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Integrable Couplings of the Kaup-Newell Soliton Hierarchy

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by

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Abstract

By enlarging the spatial and temporal spectral problems within a certain Lie algebra, a hierarchy of integrable couplings of the Kaup-Newell soliton equations is constructed. The recursion operator of the resulting hierarchy of integrable couplings is explicitly computed. The integrability of the new coupling hierarchy is exhibited by showing the existence of infinitely many commuting symmetries.

Chapter 1
Introduction

1.1 The discovery of solitons and the KdV equation

Nonlinear science is a fundamental science frontier that includes researches in the common properties of nonlinear phenomena. Since it first appeared in the 60s of the last century, the studies in this area have been widely involved in many subjects of natural and social sciences. Solitons, known as an important branch of the nonlinear science, together with Fractals and Chaos, have developed rapidly in recent decades. It has become quite useful in describing and explaining certain stable natural phenomena, like water waves and light waves traveling in optical fiber, and in finding solutions of nonlinear partial differential equations.

The discovery of solitons dates back to 1834, when British engineer J. Scott Russell was observing the motion of a boat that's drawn along a narrow channel by horses; he noticed that after the boat stopped suddenly, there was a "large solitary elevation in the form of rounded, smooth and well-defined heap of water" stimulated by the boat, which kept moving along the channel, without change of form or slowing down. He chased the wave for almost two miles till it stopped in the winding of the channel. Shortly after, he verified the existence of this type of solitary waves by recreating some long, shallow water waves of constant form in a laboratory scale wave tank [1].

Since then, many other scientists, including Stokes, Airy, Boussinesq and Rayleigh, also started doing researches on this phenomenon. Among these researchers, Boussinesq and Rayleigh both acquired the approximate descriptions of the solitary wave, and Boussinesq derived a one-dimensional evolution equation.

Later on, in 1895, while studying the traveling of shallow water waves created in a shallow channel, D. Korteweg and G. de Vries [2] obtained a new one-dimensional, nonlinear equation describing this specific type of water waves. This equation is now known as the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.1}$$

Later, in 1965, N. J. Zabusky and M. D. Kruskal [17] discovered some interesting results while analyzing the nonlinear interacting process of the collision between solitary waves in plasma. After some numerical simulations on computer, the outcome of the experiments was quite surprising: if two solitary waves traveling at different speeds collided, after the collision each of them would keep their original shape, energy and momentum, moving on toward the directions they were headed to. Because of this remarkable stability which is usually observed in elastic particles, they named the solitary wave solutions of nonlinear differential equations "solitons". Since then, solitons have been discovered in many substances other than shallow water - scientists also found them in solid state physics, plasma physics and optical experiments, etc.

Now in applied mathematics and engineering, a soliton usually refers to a localized traveling wave solution of non-linear equations, the speed and shape of which doesn't or almost doesn't change after collided with other solitons.

Since soliton theory has been established, finding solutions of nonlinear differential equations has become a very important, meaningful research topic in the field of nonlinear science.

Finding exact solutions is not only helpful in understanding the essential attributes and algebraic structures of soliton equations, but also gets to explain some related natural phenomena. The study in this field, besides numerical calculation and simulations on computer, is mainly looking for exact solutions in explicit form. And while the theory of solitons being developed, scientists have made some quite considerable progress and acquired many valuable results. Most importantly, they have developed a variety of, now widely used, methods of solving such equations, for example, the inverse scattering transformation, Lie symmetry analysis, Darboux transformation, Bäcklund transformation, Hirota bilinear method, and the Wronskian determinant method.

1.2 Inverse scattering transformation

In 1967, while studying solutions of the KdV equation, an American research team GGKM- C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura [18][19]- brought up a new analytical method- Inverse Scattering Transformation(IST), for solving the initial value problem of the KdV equation, and found the N-wave interaction solution of it.

First, they noticed that between the KdV equation (1.1) and the mKdV equation:

$$v_t + 6v^2 v_x + v_{xxx} = 0, \tag{1.2}$$

there exists a Miura transformation:

$$u = -(v_x + v^2), \tag{1.3}$$

and after having v replaced with a Cole-Hopf transformation

$$v = \psi_x / \psi, \tag{1.4}$$

they got a new equation:

$$\psi_{xx} + u(x,t)\psi = 0. \tag{1.5}$$

Since the Galilean transformation, $u \to u - \lambda$, $t \to t$, $x \to x + 6\lambda t$, does not change the KdV equation, the linear equation (1.5) is equivalent to another one-dimensional stationary Schrödinger equation:

$$\psi_{xx} + u(x,t)\psi = \lambda\psi. \tag{1.6}$$

Now if the potential function $u(x,t)$ satisfies the KdV equation (1.1), then the spectral parameter λ of the equation (1.6) is time-independent and the evolution of the eigenfunction ψ along time can satisfy the equation:

$$\psi_t + \psi_{xxx} - 3(\lambda - u)\psi_x = 0. \tag{1.7}$$

So with applying the scattering method [3] of the Schrödinger equation in quantum mechanics, they got the scattering data $S(0)$ of the potential function $u(x,0)$ from (1.6). Then they formed from (1.7) a system of ordinary differential equations, describing the evolution of scattering data along time, the solution of which at time t is just the scattering data $S(t)$. After that, through some reversion procedure, the potential function $u(x,t)$ of the Schrödinger equation was obtained, and gives a solution of the KdV equation described in (1.1).

Now IST method has been applied successfully to many other nonlinear evolution equations, and has become a systematic method for solving soliton equations. The method has both a strict physical background and a mathematical rigour, and can be used to find multiple soliton solutions for a whole hierarchy of soliton equations, which are all related to one same spectral problem.

One year after GGKM brought up the IST method, P. D. Lax [20-22] developed this method into a more generalized form. The main idea behind it is using the linear method to solve nonlinear problems and so the solution process can be simplified greatly: by introducing two new linear operators, called a Lax pair, the problem of a nonlinear differential equation can be transformed into a linear spectral problem and an auxiliary problem; and the compatible condition for these two problem generates a new equation, which is equivalent to the original and is called a Lax form [3]. From then till today, the Lax pair has stayed an important study area in soliton theory: V. E. Zakharov and A. B. Shabat got soliton solutions of the nonlinear Schrödinger equation [23,24]; Ablowitz, Kaup, Newell and Segur [13,25] applied it into many other nonlinear evolution equations, and developed a more completed and generalized analytical method. These research results have already been put into many practical applications.

1.3 The integrability of soliton equations

During the 19th century, the research in finding solutions of integrable systems ran into much difficulties, and only few examples were solved with explicit solutions. People started to realize that not all nonlinear systems are integrable, and complete integrability needs special attention.

In 1887, E. H. Brun [30] proved that the famous three-body problem is not integrable, for it doesn't have sufficiently many conserved integrals: among the eighteen second-order differential equations formed the problem, there are only ten integrals of motion- three momentum integrals, three angular momentum integrals, three integrals on motion of centroid, and one energy integral. Later on, Jules Henri Poincaré applied Bruns' results of research into cases with the existing of perturbation parameter, and gave a more generalized conclusion: in most conservative problems, the canonical equations in classical mechanics won't satisfy any analytic integral other than the energy integral. And this result indicated that integrable systems may stop being continuous after received a perturbation, and even the slightest alteration in the parameters or initial conditions could cause some quite complicated changes in their properties.

Now the study in the integrability of nonlinear equations has become an essential part of soliton theory. During the recent decades, researches in this field have been quite active, and produced lots of insightful results.

For Hamiltonian systems of finite dimensions, a complete geometrical theoretical framework was well constructed. The well-known Liouville-Arnold theorem [28], for instance, states a sufficient condition for complete integrability: for an N-dimensional Hamiltonian system, if there exists N independent and mutually commuting conserved integrals, then it is an integrable system, usually called Liouville integrable.

In the case of infinite-dimensional systems, however, the situation gets much more complicated: the existence of infinitely many involutive integrals of motion is not sufficient for getting a solution in explicit form; therefore, for integrable systems of infinite dimensions, no exact similar integrability theorem was established yet. As a result, some other definitions for integrability were adopted: for example, according to the inverse scattering transformation (IST), if a nonlinear soliton equation can be reformulated to its linear Lax form [28], this equation, called Lax integrable, can usually be solved; and if there exist infinitely many symmetries for a system, it is also regarded as an integrable system [29].

