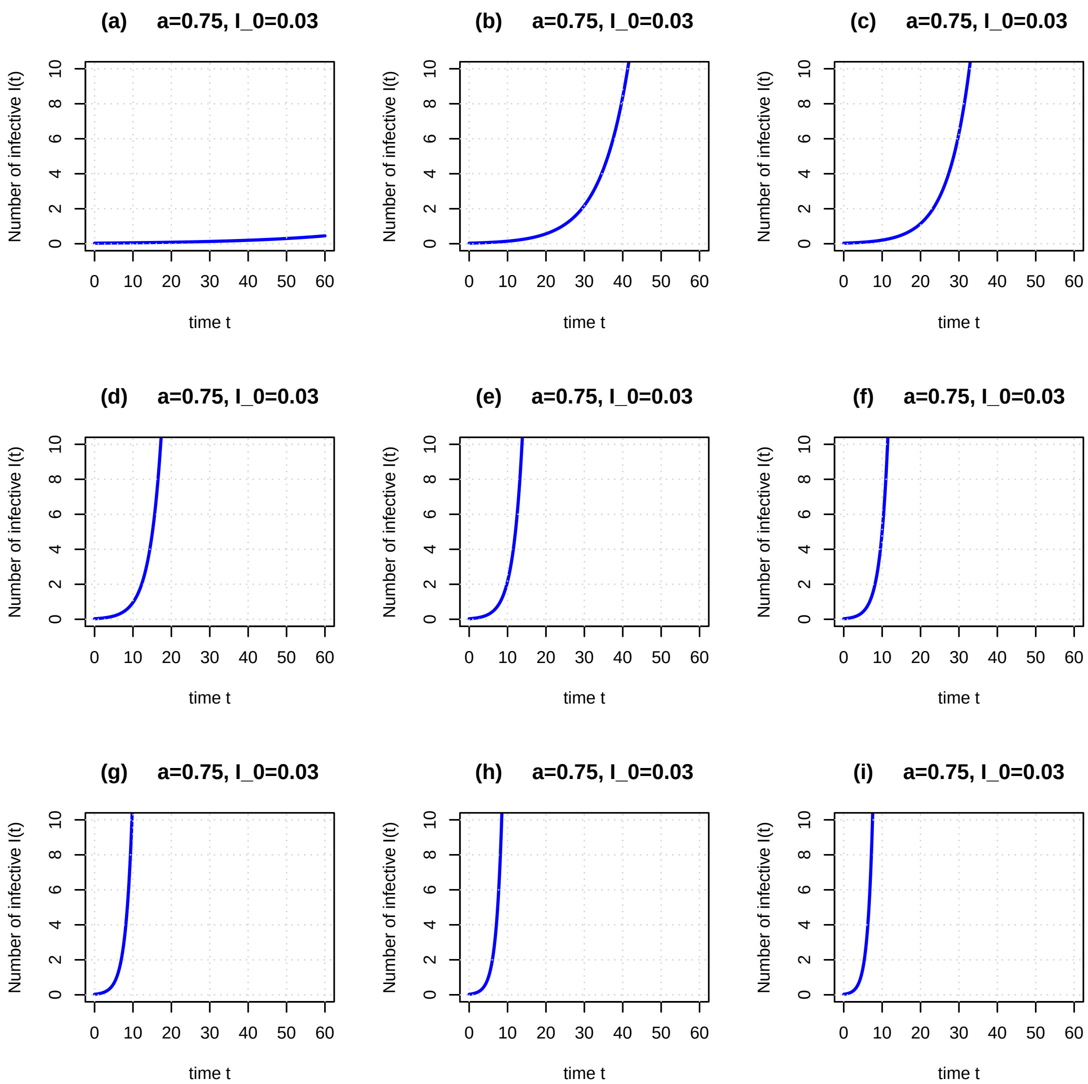

Figure 5.10: Graphs of the solution $I(t)$ with $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0$ with initial value $I_0 = 0.03$ and $\alpha = 0.75$. $\sigma_2 = 0.1$ (a), $\sigma_2 = 0.2$ (b), $\sigma_2 = 0.3$ (c), $\sigma_2 = 0.5$ (d), $\sigma_2 = 0.6$ (e), and $\sigma_2 = 0.7$ (f), $\sigma_2 = 0.8$ (g), $\sigma_2 = 0.9$ (h), and $\sigma_2 = 1$ (i).

Observations: Similar remark is made here as in Figure 5.9 above, except that $I(t)$ grows more rapidly in this case.

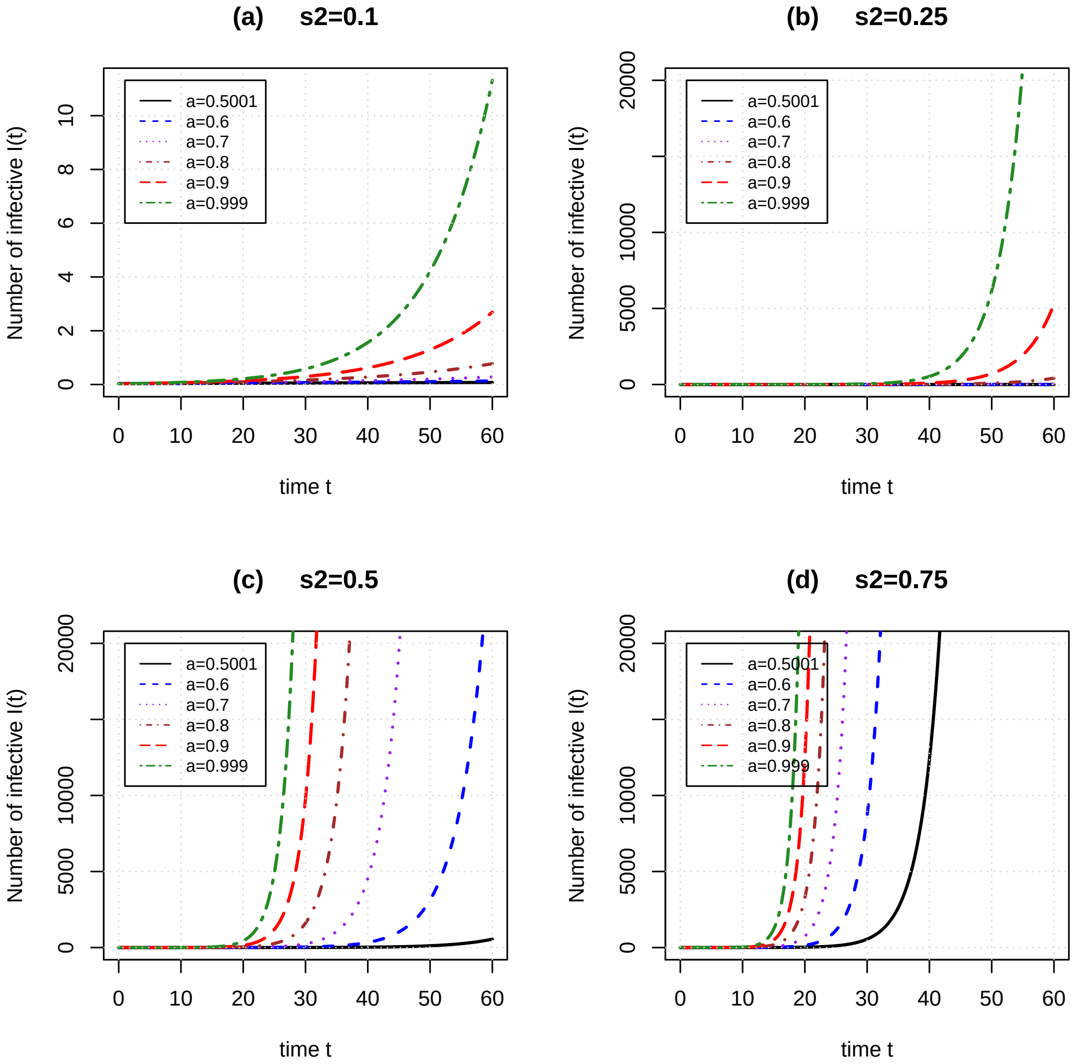

Figure 5.11: Graphs of the solution $I(t)$ with $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0$, $\alpha = 0.5001$ initial value $I_0 = 0.03$. $\sigma_2 = 0.1$ (a), $\sigma_2 = 0.25$ (b), $\sigma_2 = 0.5$ (c), and $\sigma_2 = 0.75$ (d).

Observations: We note that in the absence of noise and ordinary time-scale t, $I(t)$ grows exponentially fast with t and the speed of growth is faster with higher value of α.