Now, a crucial part of soliton theory is to find the soliton equations based on Lax pairs, by applying structures of Lie algebras, extensions in differential geometry, etc. During the past 30 years, the research in this area has been quite fruitful: scientists found many new Lax pairs and got new soliton equations. Besides the KdV hierarchy, some other well-known examples of soliton hierarchies are the MKdV hierarchy [3], the AKNS hierarchy, the Kaup-Newell hierarchy [16], the Benjamin-Ono hierarchy [4], the Dirac hierarchy [5].

1.4 Integrable couplings

By now, most of the 2×2 matrix Lax pairs for soliton equations have already been considered, and constructing new Lax pairs, as a method to find new soliton equations, could be quite cumbersome, especially for those of 3×3 or higher orders.

As a result, a new method to form Lax pairs- constructing new soliton hierarchies by coupling the existing systems of the same type- was developed in the 1990s [12] [36]. And it is now known as integrable couplings [7] [36]: when an integrable system is given, we can construct a new non-trivial integrable differential equation system, which contains the given equations as a sub-system.

By using this method, the workload in computations is greatly reduced: constructing Lax pairs of 4×4 matrices, for instance, would become much easier by coupling 2×2 matrices, since the calculations mostly happen in the 2×2 matrix blocks. So based on the Lax pairs and zero curvature representations of lower orders that have already been found, we can build many more new soliton hierarchies with similar properties yet more components.

There are several techniques adopted to construct such integrable couplings, for example, enlarging the spectral problem, using perturbations, creating new loop algebras and semi-direct sums of

Lie algebra [8].

During the last 16 years, lots of work has been done in this area, and many integrable couplings were formed from well-known soliton equations, for instance, the coupled KdV hierarchies [9,10], the coupled Harry-Dym hierarchies [11], and the coupled Burgers hierarchies [12].

In this thesis, we are going to present the procedure of integrable couplings, and construct an integrable coupling hierarchy of the Kaup-Newell hierarchy, by coupling the existing evolution equations, through the enlarged 4×4 matrices Lax pairs and zero curvature representations constructed from a non-semi-simple Lie algebra. And after that, we will verify that this new hierarchy of soliton equations is integrable like the Kaup-Newell hierarchy.

Chapter 2

Lax integrability and integrable couplings

2.1 Lax integrability and infinitely many symmetries

In order to solve a nonlinear soliton equation, we now introduce two auxiliary spectral problems:

$$\begin{cases} \varphi_x = U\varphi = U(u,\lambda)\varphi, \\ \varphi_t = V\varphi = V(u.\lambda)\varphi, \end{cases} \tag{2.1}$$

where the spatial spectral matrix U and the temporal matrix V form a Lax pair for the soliton equation.

Assuming that there exists an eigenfunction $\varphi(x,t)$, and then we have:

$$\varphi_{xt} = \varphi_{tx}, \tag{2.2}$$

and according to (2.1),

$$\begin{aligned} \varphi_{xt} &= (U\varphi)_t = U_t\varphi + U\varphi_t \\ &= (U_t + UV)\varphi, \end{aligned}$$

$$\begin{aligned} \varphi_{tx} &= (V\varphi)_x = V_x\varphi + V\varphi_x \\ &= (V_x + VU)\varphi. \end{aligned}$$

Therefore, the compatible condition for the Lax system will be:

$$U_t - V_x + [U,V] = 0, \tag{2.3}$$

where the binary product [,] between matrices U, V is defined as:

$$[U,V] = UV - VU.$$

The equation (2.3), called the zero curvature representation equation, is the linear Lax form of the original nonlinear soliton equation. And every Lax integrable soliton equation normally has such a zero-curvature representation.

For a differential equation that does or doesn't have any explicit solution, there are many different types of definitions on its integrability. In this thesis, we are applying the sufficient condition for Lax integrability and the mutual commutativity of the symmetries.

Now, in order to define the Lie bracket [,], we first give the definition of the Gateaux derivative of an object K in direction η:

$$K'[\eta] = K'(u)[\eta] = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} K(u + \epsilon\eta); \tag{2.4}$$

then the Lie bracket [,] between two given vector fields $K = K(x, t, u)$ and $S = S(x, t, u)$ reads as:

$$[K, S] = K'(u)[S] - S'(u)[K]. \tag{2.5}$$

A vector field $S(u)$, independent of t explicitly, is called a symmetry of an equation $u_t = K(u)$, if it satisfies the equation $[K, S] = 0$.

Given an evolution equation

$$u_t = K(u) = K(x, t; u, u_x, u_{xx}, ...), \tag{2.6a}$$

where u is a column vector of dependent variables, one can form a hierarchy of equations, called a soliton hierarchy:

$$u_{t_m} = K_m(u) = K_m(x, t; u, u_x, u_{xx}, ...), \tag{2.6b}$$

by using the zero curvature equations. Any member in this hierarchy is a symmetry of the equation (2.6a), and commutes with any other member..

This means that the symmetries of each equation in the hierarchy consist of vector fields $\{K_m : m = 0, 1, 2, ...\}$, which have the mutual commutativity:

$$[K_m, K_n] = 0, \qquad m, n = 0, 1, 2... \tag{2.7}$$

2.2 Integrable couplings

Now based on a given integrable soliton hierarchy, we can build a new enlarged integrable soliton hierarchy by constructing integrable couplings. Some widely used techniques in this procedure are enlarging spectral problems, creating new loop Lie algebras, and applying perturbations.

For the soliton equation (2.6a), we add another equation

$$u_{1,t} = S(u, u_1), \tag{2.8}$$

to build a new system [8]:

$$\bar{u}_t = \begin{pmatrix} K(u) \\ S(u, u_1) \end{pmatrix}, \tag{2.9}$$

where the new dependent variable is

$$\bar{u} = \begin{pmatrix} u \\ u_1 \end{pmatrix}. \tag{2.10}$$

If this new triangular system is still integrable, then we say that the enlarged integrable system (2.9) is an integrable coupling of (2.6a).

The expression of such an $S(u, u_1)$ is based on the following new zero curvature representation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0, \tag{2.11}$$

with enlarged Lax pair matrices $\bar{U}, \bar{V}$.

Any Lie algebra can be written as the semidirect sum of a semisimple algebra and a solvable algebra, and so we can use semidirect sums of Lie algebras of matrices [8] to construct such enlarged Lax pair matrices $\bar{U}, \bar{V}$.

Assume that the Lax pair (U, V) belongs to a matrix Lie algebra g, and the standard Lie product between square matrices is defined by

$$[A, B] = AB - BA.$$

In order to construct an integrable coupling of the soliton hierarchy (2.6), we can try to enlarge the original Lie algebra g.

If we are able to find a solvable matrix Lie algebra g_c, which satisfies the closure property between g and g_c under matrix multiplication:

$$g g_c, g_c g \subseteq g_c,$$

then we can enlarge the original Lie algebra by taking the semidirect sum of the two algebras g and g_c:

$$\bar{g} = g \in\!\!\!\!\!+ \; g_c,$$

where the semidirect sum means

$$[g, g_c] \subseteq g_c,$$

i.e.,

$$\forall A \in g, B \in g_c, \quad [A, B] \in g_c.$$

Apparently, g_c here is an ideal Lie sub-algebra of $\bar{g}$. The new enlarged Lax pair matrices will be taken from the semidirect sum of Lie algebras $\bar{g}$, and they have the form:

$$\bar{U} = \begin{pmatrix} U & U_1 \\ 0 & U \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} + \begin{pmatrix} 0 & U_1 \\ 0 & 0 \end{pmatrix},$$

$$\bar{V} = \begin{pmatrix} V & V_1 \\ 0 & V \end{pmatrix} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} + \begin{pmatrix} 0 & V_1 \\ 0 & 0 \end{pmatrix},$$

where the matrices $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ and $\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix}$ belong to a semi-simple loop algebra g, while $\begin{pmatrix} 0 & U_1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & V_1 \\ 0 & 0 \end{pmatrix}$ are elements of a solvable Lie algebra g_c.

Now we can form a pair of enlarged matrix spectral problems with $\bar{U}$ and $\bar{V}$:

$$\begin{cases} \bar{\varphi}_x = \bar{U}\bar{\varphi} = \bar{U}(\bar{u}, \lambda)\bar{\varphi}, \\ \bar{\varphi}_t = \bar{V}\bar{\varphi} = \bar{V}(\bar{u}, \lambda)\bar{\varphi}, \end{cases} \tag{2.12}$$

while the compatibility of these two spectral problems yields the new enlarged zero curvature equation (2.11).

The example in the next section will show more clearly how we generate such integrable couplings by the above scheme.

2.3 The KdV and dark KdV hierarchies

Now we take the KdV hierarchy, for example, to discuss about zero curvature representations and integrable couplings.

As mentioned before in the introduction, we have the standard form of the KdV equation (1.1)

$$u_t + 6uu_x + u_{xxx} = 0.$$

When using the IST method, it has the eigenvalue problems, i.e., the Lax pair [3]:

$$P\psi = \lambda\psi, \qquad P = \frac{\partial^2}{\partial x^2} + u, \tag{2.13a}$$

$$\psi_t = B\psi, \qquad B = \frac{\partial^3}{\partial x^3} + m_1\frac{\partial}{\partial x} + m_2. \tag{2.13b}$$

From the compatibility condition, the Lax form of the KdV equation is:

$$\frac{\partial P}{\partial t} + [P, B] = 0,$$

which leads to a solution for operator B:

$$m_1 = \frac{3}{2}u, \qquad m_2 = \frac{3}{4}u_x,$$

and a equation

$$\frac{\partial u}{\partial t} = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}. \tag{2.14}$$

According to (2.13a), we have

$$\psi_{xx} - \lambda\psi = u\psi.$$

Therefore when we set

$$\begin{cases} \varphi_1 &= \psi, \\ \varphi_2 &= \varphi_{1,x}, \end{cases}$$

and

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

we get:

$$\varphi_x = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_x = \begin{pmatrix} \varphi_2 \\ \varphi_{1,xx} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{2.15}$$

which shows that the spacial spectral matrix operator for the KdV equation [33] is:

$$U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}. \tag{2.16}$$

For the temporal problem, we set

$$\varphi_t = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \varphi, \tag{2.17}$$

and unfold the matrix operator into a Lauren series:

$$V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{i=0}^{\infty} V_i \lambda^{-i} = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i}. \tag{2.18}$$

First, when considering the stationary-curvature equation $V_x = [U, V]$, we have the following results:

$$\begin{cases} a_x &= c - (\lambda - u)b, \\ b_x &= -2a, \\ c_x &= 2(\lambda - u)a, \end{cases} \tag{2.19}$$

and the recurrence relation:

$$\begin{cases} b_{0,x} = 0, b_{i+1} = Lb_i, \\ a_i = -\frac{1}{2} b_{i,x}, \\ c_i = -\frac{1}{2} b_{i,xx} + b_{i+1} - ub_i, \\ L = \frac{1}{4}\partial^2 + u - \frac{1}{2}\partial^{-1}u_x, \end{cases} \qquad i \geqslant 0, \tag{2.20}$$

where ∂ stands for the partial differentiation with respect to x.

Now we set the initial conditions as

$$b_0 = 0, \quad b_1 = 1;$$

also

$$c_0 = 1, \quad c_1 = -\frac{1}{2}u;$$

$$a_0 = a_1 = 0; \tag{2.21}$$

and take all the constants produced in the integration as zero, i.e.,

$$b_i|_{u=0} = c_i|_{u=0} = a_i|_{u=0} = 0, \quad i = 2, 3, 4, ...$$

Upon calculating, we obtain the expression of the coefficients in the Lauren series (2.18). For example,

$$b_2 = \frac{1}{2}u, \quad b_3 = \frac{1}{8}u_{xx} + \frac{3}{8}u^2,$$

$$b_4 = \frac{1}{32}u_{xxxx} + \frac{5}{32}u_x^2 + \frac{5}{16}uu_{xx} + \frac{5}{16}u^3.$$

Then in order to make U, V fit in the compatibility condition of the problems (2.15) and (2.17), i.e., the zero-curvature equation:

$$U_t - V_x + [U, V] = 0,$$

we introduce a modification term:

$$\Delta_m = \begin{pmatrix} 0 & 0 \\ -b_{m+2} & 0 \end{pmatrix}, \quad m \geqslant 0,$$

and define $V^{[m]}$ as

$$V^{[m]} = (\lambda^{m+1} V)_+ + \Delta_m, \quad m \geqslant 0, \tag{2.22}$$

which is the operator of the m-th temporal spectral problem $\varphi_{t_m} = V^{[m]}\varphi$ in the soliton hierarchy.

As a result, the hierarchy of the KdV soliton equations has the form:

$$\begin{aligned} u_{t_m} &= K_m = 2b_{m+2,x} \\ &= 2\partial L b_{m+1} = ... = 2\partial L^m b_2, \quad m \geqslant 0. \end{aligned} \tag{2.23}$$

And the first equation in the hierarchy is just the KdV equation:

$$u_{t_1} = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx},$$

which is the same result as we got before, in (2.14).

Now with the results we have obtained about the spectral problems (2.15) and (2.17), we can construct integrable couplings of the KdV hierarchy, by introducing:

$$\bar{U}(\bar{u}, \lambda) = \begin{pmatrix} U & U_1 \\ 0 & U \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & 0 \\ -u_1 & 0 \end{pmatrix},$$

$$\bar{V}(\bar{u}, \lambda) = \begin{pmatrix} V & V_1 \\ 0 & V \end{pmatrix}, \quad V_1 = \begin{pmatrix} e & f \\ g & -e \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} e_i & f_i \\ g_i & -e_i \end{pmatrix} \lambda^{-i}, \tag{2.24}$$

where $\bar{u} = \begin{pmatrix} u \\ u_1 \end{pmatrix}$, and the spectral matrices U, V are defined in (2.16), (2.18), respectively. Then, based on the enlarged stationary zero curvature equation $\bar{V}_x = [\bar{U}, \bar{V}]$, the components of matrix V_1 satisfies:

$$\begin{cases} e_x = g - (\lambda - u) f + u_1 b, \\ f_x = -2e, \\ g_x = 2(\lambda - u) e - 2u_1 a. \end{cases} \tag{2.25}$$

So after using the expressions of a, b and c in (2.19), with some calculation, we get:

$$\begin{cases} e = -\frac{1}{2} f_x, \\ g = -\frac{1}{2} f_{xx} + (\lambda - u) f - u_1 b, \\ -\frac{1}{4} f_{xxx} - \frac{1}{2} u_x f + (\lambda - u) f_x = \frac{1}{2}(u_1 b)_x + \frac{1}{2} u_1 b_x, \end{cases} \tag{2.26a}$$

with a recursive formula for f:

$$f_{i+1} = L f_i + \frac{1}{2} u_1 b_i + \frac{1}{2} \partial^{-1} u_1 b_{i,x}, \qquad i \geqslant 0, \tag{2.26b}$$

where L is same as the recursive operator defined in (2.20).

Furthermore, if we set the initial data

$$f_0 = f_1 = 0,$$

we can get the expressions of f_i, $i = 2, 3, 4....$

For example:

$$f_2 = \frac{1}{2} u_1, \qquad f_3 = \frac{1}{8} u_{1,xx} + \frac{3}{4} u u_1,$$
$$f_4 = \frac{1}{32} u_{1,xxxx} + \frac{5}{16} u_x u_{1,x} + \frac{5}{16} u_{xx} u_1 + \frac{5}{16} u u_{1,xx} + \frac{15}{16} u^2 u_1.$$

Now we go back to the general form of zero curvature equation:

$$\bar{U}_t - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0, \qquad m \geqslant 0. \tag{2.27}$$

If we take

$$\bar{V}^{[m]} = \begin{pmatrix} V^{[m]} & V_1^{[m]} \\ 0 & V^{[m]} \end{pmatrix},$$

$$V^{[m]} = (\lambda^{m+1} V)_+ + \begin{pmatrix} 0 & 0 \\ -b_{m+2} & 0 \end{pmatrix}, \qquad V_1^{[m]} = (\lambda^{m+1} V_1)_+ + \begin{pmatrix} 0 & 0 \\ -f_{m+2} & 0 \end{pmatrix}, \tag{2.28}$$

we will get a hierarchy of integrable couplings of the KdV equations:

$$\bar{u}_{t_m} = \begin{pmatrix} u \\ u_1 \end{pmatrix}_{t_m} = \begin{pmatrix} 2b_{m+2,x} \\ 2f_{m+2,x} \end{pmatrix}, \qquad m = 0, 1, 2, ... \tag{2.29}$$