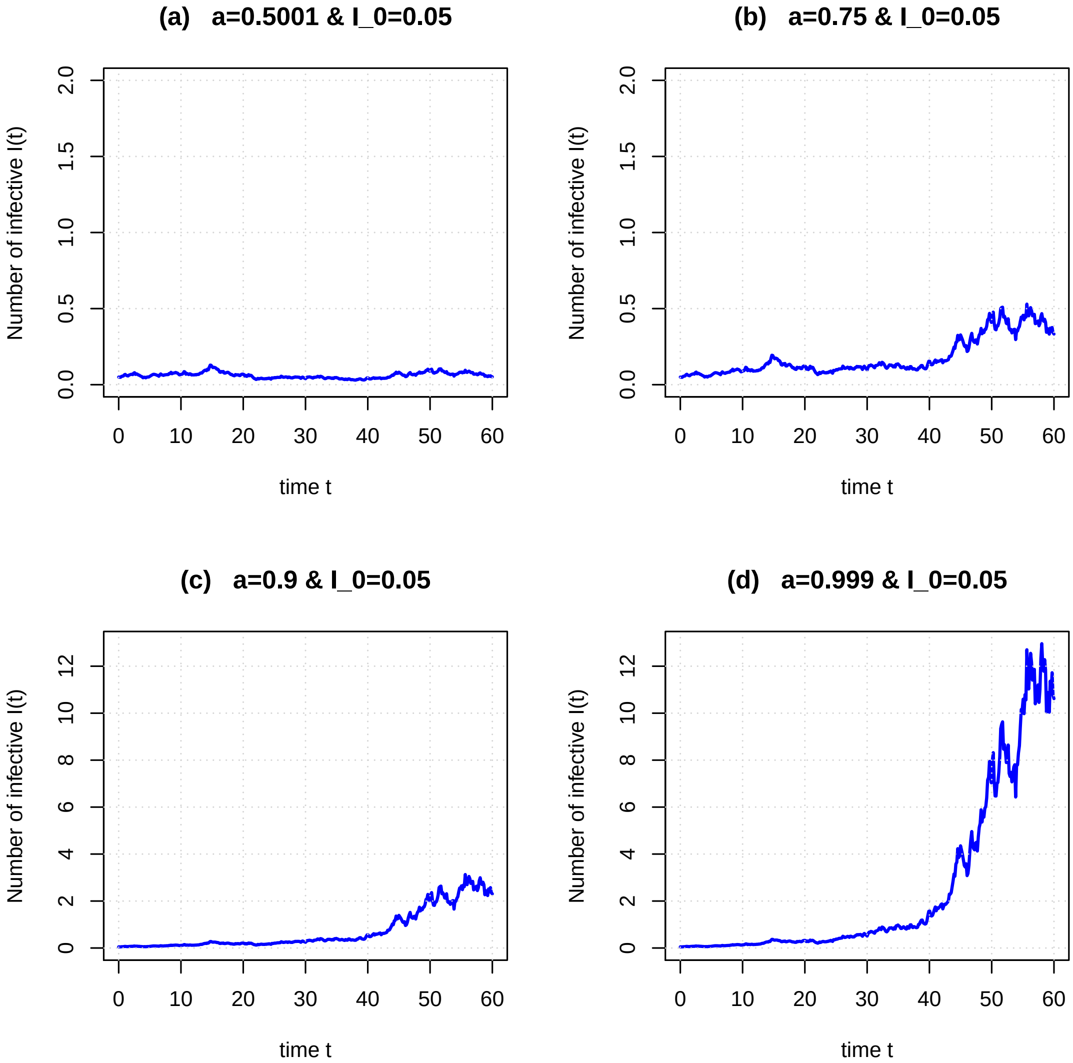

Figure 5.12: Sample paths of the solution $I(t)$ with $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$ with initial value $I_0 = 0.03$. $\alpha = 0.5001$ (a), $\alpha = 0.75$ (b), $\alpha = 0.9$ (c), and $\alpha = 0.999$ (d).

Observations: We observe that for fixed values of σ_1 and σ_2 (0.2 and 0.1 in this case, respectively) in graphs (a)-(d), α ha a significant impact of $I(t)$.

6 A Class of Higher Order Stochastic differential equations

Since the introduction of Itô-Doob calculus, the modeling and study of random dynamic phenomena have been very impressive, leading to the development of fundamental results for linear and nonlinear stochastic differential equations and their applications including science, engineering, and finance [4, 10, 53, 54, 76]. Although most of the studies are about the linear, nonlinear, and systems of stochastic differential equations, very limited explicit work on higher order stochastic differential equations with multiplicative noise is available. A treatment of higher order ordinary deterministic and stochastic differential equations can be found in Ladde et al. in [52, 53]. Several models in time series analysis are defined from higher order differential equation with random forcing process. The presented work allows to incorporate multiplicative noise.

6.1 Problem Formulation

This chapter is devoted to the study of the following higher order stochastic differential equations with constant coefficients of the form

$$dy^{(n-1)} + \sum_{i=0}^{n-1} a_i y^{(i)} dt + \sum_{i=0}^{n-1} \sigma_j y^{(j)} dw = 0, \tag{6.1.1}$$

where $n \in \mathbb{N}$, $n \geq 2$, a_i, and σ_j are constants, $i,\ j = 0, \ldots n-1$, and w is a normalized Wiener process.

Our goal is to develop a method of finding close form solutions of (6.1.1). In doing so, we will be interested in investigating conditions under which the exact solutions of such equations are feasible. For this purpose, we set $x_1(t) = y(t)$, $x_{i+1}(t) = \dot{x}_i(t)$, for $i = 1, 2, 3, \dots, n-1$, and write $x(t) = [x_1(t), \dots, x_n(t)]^T$. Under these considerations, equation (6.1.1) can be rewritten as a stochastic system of differential equations (SSDE).

$$dx = Axdt + Bxdw(t) \tag{6.1.2}$$

where matrices A (the companion matrix associated with $dy^{(n-1)} + \sum_{i=0}^{n-1} a_i y^{(i)} dt = 0$) and B (stochastic perturbations) are defined by:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix} \tag{6.1.3}$$

and

$$B = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & 0 \\ -\sigma_0 & -\sigma_1 & \dots & \dots & -\sigma_{n-2} & -\sigma_{n-1} \end{pmatrix}, \tag{6.1.4}$$

respectively.

Definition 6.1.1 Let $T > 0$ and $[t_0, t_0 + T] = J \subseteq \mathbb{R}$. A solution of the n-th ($n \geq 2$) order linear stochastic differential equation of type (6.1.1) is a stochastic process $y = y(t, w)$ defined on J, whose sample paths are is $(n-1)$ times continuously differentiable, and it satisfies (6.1.1) in the sense of Itô-Doob calculus.

In the following, we present a well-known result [52] that is useful for finding a solution process of the deterministic system of differential equations (6.1.2):

$$dx = Axdt \tag{6.1.5}$$

where A is $n \times n$ companion matrix defined in (6.1.3)

Proposition 6.1.2 *Let* $\Lambda(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda^1 + a_0$. *Then for any number* λ,

$$A \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} - \Lambda(\lambda) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{6.1.6}$$

Proof. Detailed proof is provided in [52, 53].

The following result establishes the fact that the solution processes of (6.1.1) and (6.1.2) are equivalent. In fact, a stochastic process is solution of (6.1.1) if and only if, it is a solution of (6.1.2).

■

Theorem 6.1.3 *Let* $y(t)$ *and* $x(t) = [x_1(t), \ldots, x_n(t)]^T$ *be any solutions of* (6.1.1) *and* (6.1.2), *respectively. Then,*

(a) $[x_y(t)]^T = [y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t)]$ *is solution process of* (6.1.2),

(b) $dx_1^{(n-1)} + \sum_{i=0}^{n-1} a_i x_1^{(i)} dt + \sum_{j=0}^{n-1} \sigma_j x_1^{(j)} dw = 0,$

where $x_1(t)$ *is the first component of the solution process* $x(t)$ *of* (6.1.2).

The proof is straightforward from (6.1.1) and (6.1.2). A method finding close form solution processes of (6.1.1) is presented in the following section.

6.2 Method of Solving Itô-Dood Type Homogeneous Stochastic Differential Equations

In this section, we utilize the eigenvalue type method [52, 53] to find solutions of the higher order linear homogeneous stochastic differential equations (6.1.1) with constant coefficients. The procedure is a modification of the method of solving deterministic systems of differential equations (one time scale t) with two time-scales : t and $w(t)$. By employing the procedure developed in [53], we decompose (6.1.2) into two time-scale components: deterministic component as defined in (6.1.5) and stochastic component as follows:

$$dx = Bxdw, \tag{6.2.1}$$

where B is $n \times n$ constant matrix defined in (6.1.4).

The next step consists of finding fundamental matrix solution processes $\Phi_d(t)$ and $\Phi_s(t)$ of (6.1.5) and (6.2.1), respectively. Then, create a candidate for the fundamental matrix solution process

$$\Phi(t) = \Phi_d(t)\Phi_s(t) \tag{6.2.2}$$

of (6.1.2), and test the correctness of the fundamental matrix $\Phi(t)$.

It is known [52, 53] that under the following assumptions:

(i) at least one of the matrices $\Phi_d(t)$ and $\Phi_s(t)$ is normalized fundamental matrix, and

(ii) $AB = BA$,

the matrix $\Phi(t)$ defined in (6.2.2) is the fundamental matrix solution process of (6.1.2). However, for the matrices A and B in (6.1.2), we have $AB \neq BA$, unless $B \equiv 0$, where 0 is the zero matrix. But, $B = 0$ if and only if, systems (6.1.2) and (6.1.5) are identical, i.e. the problem is reduced to deterministic system of differential equations. Therefore, in order to find the solution process of the non-trivial stochastic differential equations (6.1.1), we need to modify the above described procedure. For this purpose, in the sequel, we assume that $B \neq 0$. The procedure of finding solutions of (6.1.2) is based on the method of variation of constant parameters. For easy reference, we state and prove the following result which is a special case of the result in [53].

Theorem 6.2.1 (Method of Variation of Constant Parameter): *Let us assume that*

(H_1) $\Phi_d(t)$ *is the fundamental matrix solution of* (6.1.5) *, and*

(H_2) *let* $x(t) = \Phi_d(t)c(t)$, *where* $c(t)$ *is an* n*-dimensional unknown vector function.*

Then $x(t)$ *is a solution process of* (6.1.2) *if, and only if,* $c(t)$ *is a solution process of the stochastic system of linear differential equations with time-varying coefficients*

$$dc = \Phi_d^{-1}(t)B\Phi_d(t)c\, dw(t) \tag{6.2.3}$$

Proof. From hypotheses (H_1) and (H_2), we have

$$\begin{aligned} dx(t) &= d(\Phi_d(t)c(t)) \\ &= d\Phi_d(t)c(t) + \Phi_d(t)dc(t) \\ &= A\Phi_d(t)dtc(t) + \Phi_d(t)dc(t) \quad \text{(by using the assumption on } \Phi_d(t)) \\ &= A\Phi_d(t)c(t)dt + \Phi_d(t)dc(t) \qquad (6.2.4) \end{aligned}$$

Now, assume that $\boldsymbol{x}(t) = \Phi_d(t)c(t)$ is a solution of (6.1.2). Then, it satisfies

$$\begin{aligned} dx &= Axdt + Bxdw(t) \quad \text{(from (6.1.2))} \\ &= A\Phi_d(t)c(t)dt + B\Phi_d(t)c(t)dw(t) \quad \text{(from the hypothesis } (H_2)) \end{aligned} \tag{6.2.5}$$

Equating the right hand sides of (6.2.4) to the right hand side of (6.2.5) leads to

$$A\Phi_d(t)c(t)dt + \Phi_d(t)dc(t) = A\Phi_d(t)c(t)dt + B\Phi_d(t)c(t)dw(t) \tag{6.2.6}$$

which yields

$$\Phi_d(t)dc(t) = B\Phi_d(t)c(t)dw(t). \tag{6.2.7}$$

Hence applying $\Phi_d^{-1}(t)$ to both sides of (6.2.7), we obtain the equation (6.2.3).

Conversely, let us suppose that $c(t)$ is solution of (6.2.3). Then, replacing $dc(t)$ with $\Phi_d^{-1}(t)B\Phi_d(t)cdw(t)$ in (6.2.4) shows that $x(t) = \Phi_d(t)c(t)$ is solution process of (6.1.2).