This is known as the hierarchy of dark KdV equations [33], a special hierarchy of integrable couplings. As a result, the first integrable coupling in this hierarchy has the form:

$$\bar{u}_{t_1} = \begin{pmatrix} u \\ u_1 \end{pmatrix}_{t_1} = \begin{pmatrix} 2b_{3,x} \\ 2f_{3,x} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} u u_x + \frac{1}{4} u_{xxx} \\ \frac{3}{2}(u u_1)_x + \frac{1}{4} u_{1,xxx} \end{pmatrix}. \tag{2.30}$$

Chapter 3

Integrable couplings of the Kaup-Newell hierarchy

3.1 The Kaup-Newell soliton hierarchy

The Kaup-Newell soliton hierarchy [3] was first introduced by David J. Kaup and Alan C. Newell in 1977. In one of their research papers [16], they applied the inverse scattering transformation [13,14] to find an exact solution for the derivative nonlinear Schrödinger equation (DNLS)

$$ip_t = -p_{xx} \pm i(p^* p^2)_x,$$

with vanishing boundary conditions, which means that the function p vanishes when $x \to \pm\infty$.

The spectral problems they were using in this procedure are:

$$\varphi_x = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_x = \begin{pmatrix} -i\lambda^2 & p\lambda \\ \lambda q & i\lambda^2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

and

$$i\varphi_t = i \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

And when $q = \pm p^*$, the components of the temporal spectral matrix are

$$\begin{aligned} A &= 2\lambda^4 + \lambda^2 pq, \\ B &= 2i\lambda^3 p - \lambda p_x + i\lambda p^2 q, \\ C &= 2i\lambda^3 q + \lambda q_x + i\lambda pq^2. \end{aligned}$$

By using these Lax pairs, a whole new integrable soliton hierarchy was generated, now we call it the Kaup-Newell soliton hierarchy.

In 1996, Ma et al. reconstructed the Kaup-Newell soliton hierarchy into a much simpler form [15], by replacing the original spectral problem with a particular spectral problem:

$$\varphi_x = U\varphi,$$

where the spatial spectral matrix

$$U = U(u,\lambda) = \begin{pmatrix} \lambda & p \\ \lambda q & -\lambda \end{pmatrix}, \tag{3.1a}$$

where λ is the spectral parameter. And the temporal spectral problem in the isospectral system is

$$\varphi_t = V\varphi,$$
$$V = V(u,\lambda) = \begin{pmatrix} a & \lambda^{-1}b \\ c & -a \end{pmatrix}, \tag{3.1b}$$

Moreover, by solving the adjoint representation equation $V_x = [U, V]$ of $\varphi_x = U\varphi$, with a, b and c presented in Lauren series:

$$a = \sum_{i=0}^{\infty} a_i \lambda^{-i}, \quad b = \sum_{i=0}^{\infty} b_i \lambda^{-i}, \quad c = \sum_{i=0}^{\infty} c_i \lambda^{-i},$$

they obtained a recursion relation for determining a_i, b_i, c_i:

$$\begin{cases} a_{i+1} &= -\frac{1}{2}(\partial^{-1}q\partial b_i + \partial^{-1}p\partial c_i), \\ b_{i+1} &= \frac{1}{2}(\partial b_i - p\partial^{-1}q\partial b_i - p\partial^{-1}p\partial c_i), \qquad i \geqslant 0, \\ c_{i+1} &= -\frac{1}{2}(q\partial^{-1}q\partial b_i + \partial c_i + q\partial^{-1}p\partial c_i), \end{cases}$$

and the zero constants for integration were selected as:

$$a_0 = 1, \quad b_0 = p, \quad c_0 = q,$$

with the assumption $a_i|_{[u]=0} = b_i|_{[u]=0} = c_i|_{[u]=0} = 0$, $i \geqslant 0$, where $[u] = (u, u_x, u_{xx}, ...)$.

Furthermore, by using the zero curvature representation

$$U_t - V_x + [U, V] = 0,$$

and setting

$$u = u(x,t) = \begin{pmatrix} p \\ q \end{pmatrix},$$

they obtained the Kaup-Newell hierarchy of nonlinear integrable evolution equations

$$u_{t_m} = K_m(u) = K_m(u, u_x, u_{xx}, ...), \tag{3.2}$$

where a recursive relation between the vector fields is

$$K_{m+1} = \Phi K_m, \quad m = 0, 1, 2...$$

with the recursion operator

$$\Phi = \begin{pmatrix} \frac{1}{2}\partial - \frac{1}{2}\partial p \partial^{-1} q & -\frac{1}{2}\partial p \partial^{-1} p \\ -\frac{1}{2}\partial q \partial^{-1} q & -\frac{1}{2}\partial - \frac{1}{2}\partial q \partial^{-1} p \end{pmatrix}, \quad \partial = \frac{\partial}{\partial x}. \tag{3.3}$$

And the first equation in this hierarchy can be rewritten as the DNLS equation:

$$ip_t = -\frac{1}{2} p_{xx} + \frac{1}{2} i (p^* p^2)_x,$$

which shows that this reconstructed soliton hierarchy is actually the same as the soliton hierarchy generated with the original Kaup-Newell Lax pairs.

Moreover, each of the soliton equations in (3.2b) has infinitely many symmetries, since

$$[K_m, K_n] = 0, \quad m, n \geqslant 0,$$

which shows that this is an integrable soliton hierarchy.

3.2 Integrable couplings and enlarged zero curvature equations

3.2.1 Solving the enlarged stationary zero curvature equation

Now in order to construct integrable couplings of the Kaup-Newell soliton equations, we are going to add a new evolution equation

$$u_{1,t} = S(u, u_1), \quad u_1 = \begin{pmatrix} r \\ s \end{pmatrix}, \tag{3.4}$$

so we can get the enlarged evolution equation system [8]:

$$\bar{u}_t = \begin{pmatrix} K(u) \\ S(u, u_1) \end{pmatrix}, \tag{3.5a}$$

where

$$\bar{u} = \begin{pmatrix} u \\ u_1 \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}. \tag{3.5b}$$

To find such a function $S(u,u_1)$, we need to consider the enlarged zero curvature representation equation

$$\bar{U}_t - \bar{V}_x + [\bar{U},\bar{V}] = 0, \tag{3.6a}$$

with the enlarged Lax pair matrices $\bar{U},\bar{V}$:

$$\bar{U} = \begin{pmatrix} U & U_1 \\ 0 & U \end{pmatrix}, \quad \bar{V} = \begin{pmatrix} V & V_1 \\ 0 & V \end{pmatrix}. \tag{3.6b}$$

Now, in order to construct this new Lax pair, we first consider the enlarged zero curvature equation in the stationary case:

$$\bar{W}_x = [\bar{U},\bar{W}] \tag{3.7a}$$

to find $\bar{W}$. Then with certain modifications on the operator $\bar{W}$, we can get the matrix $\bar{V}$.

Set the new enlarged Lax pair matrices as:

$$\bar{U} = \begin{pmatrix} U & U_1 \\ 0 & U \end{pmatrix}, \quad \bar{W} = \begin{pmatrix} W & W_1 \\ 0 & W \end{pmatrix}, \tag{3.7b}$$

where $U, W = V$ are defined in (3.1), and the new matrix blocks added to form a bigger integrable system are

$$U_1 = \begin{pmatrix} 0 & r \\ \lambda s & 0 \end{pmatrix}, \quad W_1 = \begin{pmatrix} e & \lambda^{-1} f \\ g & -e \end{pmatrix}. \tag{3.8}$$

Now we start the calculation on the enlarged Lax pair.