■

Remark 6.2.2 *(*i*) From Theorems 6.1.3 and 6.2.1, we note that finding a general solution process of stochastic system* (6.2.3) *is key to developing general solution processes of* (6.1.2)*.*
*(*ii*) To find close form or exact solution processes of linear system of time varying coefficient matrix* (6.2.3)*, we need to examine the algebraic structure of matrix* $\Phi_d^{-1}(t)B\Phi_d(t)$*.*

For the sake of examining the algebraic structure of matrix $\Phi_d^{-1}(t)B\Phi_d(t)$, let's denote by B_j^r the j-th row of matrix B in (6.1.4) for each $j = 1,\ 2,\ \ldots,\ n$. We observe that B_j^r is the zero vector for $j = 1,\ 2,\ \ldots,\ n-1$. The matrix $B\Phi_d(t)$ can be rewritten as

$$B\Phi_d(t) \quad = \quad \begin{bmatrix} B_1^r \\ \vdots \\ B_j^r \\ \vdots \\ B_n^r \end{bmatrix} [\Phi_{d1}^c(t)\ \ldots\ \Phi_{dk}^c(t)\ \ldots\ \Phi_{dn}^c(t)]$$

$$\text{i.e.} \qquad B\Phi_d(t) \quad = \quad \begin{bmatrix} B_1^r\Phi_{d1}^c(t) & \ldots & B_1^r\Phi_{dk}^c(t) & \ldots & B_1^r\Phi_{dn}^c(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_j^r\Phi_{d1}^c(t) & \ldots & B_j^r\Phi_{dk}^c(t) & \ldots & B_j^r\Phi_{dn}^c(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_n^r\Phi_{d1}^c(t) & \ldots & B_n^r\Phi_{dk}^c(t) & \ldots & B_n^r\Phi_{dn}^c(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \\ B_n^r\Phi_{d1}^c(t) & \ldots & B_n^r\Phi_{dk}^c(t) & \ldots & B_n^r\Phi_{dn}^c(t) \end{bmatrix}, \qquad (6.2.8)$$

where $\Phi_{dk}^c(t)$ denotes the k-th column vector of matrix $\Phi_d(t)$.

Now, by denoting $\Phi_d^{-1}(t) = (\phi_{ij})_{n\times n}$ and using (6.2.8), the matrix $M = \Phi_d^{-1}(t)B\Phi_d(t) = \big([\Phi_d^{-1}(t)B\Phi_d(t)]_{ij}\big)_{n\times n}$ is written as:

$$M \;=\; \begin{bmatrix} \phi_{1n}(t)B_n^r\Phi_{d1}^c(t) & \ldots & \phi_{1n}(t)B_n^r\Phi_{dk}^c(t) & \ldots & \phi_{1n}(t)B_n^r\Phi_{dn}^c(t) \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ \phi_{in}(t)B_n^r\Phi_{d1}^c(t) & \ldots & \phi_{in}(t)B_n^r\Phi_{dk}^c(t) & \ldots & \phi_{in}(t)B_n^r\Phi_{dn}^c(t) \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ \phi_{nn}(t)B_n^r\Phi_{d1}^c(t) & \ldots & \phi_{nn}(t)B_n^r\Phi_{dk}^c(t) & \ldots & \phi_{nn}(t)B_n^r\Phi_{dn}^c(t) \end{bmatrix}. \qquad (6.2.9)$$

From (6.2.9), we are ready to state and prove the following result. The result provides a tool for the classifications of nth order solvable linear Itô-Doob type stochastic differential equations with constant coefficients.

Lemma 6.2.3 *Let the hypotheses of Theorem 6.2.1 be satisfied. Then, all but one column vectors of matrix* $\Phi_d^{-1}(t)B\Phi_d(t)$ *in* (6.2.3) *are zeroes if and only if for any given* $1 \le k \le n$,

$$B_n^r \Phi_{dk}^c(t) \neq 0 \quad \textit{and} \quad B_n^r \Phi_{dj}^c(t) = 0 \qquad \textit{for all } j \neq k,\ j = 1,\ 2, ...,\ n \tag{6.2.10}$$

Proof. The validity of the necessary condition follows from

$$\Phi_d^{-1}(t)B\Phi_d(t) \;=\; \begin{bmatrix} 0 & \dots & 0 & \phi_{1n}(t)B_n^r\Phi_{dk}^c(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \phi_{in}(t)B_n^r\Phi_{dk}^c(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \phi_{nn}(t)B_n^r\Phi_{dk}^c(t) & 0 & \dots & 0 \end{bmatrix}, \tag{6.2.11}$$

for any given $k = 1,\ 2, ...,\ n$.

For the sufficient condition, we note that if (6.2.10) is true for any $j = 1, 2, ..., n$ and $j \neq k$ for $1 \le k \le n$, the entire j-th column is zero because $B_n^r\Phi_{dj}^c(t)$ is a factor of each entry of the j-th column. This completes the proof.

■

The following lemma provides some important information on condition (6.2.10).

Lemma 6.2.4 *Condition* (6.2.10) *is equivalent to the following:*

(a) $\sum_{i=1}^{n}\sigma_{i-1}\lambda_j^{i-1} = 0$ *for all* $j = 1,\ 2, ...,\ n,\ j \neq k$, *and* $\sum_{i=1}^{n}\sigma_{i-1}\lambda_k^{i-1} \neq 0$

(b) $\Sigma^T\Lambda_j = 0$ *for all* $j = 1,\ 2, ...,\ n,\ j \neq k$ *and* $\Sigma^T\Lambda_k \neq 0$ *where* $\Sigma^T = [\sigma_0, \sigma_1, \dots, \sigma_{n-1}]$ *and* $\Lambda_j^T = [1, \lambda_j, \lambda_j^2, \dots, \lambda_j^{n-1}]$

(c) *For all* $j \neq k$, *eigenvectors* Λ_j *corresponding to* λ_j *are orthogonal to random environmental parameter vector* $\sum$ *and it belongs to the span of the eigenvector* Λ_k.

Remark 6.2.5 *The matrix $\Phi_d^{-1}(t)B\Phi_d(t)$ is a function of matrices B and $\Phi_d(t)$, and the latter depends of the eigenvalues of A. Therefore, the algebraic condition* (6.2.10) *depends on the coefficients of matrices A and B. Moreover, close form solution processes of a solvable n-th order linear Itô-Doob type stochastic differential equations are classified into n classes.*

In the following, we present the main result. It deals with a general procedure of finding close form solution process of (6.1.2).

Theorem 6.2.6 *Let the hypotheses of Lemma 6.2.3 be satisfied. Then,*

(a) *the higher order stochastic differential equation* (6.1.1) *is solvable;*

(b) *n solutions of* (6.1.1) *are represented by:*

$$\begin{cases} y_j(t) = \psi_{1j}(t), \ \ for \ \ j \neq k, j, k = 1, 2, \ldots, n, \\ y_k(t) = \sum\limits_{j \neq k}^{n} \psi_{1j}(t) \int_0^t \phi_{jn}(s) B_n^r \Phi_{dk}^c(s) \exp[\nu_k(s, w(s)) dw(s)] \\ \qquad\qquad + \psi_{1k}(t) \exp[\nu_k(t, w(t))], \ \ for \ \ j = k, \end{cases} \tag{6.2.12}$$

where $\psi_{1j}(t)$ is an entry of the matrix $\Phi_d(t) = (\psi_{ij}(t))_{n \times n}$ and

$$\nu_k(t, w(t)) = -\frac{1}{2} \int_0^t [\phi_{kn}(s) B_n^r \Phi_{dk}^c(s)]^2 \, ds + \int_0^t \phi_{kn}(s) B_n^r \Phi_{dk}^c(s) dw(s)$$

(c) *a close form general solution of* (6.1.1) *is*

$$y(t) \ \ = \ \ \sum_{j=1}^{n} c_j y_j(t) \tag{6.2.13}$$

where c_j's are arbitrary constants with $c_k \neq 0$.

Proof. From (6.2.3) and (6.2.12), we have

$$dc_j \ \ = \ \ \phi_{jn}(t) B_n^r \Phi_{dk}^c(t) c_k dw(t), \ \ \text{for} \ \ j \neq k, j, k = 1, 2, \ldots, n, \tag{6.2.14}$$

$$dc_k \ \ = \ \ \phi_{kn}(t) B_n^r \Phi_{dk}^c(t) c_k dw(t), \ \ \text{for some} \ \ k = 1, 2, \ldots, n. \tag{6.2.15}$$

Solving (6.2.15) for c_k yields

$$c_k(t) = c_{k0}\exp[\nu_k(t, w(t))] = c_{k0}\rho_{kk}(t) \quad \text{(by notation)} \tag{6.2.16}$$

where c_{k0} is an arbitrary constant.

Substituting $c_k(t)$ in (6.2.16) into (6.2.14) and solving for c_j, we have

$$\begin{aligned} c_j(t) &= c_{j0} + c_{k0}\int_0^t \phi_{jn}(s)B_n^r\Phi_{dk}^c(s)\exp[\nu_k(t, w(t))]dw(s) \\ &= c_{j0} + c_{k0}\rho_{jk}(t) \quad \text{(by notation)} \end{aligned} \tag{6.2.17}$$

where c_{j0}'s are arbitrary constants $j \neq k$, $j, k = 1, 2, \ldots, n$.

From (6.2.14), (6.2.14) and (6.2.16), the fundamental solution process of transformed system (6.2.3), denoted by $\Phi_T(t) \equiv \Phi_T(t, w(t))$, in the context of (6.2.11) is

$$\Phi_T(t) = \begin{bmatrix} 1 & 0 & \ldots & 0 & \varphi_{1k}(t) & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \varphi_{2k}(t) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \varphi_{(k-1)k}(t) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \varphi_{kk}(t) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \varphi_{(k+1)k}(t) & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \varphi_{nk}(t) & 0 & \ldots & 1 \end{bmatrix}, \tag{6.2.18}$$

where

$$\rho_{jk}(t) = \begin{cases} \int_0^t \phi_{jn}(s)B_n^r\Phi_{dk}^c(s)\exp[\nu_k(t, w(t))]dw(s), & \text{for } j \neq k; \\ \exp[\nu_k(t, w(t))], & \text{for } j = k \end{cases} \tag{6.2.19}$$

From (6.1.2), Lemma 6.2.3, (6.2.16)-(6.2.19), Theorem 6.2.1, we conclude that the following matrix $\Phi(t)$ is a fundamental matrix solution process of (6.1.2).

$$\Phi(t) := \Phi_d(t)\Phi_T(t)$$
$$= \begin{bmatrix} \psi_{11}(t) & \dots & \psi_{1(k-1)}(t) & \sum_{i=1}^{n}\psi_{1i}(t)\varphi_{ik}(t) & \psi_{1(k+1)}(t) & \dots & \psi_{1n}(t) \\ \psi_{21}(t) & \dots & \psi_{2(k-1)}(t) & \sum_{i=1}^{n}\psi_{2i}(t)\varphi_{ik}(t) & \psi_{2(k+1)}(t) & \dots & \psi_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{j1}(t) & \dots & \psi_{j(k-1)}(t) & \sum_{i=1}^{n}\psi_{ji}(t)\varphi_{ik}(t) & \psi_{j(k+1)}(t) & \dots & \psi_{jn}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{n1}(t) & \dots & \psi_{n(k-1)}(t) & \sum_{i=1}^{n}\psi_{ni}(t)\varphi_{ik}(t) & \psi_{n(k+1)}(t) & \dots & \psi_{nn}(t) \end{bmatrix}, \quad (6.2.20)$$

where $\Phi_d(t) = (\psi_{ij}(t))_{n\times n}$ is a fundamental matrix solution of (6.1.5). Therefore, applying Theorem 6.1.3, we conclude that the general solution of the higher order stochastic differential equation (6.1.1) is

$$\begin{aligned} y(t) &= \sum_{j\neq k}^{n} c_j\psi_{1j}(t) + c_k\sum_{i=1}^{n}\psi_{1i}\varphi_{ik}(t) \\ &= \sum_{j=1}^{n} c_j y_j(t) \end{aligned} \quad (6.2.21)$$

where c_j's are arbitrary constants, for $j \neq k$, $y_j(t) = \psi_{1j}(t)$ and $y_k(t) = c_k\sum_{i=1}^{n}\psi_{1i}\varphi_{ik}(t)$. This establishes (a), (b) and (c) and completes the proof of the theorem.

■

Next, we utilize the procedure developed in this section to find close form or exact solution processes of (6.1.2).

6.3 Close Form Solution Procedure

In applying the procedure developed in Section 6.2 for finding general solution processes of classes of (6.1.2) in the context of eigenvalues of A, we identify the following three cases: (i) distinct eigenvalues, (ii) repeated eigenvalues, and (iii) complex eigenvalues of the companion matrix A.

Let us consider the case where the companion matrix A has n distinct real eigenvalues.

Case 1: Matrix A has n distinct real eigenvalues

Here we assume that the companion matrix A has n distinct real eigenvalues (i.e. $m = n$) λ_i, $i = 1, 2, \ldots, n$. In this case, $n_j = 1$ and $f_{jk}(t) = f_j(t) = t^k e^{\lambda_j t} = e^{\lambda_j t}$ for all $j = 1, 2, \ldots, m(= n)$ since $k = 0$. Then, the fundamental matrix solution $\Phi_d(t)$ of (6.1.5) is given by

$$\Phi_d(t) = \begin{bmatrix} e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \\ \vdots \\ \lambda_1^{n-2} \\ \lambda_1^{n-1} \end{bmatrix} & e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \\ \vdots \\ \lambda_2^{n-2} \\ \lambda_2^{n-1} \end{bmatrix} & \ldots & e^{\lambda_n t} \begin{bmatrix} 1 \\ \lambda_n \\ \vdots \\ \lambda_n^{n-2} \\ \lambda_n^{n-1} \end{bmatrix} \end{bmatrix}. \qquad (6.3.1)$$

The following theorem provides the condition of existence of a k-th class ($k = 1, 2, \ldots, n$) of equation (6.1.2) when matrix A has distinct real eigenvalues.

Theorem 6.3.1 *Let the hypotheses of Theorem 6.2.6 be satisfied. Furthermore, assume that the companion matrix A in* (6.1.3) *has n distinct real eigenvalues and that condition* (a) *of Lemma 6.2.4 holds for any $k = 1, 2, 3, \ldots, n$. Then, there exist solution processes for the stochastic system of differential equations* (6.2.3).

Proof. Suppose that the condition in part (a) of Lemma (6.2.4) holds for some column k ($k = 1, 2, \ldots, n$). We note that $\phi_{kn} B_n^r \Phi_{dk}^c(t) = -\phi_{kn}(t) e^{\lambda_k t} \sum_{i=1}^n \sigma_{i-1} \lambda_k^{i-1}$. Then, (6.2.15) becomes

$$dc_k = -\phi_{kn}(t) e^{\lambda_k t} \sum_{i=1}^n \sigma_{i-1} \lambda_k^{i-1} c_k dw(t). \qquad (6.3.2)$$

Solving this equation for c_k yields

$$c_k(t) = c_{k0} e^{\left[-\frac{1}{2}\int_0^t \left(\phi_{kn}(s)e^{\lambda_k s}\sum_{i=1}^n \sigma_{i-1}\lambda_k^{i-1}\right)^2 ds - \int_0^t \phi_{kn}(s)e^{\lambda_k s}\sum_{i=1}^n \sigma_{i-1}\lambda_k^{i-1} dw(s)\right]}. \quad (6.3.3)$$

and for each $j = 1,\ 2, \dots,\ k-1,\ k+1, \dots,\ n-1,\ n,$

$$c_j(t) = c_{j0} - \int_0^t \phi_{jn}(s)e^{\lambda_k s}\sum_{i=1}^n \sigma_{i-1}\lambda_k^{i-1} c_k(s)dw(s) \quad (6.3.4)$$

where c_{j0} is constant for $j = 1, 2, \dots, k-1, k+1, \dots, n$, and $c_k(t)$ is as in (6.3.3).

The general solution process of equation (6.1.1) in this case is therefore given in the form

$$\begin{aligned} y(t) &= \Phi_{d1}^r(t)c(t) \\ &= c_k(t)e^{\lambda_k t} + \sum_{j=1, j\neq k}^n \left(c_{j0} - \int_0^t \phi_{jn}(s)e^{\lambda_k s}\sum_{i=1}^n \sigma_{i-1}\lambda_k^{i-1}c_k(s)dw(s)\right)e^{\lambda_j t}, \end{aligned} \quad (6.3.5)$$

where $c_k(t)$ is as in (6.3.3).

■

Illustration 1: Let us consider a general 2-nd order stochastic differential equations

$$d\dot{y} + a_1\dot{y}dt + a_0 y dt + \sigma_1\dot{y}dw(t) + \sigma_0 y dw(t) = 0, \quad (6.3.6)$$

where $a_0,\ a_1,\ \sigma_1$, and σ_0 are constants real numbers, and w is a Wiener process. If $a_1^2 - 4a_0 > 0$, the associated companion matrix A has distinct real eigenvalues $\lambda_1 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}$ and $\lambda_2 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}$. Then, after computing the matrix $\Phi_d(t)$ and its inverse, the matrix $\Phi_d^{-1}(t)B\Phi_d(t)$ in (6.2.9) reduces to

$$\Phi_d^{-1}(t)B\Phi_d(t) = \begin{bmatrix} \frac{(\sigma_0+\sigma_1\lambda_1)}{\lambda_2-\lambda_1} & \frac{(\sigma_0+\sigma_1\lambda_2)e^{-(\lambda_1-\lambda_2)t}}{\lambda_2-\lambda_1} \\ -\frac{(\sigma_0+\sigma_1\lambda_1)e^{(\lambda_1-\lambda_2)t}}{\lambda_2-\lambda_1} & -\frac{(\sigma_0+\sigma_1\lambda_2)}{\lambda_2-\lambda_1} \end{bmatrix}. \quad (6.3.7)$$

Moreover, equation (6.2.3) becomes

$$dc = \begin{bmatrix} \frac{\sigma_0+\sigma_1\lambda_1}{(\lambda_2-\lambda_1)} & \frac{\sigma_0+\sigma_1\lambda_2}{(\lambda_2-\lambda_1)} e^{-(\lambda_1-\lambda_2)t} \\ \frac{-(\sigma_0+\sigma_1\lambda_1)}{(\lambda_2-\lambda_1)} e^{(\lambda_1-\lambda_2)t} & \frac{-(\sigma_0+\sigma_1\lambda_2)}{(\lambda_2-\lambda_1)} \end{bmatrix} cdw(t) \tag{6.3.8}$$

Our procedure the yields two conditions: $\sigma_0 + \sigma_1\lambda_2 = 0$ and $\sigma_0 + \sigma_1\lambda_1 = 0$. From the application of Theorem 6.2.6, the general solution of (6.3.6) corresponding to condition $\sigma_0 + \sigma_1\lambda_1 = 0$ is:

$$\begin{aligned} y(t) &= \Phi^r_{d1}c(t) \;=\; c_1(t)e^{\lambda_1 t} \;+\; c_2(t)e^{\lambda_2 t} \\ &= c_{10}\exp\left[-\frac{\sigma_0 t}{\sigma_1}\right] + c_{20}\exp\left[\left(-a_1 + \frac{\sigma_0}{\sigma_1} - \frac{\sigma_1^2}{2}\right)t - \sigma_1 w(t)\right] \\ &\quad + c_{20}\sigma_1\exp\left[-\frac{\sigma_0 t}{\sigma_1}\right]\int_0^t \exp\left[\left(-a_1 + \frac{2\sigma_0}{\sigma_1} - \frac{\sigma_1^2}{2}\right)s - \sigma_1 w(s)\right]dw(s) \\ &= c_{10}e^{-\sigma_0 t/\sigma_1} + c_{20}(\xi + \sigma_1^2/2)e^{-\sigma_0 t/\sigma_1}\int_0^t \exp\left[\xi s - \sigma_1 w(s)\right]ds, \end{aligned} \tag{6.3.9}$$

where $\xi = -a_1 + 2\sigma_0/\sigma_1 - \sigma_1^2/2$, and c_{10} and $c_{20} \neq 0$ are arbitrary constants.

Similarly, the general solution of (6.3.6) corresponding to condition $\sigma_0 + \sigma_1\lambda_2 = 0$ is:

$$\begin{aligned} y(t) &= c_1(t)e^{\lambda_1 t} \;+\; c_2(t)e^{\lambda_2 t} \\ &= c_{20}e^{-\sigma_0 t/\sigma_1} + c_{10}(\xi + \sigma_1^2/2)e^{-\sigma_0 t/\sigma_1}\int_0^t \exp\left[\xi s - \sigma_1 w(s)\right]ds, \end{aligned} \tag{6.3.10}$$

where $\xi = -a_1 + 2\sigma_0/\sigma_1 - \sigma_1^2/2$, and c_{20} and $c_{10} \neq 0$ are arbitrary constants.

Example 6.3.2 Condition $\sigma_0 + \sigma_1\lambda_1 = 0$.

Let us assume that $\sigma_0 = -\sigma_1\lambda_1$ with $\sigma_1 \neq 0$. Then equation (6.3.6) becomes

$$d\dot{y} \;+\; a_1\dot{y}dt \;+\; a_0 y dt \;+\; \sigma_1\dot{y}dw(t) \;-\; \sigma_1\lambda_1 y dw(t) \;=\; 0, \tag{6.3.11}$$

where $\lambda_1 = \frac{-a_1-\sqrt{a_1^2-4a_0}}{2} \;=\; -\sigma_0/\sigma_1$.