From

$$\begin{aligned} [\bar{U},\bar{W}] &= \bar{U}\bar{W} - \bar{W}\bar{U} \\ &= \begin{pmatrix} U & U_1 \\ 0 & U \end{pmatrix}\begin{pmatrix} W & W_1 \\ 0 & W \end{pmatrix} - \begin{pmatrix} W & W_1 \\ 0 & W \end{pmatrix}\begin{pmatrix} U & U_1 \\ 0 & U \end{pmatrix} \\ &= \begin{pmatrix} [U,W] & [U,W_1]+[U_1,W] \\ 0 & [U,W] \end{pmatrix}, \end{aligned}$$

we have

$$\bar{W}_x = [\bar{U},\bar{W}] \Leftrightarrow \begin{cases} W_x = [U,W], \\ W_{1x} = [U,W_1]+[U_1,W], \end{cases} \tag{3.9}$$

with the definitions of matrix blocks U, U_1, W, W_1 in (3.1), (3.8), we have:

$$\begin{cases} a_x &= pc - qb, \\ b_x &= 2\lambda b - 2\lambda pa, \\ c_x &= 2\lambda qa - 2\lambda c; \\ e_x &= pg - qf + rc - sb, \\ f_x &= 2\lambda f - 2\lambda pe - 2\lambda ra, \\ g_x &= 2\lambda qe - 2\lambda g + 2\lambda sa. \end{cases} \tag{3.10}$$

Since the matrix blocks in the operator $\bar{W}$ contain a spectral parameter λ, we can try a solution by Laurent series:

$$\bar{W} = \sum_{i=0}^{\infty} \bar{W}_i \lambda^{-i},$$

$$W = \sum_{i=0}^{\infty} W_i \lambda^{-i} = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & \lambda^{-1} b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i},$$

$$W_1 = \sum_{i=0}^{\infty} W_{1,i} \lambda^{-i} = \sum_{i=0}^{\infty} \begin{pmatrix} e_i & \lambda^{-1} f_i \\ g_i & -e_i \end{pmatrix} \lambda^{-i}, \tag{3.11}$$

with the components of the matrix $\bar{W}$ defined by:

$$\begin{cases} a = \sum_{i=0}^{\infty} a_i \lambda^{-i}, \\ b = \sum_{i=0}^{\infty} b_i \lambda^{-i}, \\ c = \sum_{i=0}^{\infty} c_i \lambda^{-i}; \\ e = \sum_{i=0}^{\infty} e_i \lambda^{-i}, \\ f = \sum_{i=0}^{\infty} f_i \lambda^{-i}, \\ g = \sum_{i=0}^{\infty} g_i \lambda^{-i}. \end{cases}$$

After combining the coefficients of λ^{-i}, $i \geqslant 0$, in the above equations, we get:

$$\left\{\begin{array}{l} 2b_0 - 2pa_0 = 2qa_0 - 2c_0 = 0, \\ a_{i,x} - pc_i + qb_i = 0, \\ b_{i,x} + 2pa_{i+1} - 2b_{i+1} = 0, \\ c_{i,x} - 2qa_{i+1} + 2c_{i+1} = 0; \\ -2f_0 + 2pe_0 + 2ra_0 = -2qe_0 + 2g_0 - 2sa_0 = 0, \\ e_{i,x} - pg_i + qf_i - rc_i + sb_i = 0, \\ f_{i,x} - 2f_{i+1} + 2pe_{i+1} + 2ra_{i+1} = 0, \\ g_{i,x} - 2qe_{i+1} + 2g_{i+1} - 2sa_{i+1} = 0, \end{array}\right. \quad i \geqslant 0. \tag{3.12}$$

Furthermore, assuming:

$$\left\{\begin{array}{lcl} a_0 & = & \text{const.}, \\ b_0 & = & a_0 p, \\ c_0 & = & a_0 q; \\ e_0 & = & \text{const.}, \\ f_0 & = & a_0 r + e_0 p, \\ g_0 & = & a_0 s + e_0 q, \end{array}\right. \tag{3.13a}$$

there exist recursion relations

$$\left\{\begin{array}{lcl} a_{i+1} & = & -\frac{1}{2}(\partial^{-1}q\partial b_i + \partial^{-1}p\partial c_i), \\ b_{i+1} & = & \frac{1}{2}(\partial b_i - p\partial^{-1}q\partial b_i - p\partial^{-1}p\partial c_i), \\ c_{i+1} & = & -\frac{1}{2}(q\partial^{-1}q\partial b_i + \partial c_i + q\partial^{-1}p\partial c_i); \\ e_{i+1} & = & -\frac{1}{2}(\partial^{-1}s\partial b_i + \partial^{-1}r\partial c_i + \partial^{-1}q\partial f_i + \partial^{-1}p\partial g_i), \\ f_{i+1} & = & -\frac{1}{2}(p\partial^{-1}s\partial + r\partial^{-1}q\partial)b_i - \frac{1}{2}(p\partial^{-1}r\partial + r\partial^{-1}p\partial)c_i, \\ & & +\frac{1}{2}(\partial - p\partial^{-1}q\partial)f_i - \frac{1}{2}p\partial^{-1}p\partial g_i, \\ g_{i+1} & = & -\frac{1}{2}(q\partial^{-1}s\partial + s\partial^{-1}q\partial)b_i - \frac{1}{2}(q\partial^{-1}r\partial + s\partial^{-1}p\partial)c_i, \\ & & -\frac{1}{2}q\partial^{-1}q\partial f_i - \frac{1}{2}(\partial + q\partial^{-1}p\partial)g_i, \end{array}\right. \quad i \geqslant 0. \tag{3.13b}$$

For example, when setting $i = 0$ in equation (3.13b), we obtain:

$$\begin{cases} a_1 &=& -\frac{a_0}{2}pq, \\ b_1 &=& \frac{a_0}{2}p_x - \frac{a_0}{2}p^2q, \\ c_1 &=& -\frac{a_0}{2}q_x - \frac{a_0}{2}pq^2; \\ e_1 &=& -\frac{1}{2}(a_0ps + a_0qr + e_0pq), \\ f_1 &=& \frac{1}{2}(a_0r_x - a_0p^2s + e_0p_x - e_0p^2q) - a_0pqr, \\ g_1 &=& -\frac{1}{2}(a_0s_x + a_0q^2r + e_0q_x + e_0pq^2) - a_0pqs, \end{cases} \tag{3.14}$$

where a_0 and e_0 are arbitrary nonzero constants.

3.2.2 Enlarged systems

Now by using the above result, with some adjustments, we can get the enlarged hierarchy of evolution equations

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}), \quad m \geqslant 0,$$

which is related to the enlarged zero curvature equations:

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0$$
$$\Leftrightarrow \begin{cases} U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \\ U_{1,t_m} - V_{1,x}^{[m]} + [U, V_1^{[m]}] - [U_1, V^{[m]}] = 0. \end{cases} \tag{3.15}$$

To satisfy the equation (3.15), the matrix $\bar{V}^{[m]}$ should be similar to $\bar{U}$:

$$\bar{U}_{t_m} = \begin{pmatrix} U_{t_m} & U_{1,t_m} \\ 0 & U_{t_m} \end{pmatrix},$$

where

$$U_{t_m} = \begin{pmatrix} 0 & p_{t_m} \\ \lambda q_{t_m} & 0 \end{pmatrix}, \quad U_{1,t_m} = \begin{pmatrix} 0 & r_{t_m} \\ \lambda s_{t_m} & 0 \end{pmatrix}.$$

So here, by introducing a set of matrix block modifications $\{\Delta_m, \Delta_{1,m}; m = 0, 1, 2, ...\}$:

$$\Delta_m = -\begin{pmatrix} a_{m+1} & 0 \\ c_{m+1} & -a_{m+1} \end{pmatrix}, \quad \Delta_{1,m} = -\begin{pmatrix} e_{m+1} & 0 \\ g_{m+1} & -e_{m+1} \end{pmatrix},$$

we can define the adjusted Lax matrices by:

$$V^{[m]} = (\lambda^{m+1}W)_+ + \Delta_m, \quad V_1^{[m]} = (\lambda^{m+1}W_1)_+ + \Delta_{1,m},$$

and thus,

$$\bar{V}^{[m]} = \begin{pmatrix} (\lambda^{m+1}W)_+ + \Delta_m & (\lambda^{m+1}W_1)_+ + \Delta_{1,m} \\ 0 & (\lambda^{m+1}W)_+ + \Delta_m \end{pmatrix}, \tag{3.16}$$

where the subscript $+$ means that only the terms of non-negative powers of λ are included.