For instance if $a_0 = 2,\ a_1 = 3,\ \sigma_1 = -2$. The equation (6.3.6) becomes

$$d\dot{y} + 3\dot{y}dt + 2ydt - 2\dot{y}dw(t) - 4ydw(t) = 0, \tag{6.3.12}$$

$\lambda_1 = -2$, $\lambda_2 = -a_1 - \lambda_1 = -1$. Under these conditions, the general solution process of the equation (6.3.12) is given by

$$\begin{aligned} y(t) &= c_1 e^{-2t} + c_2 e^{(-3t+2w(t))} - 2c_2 e^{-2t} \int_0^t e^{(-s+2w(s))} dw(s) \\ &= c_1 e^{-2t} + c_2 e^{-2t} \int_0^t e^{(-s+2w(s))} ds \end{aligned} \tag{6.3.13}$$

where c_1 and $c_2 \neq 0$ are arbitrary constants. One sample path is displayed in Figure 6.1.

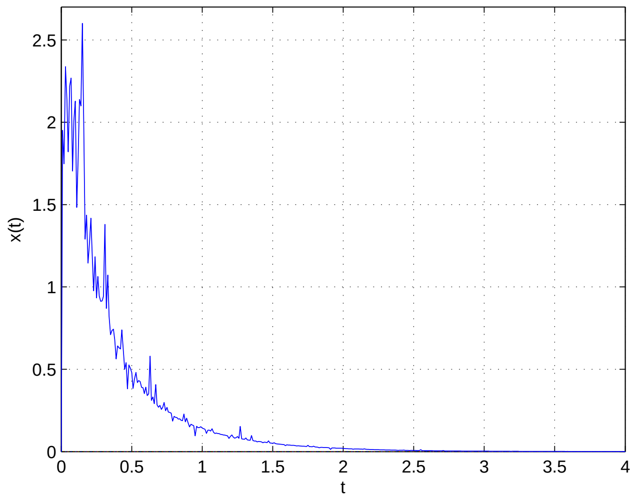

Figure 6.1: Plot of a sample path of the solution process in example 6.3.2

The mean and variance of the $y(t)$ are

$$E[y(t)] = c_1 e^{-2t} + c_2 e^{-t}$$

and

$$\text{Var}(y(t)) = 5c_2^2 e^{2t} - 2c_2^2 e^{-4t} - c_2^2 e^{-2t},$$

respectively.

Example 6.3.3 Condition $\sigma_0 + \sigma_1\lambda_2 = 0$.

Let us consider the case where $\sigma_0 + \sigma_1\lambda_2 = 0$. Then equation (6.3.6) becomes

$$d\dot{y} + a_1\dot{y}dt + a_0ydt + \sigma_1\dot{y}dw(t) - \sigma_1\lambda_2ydw(t) = 0, \tag{6.3.14}$$

For example, taking $a_0 = 6$, $a_1 = -5$, $\sigma_1 = -2$, we have $\lambda_2 = \frac{-a_1+\sqrt{a_1^2-4a_0}}{2} = 3$, $\lambda_1 = -a_1 - \lambda_2 = 2$. Then equation (6.3.14) reduces to

$$d\dot{y} - 5\dot{y}dt + 6ydt - 2\dot{y}dw + 4ydw(t)(t) = 0, \tag{6.3.15}$$

Therefore, by applying the result obtained in (6.3.10), the general solution process of (6.3.15) is

$$\begin{aligned} y(t) &= c_1e^{-3t+2w(t)} + c_2e^{3t} - 2c_1e^{3t}\int_0^t e^{(-3s+2w(s))}dw(s) && (6.3.16)\\ &= c_1e^{3t} + c_2e^{3t}\int_0^t e^{(-3s+2w(s))}ds && (6.3.17) \end{aligned}$$

where c_1 and $c_2 \neq 0$ are arbitrary constants. One sample path is displayed in Figure 6.2.

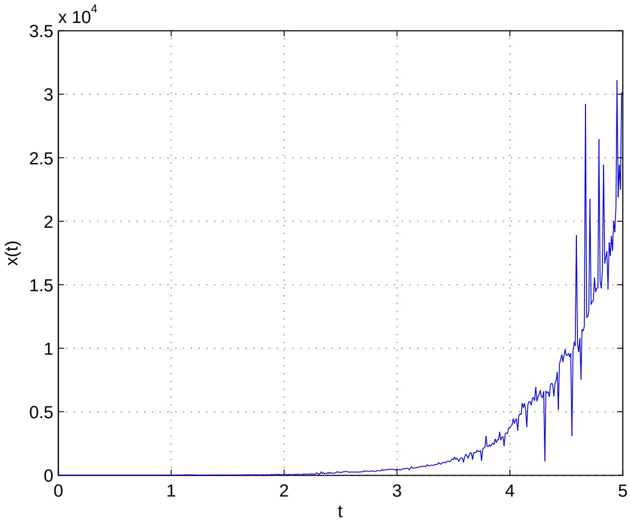

Figure 6.2: Plot of sample a path of the solution process in example 6.3.3

The mean and variance of the $y(t)$ are

$$E[y(t)] = c_1 e^{2t} + c_2 e^{3t}$$

and

$$\text{Var}(y(t)) = \frac{1}{3} c_1^2 [5e^{10t} + c_1^2 e^{4t} - 3e^{6t}],$$

respectively.

Next, we investigate the solution procedure when the companion matrix A has some eigenvalue with multiplicity greater than one.

Case 2: Matrix A has Repeated real eigenvalues

without loss of generality, we assume that the companion matrix A has an eigenvalue λ of multiplicity m, $1 < m \leq n$ (for simplicity, we take $\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_m$). Furthermore, assume that the remaining $n - m$ eigenvalues are distinct and real, and we denote them by $\lambda_{m+1},\ \lambda_{m+2},\ \ldots,\ \lambda_n$.

We recall here some useful results in the theory of deterministic higher order differential equations.

Proposition 6.3.4 [8] *For any nth-order linear homogeneous differential equation of the form*

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y \quad = \quad 0 \tag{6.3.18}$$

whose characteristic equation has a repeated root r of multiplicity k, the linearly independent solutions of the equation corresponding to r are

$$e^{rt},\ te^{rt},\ t^2 e^{rt},\ \ldots,\ t^{k-1} e^{rt}$$

Now, suppose the characteristic equation of (6.3.18) has m distinct roots r_1 (multiplicity n_1), r_2 (multiplicity n_2), ..., r_m (multiplicity n_m), where $n_1 + n_2 + \cdots + n_m = n$ ($mj \geq 1$ for all $j = 1, 2, \ldots, m$. Then the general solution of (6.3.18) is

$$y(t) = \left[f_{10}(t)\; f_{11}(t) \ldots f_{1(n_1-1)}(t)\; \ldots\; f_{m0}(t) \ldots f_{m(n_m-1)}(t)\right] [c_1\; \ldots\; c_n]^T, \tag{6.3.19}$$

where $[c_1\; c_2\; \ldots\; c_n]^T$ is an n-dimensional constant vector, $f_{jk}(t) = t^k e^{\lambda_j t}$, $1 \leq j \leq m$ and $0 \leq k \leq n_j - 1$. Furthermore, from (6.1.1), (6.3.18) and (6.3.19), the fundamental matrix solution of the deterministic system of differential equations (6.1.5) is then given by

$$\Phi_d(t) = \left[\begin{bmatrix} f_{10}(t) \\ f'_{10}(t) \\ \vdots \\ f_{10}^{(n-2)}(t) \\ f_{10}^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_{1(n_1-1)}(t) \\ f'_{1(n_1-1)}(t) \\ \vdots \\ f_{1(n_1-1)}^{(n-2)}(t) \\ f_{1(n_1-1)}^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_{m0}(t) \\ f'_{m0}(t) \\ \vdots \\ f_{m0}^{(n-2)}(t) \\ f_{m0}^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_{m(n_m-1)}(t) \\ f'_{m(n_m-1)}(t) \\ \vdots \\ f_{m(n_m-1)}^{(n-2)}(t) \\ f_{m(n_m-1)}^{(n-1)}(t) \end{bmatrix} \right].$$

We begin by discussing the case when $m = 2$. In this case the fundamental matrix solution $\Phi_d(t)$ of (6.1.5) is as follows:

$$\Phi_d(t) = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & e^{\lambda_3 t} & \ldots & e^{\lambda_(n-1)t} \\ \lambda_1 e^{\lambda_1 t} & e^{\lambda_1 t} + \lambda_1 te^{\lambda_1 t} & \lambda_3 e^{\lambda_3 t} & \ldots & \lambda_{n-1} e^{\lambda_{n-1} t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-2} e^{\lambda_1 t} & (n-2)\lambda_1^{n-3} e^{\lambda_1 t} + \lambda_1^{n-2} te^{\lambda_1 t} & \lambda_3^{n-2} e^{\lambda_3 t} & \ldots & \lambda_{n-1}^{n-2} e^{\lambda_{n-1} t} \\ \lambda_1^{n-1} e^{\lambda_1 t} & (n-1)\lambda_1^{n-2} e^{\lambda_1 t} + \lambda_1^{n-1} te^{\lambda_1 t} & \lambda_3^{n-1} e^{\lambda_3 t} & \ldots & \lambda_{n-1}^{n-1} e^{\lambda_{n-1} t} \end{bmatrix}. \tag{6.3.20}$$

And, by using the results in equations (6.2.9), we obtain

$$B_n^r \Phi_{dj}^c(t) \;=\; -\sum_{i=1}^{n} \sigma_{i-1} \lambda_j^{i-1} e^{\lambda_1 t}, \quad \text{for } j = 1,\ 3,\ 4, \ldots, n, \tag{6.3.21}$$

and

$$
\begin{aligned}
B_n^r \Phi_{d2}^c(t) &= -\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2}e^{\lambda_1 t} - \sum_{i=1}^{n}\sigma_{i-1}\lambda_1^{i-1}te^{\lambda_1 t} && (6.3.22)\\
&= -\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2}e^{\lambda_1 t} + tB_n^r\Phi_{d1}^c(t) && (6.3.23)
\end{aligned}
$$

We note that (6.2.10) of Lemma 6.2.3 holds only for $k = 2$. This means that, in the case of exactly one repeated real eigenvalue of multiplicity 2 of companion matrix A, the technique utilized to find solution processes of classes of equation (6.1.1) yields solutions only for the 2nd class equation.

Theorem 6.3.5 *Let the hypotheses of Theorem 6.2.6 be satisfied. Furthermore, assume that the companion matrix A in* (6.1.3) *has $n-1$ distinct real eigenvalues, one of them, say without loss of generality λ_1, with multiplicity 2. Then, the stochastic system of differential equations* (6.2.3) *has at least one solution process, provided that*

$$
\sum_{i=1}^{n}\sigma_{i-1}\lambda_j^{i-1} = 0 \quad \textit{for all } j = 1,\, 3, 4, ..., n. \qquad (6.3.24)
$$

Proof. Assume that the hypotheses of Theorem 6.3.5 hold and that condition (6.3.24) is satisfied. Then equation (6.2.3) reduces to

$$
\begin{aligned}
dc_j &= \phi_{jn}(t)B_n^r\Phi_{d2}^c(t)c_2dw(t) \quad \text{for } j = 1,\, 2\,, \ldots,\, n,\\
&= -\,\phi_{jn}(t)e^{\lambda_1 t}\left(\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2} + \sum_{i=1}^{n}\sigma_{i-1}\lambda_1^{i-1}t\right)c_2dw(t), && (6.3.25)
\end{aligned}
$$

and for $j = 2$ we solve the following stochastic differential equation:

$$
dc_2 = -\,\phi_{2n}(t)e^{\lambda_1 t}\left(\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2} + \sum_{i=1}^{n}\sigma_{i-1}\lambda_1^{i-1}t\right)c_2dw(t). \qquad (6.3.26)
$$

Let's define

$$G_j(t) = -\,\phi_{jn}(t)e^{\lambda_1 t}\left(\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2} \;+\; \sum_{i=1}^{n}\sigma_{i-1}\lambda_1^{i-1}t\right) \qquad \text{for all } j = 1, 2, \ldots, n.$$

Then the solution of equation (6.3.26) is given by

$$c_2(t) \;=\; c_{20}\exp\left(-\frac{1}{2}\int_0^t G_2^2(s)ds + \int_0^t G_2(s)dw(s)\right). \tag{6.3.27}$$

By using this solution, we obtain the solution process of (6.3.25) as

$$c_j(t) \;=\; c_{j0} + \int_0^t G_j(s)c_2(s)dw(s) \text{ for all } j = 1, 3, 4, \ldots, n. \tag{6.3.28}$$

Then, the general solution process of (6.1.1) is given by

$$\begin{aligned} y(t) &= \Phi_{d1}^r(t)c(t) \\ &= [c_1(t) + tc_2(t)]e^{\lambda_1 t} + \sum_{j=3}^{n} e^{\lambda_j t}c_j(t). \end{aligned} \tag{6.3.29}$$

where $c_j(t)$ are provided in (6.3.28).

Now, let's consider the situation where $m > 2$. In this case, a fundamental matrix solution $\Phi_d(t)$ of (6.1.5) is as follows:

$$\Phi_d(t) = \left[\begin{bmatrix} f_1(t) \\ f_1^{(1)}(t) \\ \vdots \\ f_1^{(n-2)}(t) \\ f_1^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_m(t) \\ f_m^{(1)}(t) \\ \vdots \\ f_m^{(n-2)}(t) \\ f_m^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} f_{m+1}(t) \\ f_{m+1}^{(1)}(t) \\ \vdots \\ f_{m+1}^{(n-2)}(t) \\ f_{m+1}^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_n(t) \\ f_n^{(1)}(t) \\ \vdots \\ f_n^{(n-2)}(t) \\ f_n^{(n-1)}(t) \end{bmatrix}\right], \tag{6.3.30}$$

where $f_p(t) = f_p(\lambda_1, t) := t^{p-1}e^{\lambda_1 t}$ and

$$f_p^{(k)}(t) = \frac{d^k}{dt^k} f_p(t) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) \frac{d^j}{dt^j}[t^{p-1}] \frac{d^{k-j}}{dt^{k-j}}[e^{\lambda_1 t}], \tag{6.3.31}$$

for $p = 1, 2, \ldots, m$ and $k = 0, 1, 2, \ldots, n-1$.

In this case, we have

$$B_n^r \Phi_{dj}^c(t) \ = \ - \sum_{i=1}^{n} \sigma_{i-1} f_j^{(i-1)}(t) \tag{6.3.32}$$

for $j = 2, \ldots,\ m$, and

$$B_n^r \Phi_{dj}^c(t) \ = \ - \sum_{i=1}^{n} \sigma_{i-1} \lambda_j^{i-1} e^{\lambda_j t} \ = \ - e^{\lambda_j t} \sum_{i=1}^{n} \sigma_{i-1} \lambda_j^{i-1}, \tag{6.3.33}$$

for $j = 1,\ m+1,\ m+2, \ldots,\ n$.

■

Remark 6.3.6 *(*i*) From* (6.3.32)*, we observe that* $B_n^r \Phi_{dj}^c(t) \neq 0$ *for any* $j = 2, 3, \ldots, m$. *Therefore condition* (6.2.10) *does not hold here.*

*(*ii*) Theorem 6.2.6 is not applicable since Lemma 6.2.3 failed to hold.*

Illustration 2: Consider the second degree equation (6.3.6) introduced in the previous case. When $a_1^2 - 4a_0 = 0$, matrix A has one repeated eigenvalue $\lambda_1 = \lambda_2 = -a_1/2$.

Under these conditions, we have

$$\Phi_d^{-1}(t) B \Phi_d(t) \ = \ \begin{bmatrix} -\left(-\sigma_0 + \frac{a_1\sigma_1}{2}\right) t & \sigma_0 t^2 + \sigma_1 \left(1 - \frac{a_1 t}{2}\right) t \\ -\sigma_0 + \frac{a_1\sigma_1}{2} & -\sigma_0 t - \sigma_1 (1 - \frac{a_1 t}{2}) \end{bmatrix}. \tag{6.3.34}$$

Therefore,

$$\begin{bmatrix} dc_1 \\ dc_2 \end{bmatrix} = \begin{bmatrix} -\left(-\sigma_0 + \frac{a_1\sigma_1}{2}\right) t & \sigma_1 t \\ -\sigma_0 + \frac{a_1\sigma_1}{2} & -\sigma_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} dw(t). \tag{6.3.35}$$

Our procedure the yields two conditions: $-\sigma_0 + \frac{a_1\sigma_1}{2} = 0$ and $\sigma_1 = 0$. If $-\sigma_0 + \frac{a_1\sigma_1}{2} = 0$ i.e. $\sigma_0 = \frac{a_1\sigma_1}{2}$, (6.3.35) becomes

$$\begin{bmatrix} dc_1 \\ dc_2 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_1 t \\ 0 & -\sigma_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} dw(t) = \begin{bmatrix} \sigma_1 t c_2 dw(t) \\ -\sigma_1 c_2 dw(t) \end{bmatrix}. \tag{6.3.36}$$

Solving $dc_2 = -\sigma_1 c_2 dw(t)$ yields

$$c_2(t) = c_{20} \exp\left(-\frac{1}{2}\sigma_1^2 t - \sigma_1 w(t)\right) \tag{6.3.37}$$

and substituting the quantity on the right hand side of (6.3.37) for $c_2(t)$ in $dc_1(t) = \sigma_1 t c_2 dw$, and solving for $c_1(t)$ we obtain

$$c_1(t) = c_{10} + c_{20}\sigma_1 \int_0^t s \exp\left(-\frac{1}{2}\sigma_1^2 s - \sigma_1 w(s)\right) dw(s). \tag{6.3.38}$$

And the general solution process of (6.3.6) is given by

$$\begin{aligned} y(t) &= c_{20}\sigma_1 e^{-\frac{a_1}{2}t} \int_0^t s \exp\left(-\frac{1}{2}\sigma_1^2 s - \sigma_1 w(s)\right) dw(s) \\ &\quad + c_{20} t \exp\left((-a_1/2 - \sigma_1^2/2)t - \sigma_1 w(t)\right) + \ c_{10} e^{-\frac{a_1}{2}t} \\ &= c_{10} e^{-\frac{a_1}{2}t} + c_{20} e^{-\frac{a_1}{2}t} \int_0^t \exp\left(-\frac{1}{2}\sigma_1^2 s - \sigma_1 w(s)\right) ds \end{aligned} \tag{6.3.39}$$

where c_{10} and $c_{20} \neq 0$ are arbitrary constants.

Next, we present an example with a simulated sample path that illustrates the case of repeated eigenvalue of matrix A.

Example 6.3.7 Let $a_0 = 1,\ a_1 = -2,\ \sigma_1 = 2$, and $\sigma_0 = -2$. The equation (6.3.6) becomes

$$d\dot{y} \ - \ 2\dot{y}dt \ + \ ydt \ + \ 2\dot{y}dw(t) \ - \ 2ydw(t) \ = \ 0, \tag{6.3.40}$$

$\lambda_1 = \lambda_2 = 1$, and $-\sigma_0 + \frac{a_1\sigma_1}{2} = 2 + \frac{(-2)(2)}{2} = 0$. Therefore, from (6.3.39) the general solution solution of (6.3.40) is

$$y(t) \;=\; c_1 e^t \;+\; c_2 e^t \int_0^t \exp[-2s - 2w(s)]ds \tag{6.3.41}$$

where c_1 and c_2 are arbitrary constants ($c_2 \neq 0$). A sample path of this solution process is shown in Figure 6.3.

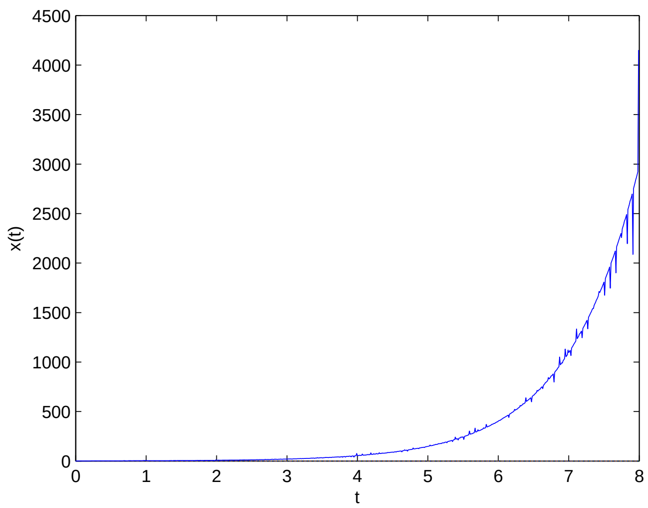

Figure 6.3: Plot of a sample path of the solution process in example 6.3.7

For obvious reasons, we emphasize that Theorem 6.2.6 is not applicable to condition $\sigma_1 = 0$. Moreover, the coefficient rate matrix $\Phi_d^{-1}(t)B\Phi_d(t)$ under this condition is a time-varying matrix, and its structure exhibits non-trivial coupled interactions with the components of state vector c. Because of this, one is not able to find the close form solution of this types of system. This, it is not feasible to find close form solution of (6.3.6) corresponding to this condition.