Therefore by (3.15), the enlarged hierarchy of integrable couplings of the Kaup-Newell soliton hierarchy is given by:

$$\begin{cases} u_{t_m} = \begin{pmatrix} p \\ q \end{pmatrix}_{t_m} = \begin{pmatrix} b_{m,x} \\ c_{m,x} \end{pmatrix} = K_m, \\ \\ u_{1,t_m} = \begin{pmatrix} r \\ s \end{pmatrix}_{t_m} = \begin{pmatrix} f_{m,x} \\ g_{m,x} \end{pmatrix} = K_{1,m}, \end{cases} \quad m \geqslant 0.$$

Namely,

$$\bar{u}_{t_m} = \bar{K}_m = \begin{pmatrix} b_{m,x} \\ c_{m,x} \\ f_{m,x} \\ g_{m,x} \end{pmatrix}, \quad m \geqslant 0. \tag{3.17}$$

3.2.3 Recursive structure and examples

After taking the partial differentiation with respect to x on the equations in (3.13), we get an recursion relation:

$$\begin{pmatrix} b_{i+1,x} \\ c_{i+1,x} \\ f_{i+1,x} \\ g_{i+1,x} \end{pmatrix} = \bar{\Phi} \begin{pmatrix} b_{i,x} \\ c_{i,x} \\ f_{i,x} \\ g_{i,x} \end{pmatrix}, \quad i \geqslant 0, \tag{3.18}$$

where the recursion operator is

$$\bar{\Phi} = \begin{pmatrix} \frac{1}{2}\partial - \frac{1}{2}\partial p \partial^{-1} q & -\frac{1}{2}\partial p \partial^{-1} p & 0 & 0 \\ -\frac{1}{2}\partial q \partial^{-1} q & -\frac{1}{2}\partial - \frac{1}{2}\partial q \partial^{-1} p & 0 & 0 \\ -\frac{1}{2}\partial p \partial^{-1} s - \frac{1}{2}\partial r \partial^{-1} q & -\frac{1}{2}\partial p \partial^{-1} r - \frac{1}{2}\partial r \partial^{-1} p & \frac{1}{2}\partial - \frac{1}{2}\partial p \partial^{-1} q & -\frac{1}{2}\partial p \partial^{-1} p \\ -\frac{1}{2}\partial q \partial^{-1} s - \frac{1}{2}\partial s \partial^{-1} q & -\frac{1}{2}\partial q \partial^{-1} r - \frac{1}{2}\partial s \partial^{-1} p & -\frac{1}{2}\partial q \partial^{-1} q & -\frac{1}{2}\partial - \frac{1}{2}\partial q \partial^{-1} p \end{pmatrix}.$$

Consequently, there is a relation between the vector fields $\bar{K}_m$ and the recursion operator $\bar{\Phi}$:

$$\bar{K}_m = \begin{pmatrix} b_{m,x} \\ c_{m,x} \\ f_{m,x} \\ g_{m,x} \end{pmatrix} = \bar{\Phi}^m \begin{pmatrix} b_{0,x} \\ c_{0,x} \\ f_{0,x} \\ g_{0,x} \end{pmatrix} = \bar{\Phi}^m \begin{pmatrix} a_0 p_x \\ a_0 q_x \\ a_0 r_x + e_0 p_x \\ a_0 s_x + e_0 q_x \end{pmatrix}, \tag{3.19a}$$

i.e., the integrable couplings can be rewritten as

$$\bar{u}_{t_m} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}_{t_m} = \bar{\Phi}^m \begin{pmatrix} a_0 p_x \\ a_0 q_x \\ a_0 r_x + e_0 p_x \\ a_0 s_x + e_0 q_x \end{pmatrix}, \quad m \geqslant 0. \tag{3.19b}$$

After comparing $\bar{\Phi}$ with the recursion operator (3.3) for the Kaup-Newell hierarchy:

$$\Phi = \begin{pmatrix} \frac{1}{2}\partial - \frac{1}{2}\partial p \partial^{-1} q & -\frac{1}{2}\partial p \partial^{-1} p \\ -\frac{1}{2}\partial q \partial^{-1} q & -\frac{1}{2}\partial - \frac{1}{2}\partial q \partial^{-1} p \end{pmatrix},$$

we can see that the enlarged recursion operator $\bar{\Phi}$ has a partition form:

$$\bar{\Phi} = \begin{pmatrix} \Phi & 0 \\ \Phi_1 & \Phi \end{pmatrix}, \tag{3.20a}$$

with a newly added matrix block

$$\Phi_1 = \begin{pmatrix} -\frac{1}{2}\partial p \partial^{-1} s - \frac{1}{2}\partial r \partial^{-1} q & -\frac{1}{2}\partial p \partial^{-1} r - \frac{1}{2}\partial r \partial^{-1} p \\ -\frac{1}{2}\partial q \partial^{-1} s - \frac{1}{2}\partial s \partial^{-1} q & -\frac{1}{2}\partial q \partial^{-1} r - \frac{1}{2}\partial s \partial^{-1} p \end{pmatrix}. \tag{3.20b}$$

Furthermore, the recursive relation between $\bar{u}_{t_m} = \begin{pmatrix} u_{t_m} \\ u_{1,t_m} \end{pmatrix}$ and $\bar{u}_{t_{m+1}} = \begin{pmatrix} u_{t_{m+1}} \\ u_{1,t_{m+1}} \end{pmatrix}$ reads as

$$\begin{pmatrix} u_{t_{m+1}} \\ u_{1,t_{m+1}} \end{pmatrix} = \begin{pmatrix} \Phi & 0 \\ \Phi_1 & \Phi \end{pmatrix} \begin{pmatrix} u_{t_m} \\ u_{1,t_m} \end{pmatrix}$$

$$= \begin{pmatrix} \Phi u_{t_m} \\ \Phi_1 u_{t_m} + \Phi u_{1,t_m} \end{pmatrix} = \begin{pmatrix} \Phi^m u_{t_0} \\ \sum_{i=0}^{m-1} \Phi^{m-i-1} \Phi_1 \Phi^i u_{t_0} + \Phi^m u_{1,t_0} \end{pmatrix}.$$

According to the equation (3.19), after some quite prolix calculation, we found the first three integrable couplings in the enlarged soliton hierarchy:

$$\bar{u}_{t_0} = \bar{K}_0 = \begin{pmatrix} a_0 p_x \\ a_0 q_x \\ a_0 r_x + e_0 p_x \\ a_0 s_x + e_0 q_x \end{pmatrix}, \tag{3.21}$$

$$\bar{u}_{t_1} = \bar{K}_1 = \begin{pmatrix} p_{t_1} \\ q_{t_1} \\ r_{t_1} \\ s_{t_1} \end{pmatrix} = \bar{\Phi}\bar{K}_0$$

$$= \frac{a_0}{2}\begin{pmatrix} p_{xx} - (p^2 q)_x \\ -q_{xx} - (pq^2)_x \\ r_{xx} - (p^2 s)_x - 2(pqr)_x \\ -s_{xx} - (q^2 r)_x - 2(pqs)_x \end{pmatrix} + \frac{e_0}{2}\begin{pmatrix} 0 \\ 0 \\ p_{xx} - (p^2 q)_x \\ -q_{xx} - (pq^2)_x \end{pmatrix}, \tag{3.22}$$

$$\bar{u}_{t_2} = \bar{K}_2 = \begin{pmatrix} p_{t_2} \\ q_{t_2} \\ r_{t_2} \\ s_{t_2} \end{pmatrix} = \bar{\Phi}\bar{K}_1$$

$$= \frac{a_0}{4}\begin{pmatrix} p_{xxx} - (p^2 q)_{xx} - (pqp_x)_x + (p^2 q_x)_x + \frac{3}{2}(p^3 q^2)_x \\ q_{xxx} + (pq^2)_{xx} + (pqq_x)_x - (q^2 p_x)_x + \frac{3}{2}(p^2 q^3)_x \\ r_{xxx} - 3(psp_x)_x - 3(pqr_x)_x - 3(qrp_x)_x + 3(p^3 qs)_x + \frac{9}{2}(p^2 q^2 r)_x \\ s_{xxx} + 3(pqs_x)_x + 3(psq_x)_x + 3(qrq_x)_x + 3(pq^3 r)_x + \frac{9}{2}(p^2 q^2 s)_x \end{pmatrix}$$

$$+\frac{e_0}{4}\begin{pmatrix} 0 \\ 0 \\ p_{xxx} - 3(pqp_x)_x + \frac{3}{2}(p^3 q^2)_x \\ q_{xxx} + 3(pqq_x)_x + \frac{3}{2}(p^2 q^3)_x \end{pmatrix}. \tag{3.23}$$