Remark 6.3.8 *From* (6.3.9), (6.3.10) *and* (6.3.39), *we observe that the solution processes (when the companion matrix A has a repeated eigenvalue or when all its eigenvalues real and distinct) can be expressed in the following form:*

$$y(t) \;=\; P(t) + Q(t)\int_0^t \exp\left[\xi s - \sigma_1 w(s)\right] ds, \tag{6.3.42}$$

where $P(t) = c_1 e^{-\sigma_0 t/\sigma_1}$, $Q(t) = c_2(\xi + \sigma_1^2/2)e^{-\sigma_0 t/\sigma_1}$, $\xi = -a_1 + 2\sigma_0/\sigma_1 - \sigma_1^2/2$, *for distinct eigenvalues of* A, *and* $P(t) = c_1 e^{-\frac{a_1}{2}t}$, $Q(t) = c_2 e^{-\frac{a_1}{2}t}$, $\xi = -\frac{1}{2}\sigma_1^2$ *when* A *has a repeated eigenvalue;* c_1 *and* $c_2 \neq 0$ *are arbitrary constants,*

In the next section, we examine the situation where the companion matrix A has complex eigenvalues.

Case 3: Matrix A has distinct complex eigenvalues Without loss of generality, let us assume that A has at least one complex eigenvalue, say for simplicity, $\lambda = \lambda_1 = \alpha + i\beta$. Then, its conjugate $\overline{\lambda} = \lambda_2 = \alpha - i\beta$ is also an eigenvalue of A. Writing λ_1 in the polar form, we have $\lambda_1 = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta \in [0, 2\pi)$ is the angle of λ_1 in the polar coordinate system. Furthermore, for $k = 0, 1, 2, \ldots, n-1$, $e^{\lambda_1 t}\lambda_1^k = r^k e^{\alpha t} e^{i(\beta t + k\theta)}$, $\lambda_2 = r(\cos\theta - i\sin\theta)$, and $e^{\lambda_2 t}\lambda_2^k = r^k e^{\alpha t} e^{i(-\beta t - k\theta)}$. Note that the real and imaginary parts of the complex solution process corresponding the eigenvalue $\lambda = \lambda_1$ (or $\lambda = \lambda_2$) are also solution processes of the same equation. Furthermore, those two solution processes are linearly independent. Therefore, it is convenient to replace the solution process corresponding to λ_1 with the real part of that solution and the solution process corresponding to λ_2 with the imaginary part, respectively. In addition, if we assume that the remaining $n-2$ eigenvalues of the companion matrix A, denoted λ_i, $i = 3, 4, \ldots, n$, are real and distinct, then the fundamental matrix solution $\Phi_d(t)$ of (6.1.5) is given by

$$\Phi_d(t) = \begin{bmatrix} \text{Re}\left(e^{\lambda_1 t}\right) & \text{Im}\left(e^{\lambda_1 t}\right) & e^{\lambda_3 t} & \ldots & e^{\lambda_n t} \\ \text{Re}\left(\lambda_1 e^{\lambda_1 t}\right) & \text{Im}\left(\lambda_1 e^{\lambda_1 t}\right) & \lambda_3 e^{\lambda_3 t} & \ldots & \lambda_n e^{\lambda_n t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{Re}\left(\lambda_1^{n-2} e^{\lambda_1 t}\right) & \text{Im}\left(\lambda_1^{n-2} e^{\alpha_1 t}\right) & \lambda_3^{n-2} e^{\lambda_3 t} & \ldots & \lambda_n^{n-2} e^{\lambda_n t} \\ \text{Re}\left(\lambda_1^{n-1} e^{\lambda_1 t}\right) & \text{Im}\left(\lambda_1^{n-1} e^{\lambda_1 t}\right) & \lambda_3^{n-1} e^{\lambda_3 t} & \ldots & \lambda_n^{n-1} e^{\lambda_n t} \end{bmatrix}, \qquad (6.3.43)$$

where for $k = 0, 1, 2, \ldots, n-1$, $\text{Re}\left(\lambda_1^k e^{\lambda_1 t}\right) = r^k e^{\alpha t}\cos(k\theta + \beta t)$ and $\text{Im}\left(\lambda_1^k e^{\lambda_1 t}\right) = r^k e^{\alpha t}\sin(k\theta + \beta t)$ are the real and imaginary parts of the complex variable function $\lambda_1^k e^{\lambda_1 t}$. By using the results in equation (6.2.9), the entries of the matrix $\Phi_d^{-1} B \Phi_d$ are as follows:

for the first column,

$$\begin{aligned} \phi_{jn}(t) B_n^r \Phi_{d1}^c(t) &= -\phi_{jn}(t) \sum_{k=1}^{n} \sigma_{k-1} Re\left(\lambda_1^{k-1} e^{\lambda_1 t}\right) \\ &= -\phi_{jn}(t) e^{\alpha t} \sum_{k=1}^{n} \sigma_{k-1} r^{k-1} \cos[(k-1)\theta + \beta t], \quad j = 1, 2, ..., n \end{aligned} \tag{6.3.44}$$

for the second column,

$$\begin{aligned} \phi_{jn}(t) B_n^r \Phi_{d2}^c(t) &= -\phi_{jn}(t) \sum_{k=1}^{n} \sigma_{k-1} Im\left(\lambda_1^{k-1} e^{\lambda_1 t}\right) \\ &= -\phi_{jn}(t) e^{\alpha t} \sum_{k=1}^{n} \sigma_{k-1} r^{k-1} \sin[(k-1)\theta + \beta t], \quad j = 1, 2, ..., n \end{aligned} \tag{6.3.45}$$

and for the $l - th$ column, $l = 3, 4, \ldots, n$,

$$\phi_{jn}(t) B_n^r \Phi_{d1}^c(t) = -\phi_{jn}(t) e^{\lambda_l t} \sum_{k=1}^{n} \sigma_{k-1} \lambda_l^{k-1}, \text{ for } j = 1, 2, ..., n, \tag{6.3.46}$$

Remark 6.3.9 *(*i*) From* (6.3.44) *and* (6.3.45)*, we note that Theorem 6.3.1 is not applicable in this case. The time-varying coefficient rate matrix* $\Phi_d^{-1}(t) B_n^r \Phi_d(t)$ *of the transformed system* (6.2.3) *in Theorem 6.2.1 cannot be reduced to* (6.2.10)*. Thus, we cannot utilize the our method to find close form solution for* (6.1.1)*. To illustrate this, let's consider the second degree equation* (6.3.6) *given above and assume that* $a_1^2 - 4a_0 < 0$ *so that the companion matrix* A *has distinct complex solutions* $\lambda_1 = \alpha + i\beta$ *and* $\lambda_2 = \alpha - i\beta$ *with* $\beta \neq 0$*, then*

$$\Phi_d^{-1}(t) B \Phi_d(t) = \frac{-1}{\beta} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix},$$

where

$$\begin{aligned}
\phi_{11} &= -\sin\beta t[\sigma_0\cos\beta t+\sigma_1(\alpha\cos\beta t-\beta\sin\beta t)]\\
\phi_{12} &= -\sin\beta t[\sigma_0\sin\beta t+\sigma_1(\beta\cos\beta t+\alpha\sin\beta t)]\\
\phi_{21} &= \cos\beta t[\sigma_0\cos\beta t+\sigma_1(\alpha\cos\beta t-\beta\sin\beta t)]\\
\phi_{22} &= \cos\beta t[\sigma_0\sin\beta t+\sigma_1(\beta\cos\beta t+\alpha\sin\beta t)].
\end{aligned}$$

The matrix $\Phi_d^{-1}(t)B\Phi_d(t)$ *is singular. Therefore,* (6.2.3) *does not have a unique solution.*
(ii) *As we shall see in the following example, there are cases where the matrix* $\Phi_d^{-1}(t)B_n^r\Phi_d(t)$ *has exactly one non null column vector.*
Consider the 3rd order linear homogeneous Itô-Doob type stochastic differential equation

$$dy^{(2)}\,+\,[y^{(2)}+y^{(1)}+y]dt\,+\,[y^{(2)}+y]dw(t)\,=\,0. \tag{6.3.47}$$

In the vector form, this equation is written as

$$dx=Axdt\,+\,Bxdw(t)\,=\,0, \tag{6.3.48}$$

where $x\in\mathbb{R}^3$*;* A *and* B *are* 3×3 *deterministic and stochastic companion matrices in* (6.1.5) *and* (6.2.1) *relative to* (6.3.48), *and they are as:*

$$A\;=\begin{bmatrix}0&1&0\\0&0&1\\-1&-1&-1\end{bmatrix}\;\text{and } B\;=\begin{bmatrix}0&0&0\\0&0&0\\-1&0&-1\end{bmatrix}. \tag{6.3.49}$$

The fundamental matrix solution of the deterministic part of (6.3.48) *is*

$$\Phi_d(t)\;=\begin{bmatrix}\cos t&\sin t&e^{-t}\\\sin t&\cos t&-e^{-t}\\-\cos t&-\sin t&e^{-t}\end{bmatrix} \tag{6.3.50}$$

and its inverse is

$$\Phi_d^{-1}(t) = \frac{1}{2}\begin{bmatrix} \cos t - \sin t & -2\sin t & -(\cos t + \sin t) \\ \cos t + \sin t & 2\cos t & \cos t - \sin t \\ e^t & 0 & e^t \end{bmatrix}. \tag{6.3.51}$$

The stochastic system of differential equations (6.2.3) *in the context of* (6.3.48) *is*

$$dc = \Phi_d^{-1}(t)B\Phi_d(t)cdw(t) = \begin{bmatrix} 0 & 0 & (\cos t + \sin t)e^{-t} \\ 0 & 0 & -(\cos t - \sin t)e^{-t} \\ 0 & 0 & -1 \end{bmatrix} cdw(t). \tag{6.3.52}$$

Thus, the time varying coefficient matrix rate matrix (6.2.3) *corresponding to the given stochastic differential equation satisfies the hypothesis of Lemma 6.2.3. Therefore, the application of Theorem 6.2.6, the condition (a) of Theorem 6.2.6 assures the feasibility of a close form solution. Moreover, condition (c) of the same theorem provides the close form representation of the general solution of the given differential equation.*

Thus, we have

$$\begin{aligned} y(t) &= c_3\Big[e^{-(3/2)t-w(t)} + \cos t\int_0^t(\cos s + \sin s)e^{-(3/2)s-w(s)}dw(s) \\ &\quad - \sin t\int_0^t(\cos s - \sin s)e^{-(3/2)s-w(s)}dw(s)\Big] + c_1\cos t + c_2\sin t. \end{aligned} \tag{6.3.53}$$

The following example is a modified version of Chandrasekhar equation. This version incorporates multiplicative noise rather than the additive noise as in the original version.

Applications: A modified Chandrasekhar equation

The theory of the Brownian motion of a free particle (i.e., in the absence of an external field of force) generally starts with the Langevin's equation

$$\frac{du}{dt} = -\beta u + a(t), \tag{6.3.54}$$

where u denotes the velocity of the particle. According to this equation, the influence of the surrounding medium on the motion can be split up into two parts: first, a systematic part $-\beta u$ representing a dynamical friction experienced by the particle and second, a fluctuating part $a(t)$ which is characteristic of the Brownian motion. $a(t)$ is independent of u and varies extremely rapidly compared to the variations of u. Chandrasekhar [12] generalized the Langevin equation by considering the presence of an external field of force which leads to the following equation

$$\frac{du}{dt} = -\beta u + a(t) + K(r,t), \tag{6.3.55}$$

where $K(r,t)$ is the acceleration produced by the field. The method of solution is illustrated sufficiently by a one-dimensional harmonic oscillator describing Brownian motion.

$$\frac{du}{dt} = -\beta u + a(t) - \omega^2 t, \tag{6.3.56}$$

where ω is denotes the circular frequency of the oscillator. Alternatively, equation (6.3.55) can be written in the form

$$\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + \omega^2 y = a(t), \tag{6.3.57}$$

or as it is known today

$$d\dot{y} + (\beta\dot{y} + \nu y)dt = \sigma dw(t), \tag{6.3.58}$$

for $\sigma \neq 0$ and $\beta > 0$. If the system is also subject to (the influence of) external environmental random perturbations, the equation (6.3.58) becomes

$$d\dot{y} + (\beta_1\dot{y} + \beta_0 y)dt + (\sigma_1\dot{y} + \sigma_0 y)dw(t) = 0, \tag{6.3.59}$$

with $\beta_1,\ \beta_0,\ \sigma_1 > 0$. The companion matrices for this equation are

$$A = \begin{bmatrix} 0 & 1 \\ -\beta_0 & -\beta_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ -\sigma_0 & -\sigma_1 \end{bmatrix}. \tag{6.3.60}$$

The eigenvalues of matrix A are $\lambda_1 = \frac{-\beta_1 - \sqrt{\beta_1^2 - 4\beta_0}}{2}$ and $\lambda_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\beta_0}}{2}$. If $\beta_1^2 > 4\beta_0$, these eigenvalues are real and distinct. In this case, the solution process of (6.3.59) for is of the form

$$y(t) \;=\; c_1 e^{-\sigma_0 t/\sigma_1} + c_2(\xi + \sigma_1^2/2)e^{-\sigma_0 t/\sigma_1} \int_0^t \exp\left[\xi s - \sigma_1 w(s)\right] ds, \tag{6.3.61}$$

where $\xi = -\beta_1 + \frac{2\sigma_0}{\sigma_1} - \frac{\sigma_1^2}{2}$, c_1 and $c_2 \neq 0$ are arbitrary constants. If $\beta_1^2 = 4\beta_0$ there is one repeated eigenvalue, namely $\lambda = -\beta_1/2$. In this case, equation (6.3.59) yields the following solution

$$y(t) \;=\; c_1 e^{-\beta_1 t/2} + c_2 e^{-\beta_1 t/2} \int_0^t \exp\left(-\frac{1}{2}\sigma_1^2 s - \sigma_1 w(s)\right) ds, \tag{6.3.62}$$

where c_1 and $c_2 \neq 0$ are arbitrary constants.

If $\beta_1^2 < 4\beta_0$ the companion matrix A has two complex eigenvalues. As discussed earlier, in this case our method for finding solution of (6.3.59) is not feasible.

Remark 6.3.10 *From Remark 6.3.8, we note that when the companion matrix A has repeated or distinct real eigenvalues, the solution process of equation* (6.3.6) *is an isomorphic transformation of the exponential functional of Brownian motion*

$$B_t^{(\mu)} = \int_0^t \exp(\sigma w(s) + \mu s) ds, \tag{6.3.63}$$

where $\sigma \neq 0$ and μ are arbitrary real constants. Therefore, to find probability distribution of $y(t)$, it is enough to obtain the one of $B_t^{(\mu)}, t \geq 0$.

The distribution of the process $\{B_t^{(\mu)}, t \geq 0\}$ has been subject of extensive research in the past *two decades thanks to it applications in fields of mathematical finance, diffusion processes in random environments.*

In the following, we examine the probability distribution of the solution processes $y(t)$ of the second order ($n = 2$) stochastic differential equation (6.3.6) in the context of repeated or distinct real eigenvalues of companion matrix A.

6.4 The Probability distribution $y(t)$ when $n = 2$

The probability distribution of the solution process $y(t)$ (for $n = 2$) is identical to the that of $A_t = \int_0^t \exp\left[\xi s - \sigma_1 w(s)\right] ds$. A_t is exponential functional of Brownian motion which has been of interest to researchers in Mathematical finance, diffusion processes in random environments, stochastic analysis related to Brownian motions on hyperbolic spaces. Various approaches were adopted to determine the law of A_t. One approach by Dufresne [22] focusses on the reciprocal of the integral A_t. A partial differential equation is derived for its Laplace transform and then used to derive an expression for the density. Another very popular approach derives the law of A_t thought Bessel processes [62, 76]. Thanks to scaling properties of Brownian motion, to determine the law of A_t, it suffices to determine the law of $A_t^{(\nu)} = \int_0^t \exp\left[2(w(s) + \nu s)\right] ds$. The conversion rule between A_t and $A_t^{(\nu)}$ is given by (ref. [10], pp. 43)

$$\int_0^T \exp[\mu s + \sigma w(s)] ds \overset{\text{law}}{=} \frac{4}{\sigma^2} A_t^{(\nu)}, \quad t = \frac{\sigma^2 T}{4}, \quad \nu = \frac{2\mu}{\sigma^2}, \tag{6.4.1}$$

where $T > 0$ and $t \in [0, T]$.

The probability distribution of $A_t^{(\nu)}$, taken at a fixed time t (was obtained by Yor [75]) is determined by

$$P\left(A_t^{(\nu)} \in du | w(t) + \nu t = x\right) \overset{\text{def.}}{=} a_t(x, u) du. \tag{6.4.2}$$

In relation (6.4.2), $a_t(x,u)$ satisfies

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) a_t(x,u) = \frac{1}{u} \exp\left(-\frac{1}{2u}(1+e^{2x})\right) \theta\left(\frac{e^x}{u}, t\right) \tag{6.4.3}$$

where

$$\theta(r,t) = \frac{re^{\pi^2/2t}}{\sqrt{2\pi^3 t}} \int_0^\infty e^{-\xi^2/2t} e^{-r\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi \tag{6.4.4}$$

This means that the probability density function of $A_t^{(\nu)}$ is the integral of $a_t(\cdot,u)$ times the normal density function with mean νt and variance t. It follows [76] from (6.4.2) that

$$P\left(A_t^{(\nu)} \in du\right) = du\gamma_t^{(\nu)}(u), \tag{6.4.5}$$

where

$$\gamma_t^{(\nu)}(u) = \int_0^\infty \theta(z,t)(uz)^{\nu-1} \exp\left(\frac{-\nu^2 t}{2}\right) \exp\left[-\frac{1}{2}\left(\frac{1}{u} + uz^2\right)\right] dz \tag{6.4.6}$$

Given the complexity of the distribution of $A_t^{(\nu)}$, it is worth looking into its moments to gain some partial information about its distribution.

The Moments

Although the first two moments of $A_t^{(\nu)}$ can easily by obtained by integration, higher moments are not obtained that way. The higher moments were can be computed using Itô's Lemma, time reversal, and a recurrence argument were (Dufresne [20, 21]) or using Laplace transform (Geman and Yor [76], pp. 49-54). They obtained the formula

$$E\left[\left(\int_0^t \exp\left[\lambda(\nu s + w(s))\right] ds\right)^n\right] = \frac{n!}{\lambda^{2n}} \left\{\sum_{j=0}^n d_j^{(\nu/\lambda)} \exp\left[\left(\frac{\lambda^2 j^2}{2} + \lambda j \nu\right) t\right]\right\}, \tag{6.4.7}$$

where

$$d_j^{(\beta)} = 2^n \prod_{\substack{i \neq j \\ 0 \leq i \leq n}} \left[(\beta + j)^2 - (\beta + i)^2 \right]^{-1}, \tag{6.4.8}$$

By applying this result to the solution process $\{y(t),\ t \geq 0\}$ in its formulation presented in Remark 6.3.8 in the case of distinct and repeated eigenvalues of matrix A, we have the following formula for the moments of $y(t)$:

$$\begin{aligned} E\left[(y(t))^n\right] &= \sum_{k=0}^{n} \binom{n}{k} P^{n-k}(t) Q^k(t) E\left[\left(\int_0^t \exp\left[\lambda(\nu s + w(s))\right] ds \right)^k \right] \\ &= \sum_{k=0}^{n} \binom{n}{k} P^{n-k}(t) Q^k(t) \frac{k!}{\lambda^{2k}} \left\{ \sum_{j=0}^{k} d_j^{(\nu/\lambda)} \exp\left[\left(\frac{\lambda^2 j^2}{2} + \lambda j \nu \right) t \right] \right\}, \end{aligned} \tag{6.4.9}$$

where $d_j^{(\beta)}$ is defined in (6.4.8) and $\lambda = -\sigma_1,\quad \nu\lambda = \xi$; $P(t)$, $Q(t)$ and ξ are defined in Remark 6.3.8. This result is subject to the existence of the $E\left[|y(t)|^n\right]$.

7 Future Research Plan

In this work, by reformulating the concept of stochastic differential equations in the context of multi-time scale setting. This mathematical model reformulation incorporate dynamic effects of internal structural and external environmental perturbations, described by attributable parameters or variables. We investigated the basic fundamental properties of existence and uniqueness of solutions of this type of differential equations. We also presented methods for solving both linear and nonlinear stochastic fractional differential equations. And finally, we explored applications in ecology and epidemiology. we presented a numerical schemes for fractional stochastic differential equations and simulation results for this type of equations thereafter. In our current research project, we plan to investigate a model formulation under multiple time-scales that involves the case with $0 < \alpha < 1/2$. In the last chapter of the presented work, we provided a method of finding classes of solution processes of higher order stochastic differential equations (HOSDE). Although higher order deterministic and matrix differential equations are available in literature, we note that HOSDE remains an interesting problem. Our current research project includes applications of such equations and as well as the stability of their solution processes.

Future research work includes, but not limited to:

1. Generalize the fractional time scale to include $0 < \alpha < 1/2$.
2. Apply the concept of three time scales to economy and finance.
3. Carry out statistical inference on the the coefficient functions of three time-scale dynamic processes.

4. Determine the exact distribution of the solution of higher order stochastic differential equations presented in chapter 6

5. Find more applications for the higher order stochastic differential equations.

6. Develop theoretical and computational tools to investigate data sets drawn from the dynamic processes with multiples attributes that are described by the parameters.

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