Now if we set the constants a_0 and e_0 to certain values, for instance, $a_0 = e_0 = 2$, the second integrable coupling becomes

$$\begin{cases} p_{t_1} = p_{xx} - (p^2q)_x, \\ q_{t_1} = -q_{xx} - (pq^2)_x, \\ r_{t_1} = p_{xx} - (p^2q)_x + r_{xx} - (p^2s)_x - 2(pqr)_x, \\ s_{t_1} = -q_{xx} - (pq^2)_x - s_{xx} - (q^2r)_x - 2(pqs)_x. \end{cases}$$

Or if we let $a_0 = e_0 = 4$, we can get a special case of the third system:

$$\begin{cases} p_{t_2} &= (qp_x + p^2q_x + \frac{3}{2}p^3q^2)_x, \\ q_{t_2} &= (q_{xx} + (pq^2)_x + pqq_x - q^2p_x + \frac{3}{2}p^2q^3)_x, \\ r_{t_2} &= (qp_x + p^2q_x + \frac{3}{2}p^3q^2)_x \\ & \quad +(r_{xx} - 3psp_x - 3pqr_x - 3qrp_x + 3p^3qs + \frac{9}{2}p^2q^2r)_x, \\ s_{t_2} &= (q_{xx} + (pq^2)_x + pqq_x - q^2p_x + \frac{3}{2}p^2q^3)_x \\ & \quad +(s_{xx} + 3pqs_x + 3psq_x + 3qrq_x + \frac{9}{2}p^2q^2s + 3pq^3r)_x. \end{cases}$$

With the description in (3.11) and (3.16), we can also get the spectral matrices $\bar{V}^{[m]}$ in the Lax pairs. For example, the corresponding Lax matrix of the first integrable coupling in the enlarged soliton hierarchy is:

$$\bar{V}^{[0]} = \begin{pmatrix} V^{[0]} & V_1^{[0]} \\ 0 & V^{[0]} \end{pmatrix}, \tag{3.24a}$$

where

$$V^{[0]} = a_0 \begin{pmatrix} 1 + \frac{pq}{2} & \lambda^{-1}p \\ q + \frac{q_x}{2} + \frac{pq^2}{2} & -1 - \frac{pq}{2} \end{pmatrix},$$

$$V_1^{[0]} = e_0 \begin{pmatrix} 1 + \frac{pq}{2} & \lambda^{-1}p \\ q + \frac{q_x}{2} + \frac{pq^2}{2} & -1 - \frac{pq}{2} \end{pmatrix} + a_0 \begin{pmatrix} \frac{ps}{2} + \frac{qr}{2} & \lambda^{-1}r \\ s - \frac{s_x}{2} - \frac{q^2}{2}r + pqs & -\frac{ps}{2} - \frac{qr}{2} \end{pmatrix}. \tag{3.24b}$$

3.3 Infinitely many symmetries

In the second chapter, we gave the definition of a symmetry of an evolution equation, now we are going to prove that each new enlarged Lax integrable system we got possesses infinitely many symmetries, just like its subsystem, the Kaup-Newell system.

According to the result in [31,32], for the evolution equations (3.2b) defined by the zero-curvature representation (3.15), we have:

$$\bar{U}'[[\bar{K}_m, \bar{K}_n]] - [\![\bar{V}^{[m]}, \bar{V}^{[n]}]\!]_x + [\bar{U}, [\![\bar{V}^{[m]}, \bar{V}^{[n]}]\!]] = 0, \tag{3.25}$$

where $[\![\bar{V}_m, \bar{V}_n]\!]$ is given as:

$$[\![\bar{V}_m, \bar{V}_n]\!] := \bar{V}^{[m]\prime}(\bar{u})[\bar{K}_n] - \bar{V}^{[n]\prime}(\bar{u})[\bar{K}_m] + [\bar{V}^{[m]}, \bar{V}^{[n]}], \quad m, n = 0, 1, 2, ... \tag{3.26}$$

With the results obtained in Section 3.2, we know that

$$V^{[m]} = \begin{pmatrix} \sum_{i=0}^{\infty} a_i \lambda^{m+1-i} - a_{m+1} & \sum_{i=0}^{\infty} b_i \lambda^{m-i} \\ \sum_{i=0}^{\infty} c_i \lambda^{m+1-i} - c_{m+1} & -\sum_{i=0}^{\infty} a_i \lambda^{m+1-i} + a_{m+1} \end{pmatrix},$$

$$V_1^{[m]} = \begin{pmatrix} \sum_{i=0}^{\infty} e_i \lambda^{m+1-i} - e_{m+1} & \sum_{i=0}^{\infty} f_i \lambda^{m-i} \\ \sum_{i=0}^{\infty} g_i \lambda^{m+1-i} - g_{m+1} & -\sum_{i=0}^{\infty} e_i \lambda^{m+1-i} + e_{m+1} \end{pmatrix}.$$

According to the equations in (3.13), when

$$\bar{u} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = 0,$$

the functions $b_0 = c_0 = f_0 = g_0 = 0$, and for all $i \geqslant 0$, $a_i = b_i = c_i = e_i = f_i = g_i = 0$. Thus the temporal spectral matrix becomes

$$\bar{V}^{[m]}|_{\bar{u}=0} = \begin{pmatrix} a_0\lambda^{m+1} & 0 & e_0\lambda^{m+1} & 0 \\ 0 & -a_0\lambda^{m+1} & 0 & -e_0\lambda^{m+1} \\ 0 & 0 & a_0\lambda^{m+1} & 0 \\ 0 & 0 & 0 & -a_0\lambda^{m+1} \end{pmatrix}, \tag{3.27}$$

and furthermore, we have

$$[\bar{V}^{[m]}, \bar{V}^{[n]}]|_{\bar{u}=0} = 0. \tag{3.28}$$

And with the expression of $\bar{K}_m$ in (3.17), we also get $\bar{K}_m|_{\bar{u}=0} = 0$, for all $m = 0, 1, 2, ...$, which yields

$$\bar{V}^{[m]\prime}(\bar{u})[\bar{K}_m]|_{\bar{u}=0} = \bar{V}^{[n]\prime}(\bar{u})[\bar{K}_n]|_{\bar{u}=0} = 0. \tag{3.29}$$

So now, according to the definition of $[\![\bar{V}_m, \bar{V}_n]\!]$ in (3.26), we have

$$[\![\bar{V}_m, \bar{V}_n]\!]|_{\bar{u}=0} = 0.$$

By applying the uniqueness property of $\bar{U}$ [31], we get the conclusion that if $[\![\bar{V}_m, \bar{V}_n]\!]|_{\bar{u}=0} = 0$, then $[\![\bar{V}_m, \bar{V}_n]\!] = 0$ for any $\bar{u}$; which leaves the equation (3.25) as:

$$\bar{U}'[[\bar{K}_m, \bar{K}_n]] = 0. \tag{3.30}$$

We know from (3.1a) and (3.8) that:

$$\bar{U} = \begin{pmatrix} \lambda & p & 0 & r \\ \lambda q & -\lambda & \lambda s & 0 \\ 0 & 0 & \lambda & p \\ 0 & 0 & \lambda q & -\lambda \end{pmatrix},$$

and so the Gateaux derivative of $\bar{U}$ along a direction $S = (S_1, S_2, S_3, S_4)^T$ is

$$\bar{U}'[S] = \bar{U}'(\bar{u})[S] = \frac{\partial}{\partial \epsilon}\bar{U}(\bar{u} + \epsilon S)|_{\epsilon=0},$$

which can be computed as

$$\bar{U}(\bar{u} + \epsilon S) = \begin{pmatrix} \lambda & p + \epsilon S_1 & 0 & r + \epsilon S_3 \\ \lambda(q + \epsilon S_2) & -\lambda & \lambda(s + \epsilon S_4) & 0 \\ 0 & 0 & \lambda & p + \epsilon S_1 \\ 0 & 0 & \lambda(q + \epsilon S_2) & -\lambda \end{pmatrix},$$

and

$$\bar{U}'[S] = \begin{pmatrix} 0 & S_1 & 0 & S_3 \\ \lambda S_2 & 0 & \lambda S_4 & 0 \\ 0 & 0 & 0 & S_1 \\ 0 & 0 & \lambda S_2 & 0 \end{pmatrix}. \tag{3.31}$$

Now apparently, $\bar{U}'[S]$ is an injective operator, which means that $\bar{U}'[S]=0$ if and only if $S=0$. So the equation (3.30) holds if and only if

$$[\bar{K}_m, \bar{K}_n] = 0, \qquad m, n = 0, 1, 2, ... \tag{3.32}$$

Hence the vector field $\{\bar{K}_m : m = 0, 1, 2, ...\}$ contains elements in involutions to each other; which means that each evolution equation $\bar{K}_m$ in (3.17) has infinitely many symmetries: $\bar{K}_0, \bar{K}_1$, $\bar{K}_2, ...,$. For example, we can check the Lie products between the first 3 integrable couplings in the enlarged soliton hierarchy and get the conclusion they are mutually commuting.

Given $a_0 = e_0 = 2$, we have a special case of $\bar{K}_m = (p_{t_m}, q_{t_m}, r_{t_m}, s_{t_m})^T$:

when $m = 0$,

$$\begin{cases} p_{t_0} = 2p_x, \\ q_{t_0} = 2q_x, \\ r_{t_0} = 2p_x + 2r_x, \\ s_{t_0} = 2q_x + 2s_x; \end{cases}$$

when $m = 1$,

$$\begin{cases} p_{t_1} = p_{xx} - (p^2q)_x, \\ q_{t_1} = -q_{xx} - (pq^2)_x, \\ r_{t_1} = p_{xx} + r_{xx} - (p^2q)_x - (p^2s)_x - 2(pqr)_x, \\ s_{t_1} = -q_{xx} - s_{xx} - (pq^2)_x - (q^2r)_x - 2(pqs)_x; \end{cases}$$

and at last, when $m = 2$,

$$\begin{cases} p_{t_2} &=& \frac{1}{2}(p_{xxx} - (p^2q)_{xx} - (pqp_x)_x + (p^2q_x)_x + \frac{3}{2}(p^3q^2)_x), \\ q_{t_2} &=& \frac{1}{2}(q_{xxx} + (pq^2)_{xx} + (pqq_x)_x - (q^2p_x)_x + \frac{3}{2}(p^2q^3)_x), \\ r_{t_2} &=& \frac{1}{2}(p_{xxx} + r_{xxx} - 3(pqp_x)_x - 3(psp_x)_x - 3(pqr_x)_x - 3(qrp_x)_x \\ && +3(p^3qs)_x + \frac{3}{2}(p^3q^2)_x + \frac{9}{2}(p^2q^2r)_x), \\ s_{t_2} &=& \frac{1}{2}(q_{xxx} + s_{xxx} + 3(pqq_x)_x + 3(pqs_x)_x + 3(psq_x)_x + 3(qrq_x)_x \\ && +3(pq^3r)_x + \frac{3}{2}(p^2q^3)_x + \frac{9}{2}(p^2q^2s)_x). \end{cases}$$

Therefore, with the definition of Gateaux derivative in (2.4), we have the Lie product between $\bar{K}_0$ and $\bar{K}_1$:

$$\begin{aligned} [\bar{K}_0, \bar{K}_1] &:= \bar{K}_0{}'(\bar{u})[\bar{K}_1] - \bar{K}_1{}'(\bar{u})[\bar{K}_0] \\ &= \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0}[\bar{K}_0(\bar{u} + \epsilon\bar{K}_1) - \bar{K}_1(\bar{u} + \epsilon\bar{K}_0)] \end{aligned}$$

Put this expression in Maple, we get $[\bar{K_0}, \bar{K_1}] = 0$. Similarly, the results of $[\bar{K_2}, \bar{K_0}]$, $[\bar{K_1}, \bar{K_2}]$ are also zeros. To conclude, we have the following theorem on integrability of the enlarged soliton hierarchy.

Thereom: The nonlinear evolution equations in the coupling hierarchy of the Kaup-Newell soliton (3.17) are integrable. Namely, they have infinitely many common commuting symmetries $\{\bar{K}_m | m \geqslant 0\}$.

Chapter 4

Conclusions

In this thesis, we studied integrable couplings: first we showed how to work out the dark KdV equations, then we obtained a new enlarged soliton hierarchy of integrable couplings of the Kaup-Newell soliton hierarchy. The evolution equations in the new hierarchy we got hold the Kaup-Newell soliton equations as their subsystems; the new Lax pairs are built from the given Lax pairs of the Kaup-Newell soliton hierarchy, with some newly added matrix blocks; and these enlarged Lax matrices yield the zero curvature representation of the coupling hierarchy of the Kaup-Newell soliton hierarchy.

Then we verified the integrability of this new enlarged soliton hierarchy by proving that there exists infinitely many symmetries for each evolution equation system contained in the hierarchy, and tested the mutual involutivity of the first three evolution equation systems in the coupling hierarchy of the Kaup-Newell soliton hierarchy.

Besides the proof used in Section 3.3, there are also other methods we can apply to prove the existence of infinitely many symmetries for the new enlarged hierarchy. For example, the recursion operators have a very important property that they always map symmetries to symmetries [7]. Therefore, if we can prove that the operator $\bar{\Phi}$ in (3.19), mapping $\bar{K}_m$ to $\bar{K}_{m+1}$, is a recursion operator, and that $\bar{\Phi}$ is hereditary, then with the mutual commutativity on the first several vector fields $\bar{K}_m$ in the coupling hierarchy of the Kaup-Newell soliton hierarchy, we will obtain the result that all vector fields $\bar{K}_m : m = 0, 1, 2...$ are symmetries to the evolution equation systems in the enlarged hierarchy.

And to prove that $\bar{\Phi}$ is a recursion operator of $\bar{u}_{t_m} = \bar{K}_m$, we will need to show that it satisfies the equation [34]:

$$\frac{\partial \bar{\Phi}}{\partial t}\bar{X} + \bar{\Phi}'(u)[\bar{K}_m]\bar{X} - \bar{K}_m{}'(u)[\bar{\Phi}\bar{X}] + \bar{\Phi}\bar{K}_m{}'(u)[\bar{X}] = 0,$$

where $\bar{X} = \bar{X}(x, t, \bar{u})$ is an arbitrary vector field. And to prove that recursion operator $\bar{\Phi}$, which maps symmetry to symmetry, is hereditary [35], we need to show that equation

$$\bar{\Phi}'[\bar{\Phi}(\bar{u})\bar{X}(\bar{u})]\bar{Y}(\bar{u}) - \bar{\Phi}(\bar{u})\bar{\Phi}'(\bar{u})[\bar{X}(\bar{u})]\bar{Y}(\bar{u})$$

$$-\bar{\Phi}'(\bar{u})[\bar{\Phi}(\bar{u})\bar{Y}(\bar{u})]\bar{X}(\bar{u}) + \bar{\Phi}(\bar{u})\bar{\Phi}'(\bar{u})[\bar{Y}(\bar{u})]\bar{X}(\bar{u}) = 0$$

holds for all vector fields $\bar{X}, \bar{Y}$.

Furthermore, if we keep on studying the new enlarged soliton hierarchy $\bar{u}_{t_m} = \bar{K}_m$ defined in (3.17), we will find that it possesses many other interesting properties. For example, it is invariant under the translation:

$$x \to x + c, \quad c = \text{const},$$

or

$$t \to t + c, \quad c = \text{const},$$

i.e., the x-translation or t-translation generates a symmetry of all the equations in the enlarged soliton hierarchy.

So obviously, we can prove this property for the enlarged Kaup-Newell hierarchy only by applying the definition of symmetry: for the x-translation, we set $\sigma = \bar{u}_x$, and for the t-translation, we set $\sigma = \bar{u}_x$, then we can prove that every vector field in $\{\bar{K}_m : m = 0, 1, 2, ...\}$ satisties the equation

$$[\sigma, \bar{K}_m] = \bar{K}'_m(\bar{u})[\sigma] - \sigma'(u)[\bar{K}_m] = 0.$$

This implies that σ is a common symmetry of $\bar{u}_{t_m} = \bar{K}_m$ in (3.17), and so the coupling system $\bar{u}_{t_m} = \bar{K}_m$ possesses infinitely many symmetries $\{\bar{\Phi}^n = \bar{K}_n | n \geqslant 0\}$, since $\bar{\Phi}$ is a common recursion operator.

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