4-2-2007

Kauffman-Harary Conjecture for Virtual Knots

Mathew Williamson
University of South Florida

Kauffman-Harary Conjecture for Virtual Knots

by

Mathew Williamson

A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
Department of Mathematics
College of Arts and Sciences
University of South Florida

Major Professor: Masahiko Saito, Ph.D.
Mohamed Elhamdadi, Ph.D.
Milé Krajcevski, Ph.D.

Date of Approval:
April 2, 2007

Keywords: Fox colorings, diagrams with distinct colors, determinants, links, Jones Polynomial

Table of Contents

List of Figures

Kauffman-Harary Conjecture for Virtual Knots

Mathew Williamson

ABSTRACT

In this paper, we examine Fox colorings of virtual knots, and moves called k-swap moves defined for virtual knot diagrams. The k-swap moves induce a one-to-one correspondence between colorings before and after the move, and can be used to reduce the number of virtual crossings. For the study of colorings, we characterize families of alternating virtual knots to generalize $(2, n)$-torus knots, alternating pretzel knots, and alternating 2-bridge knots. The k-swap moves are then applied to prove a "virtualization" of the Kauffman-Harary conjecture, originally stated for classical knot diagrams, for the above families of virtual pretzel knot diagrams.

1 Background and Motivation

1.1 Introduction

In [KVKT], Kauffman defines an extension of classical knot theory, called virtual knot theory, motivated by Gauss codes and thickened surfaces. Many classical knot invariants can be generalized to virtual knot theory, including quandle knot colorings. Fox ([F]) described colorings of knot diagrams by $\mathbb{Z}_n$ and related them to Alexander polynomials and homomorphisms from the knot group to dihedral groups. Since then, colorings of knot diagrams by quandles have been extensively studied. This paper considers virtual knot colorings, along with a virtualization of a classical knot conjecture by Kauffman and Harary ([KH]).

Conjecture 1.1.1 (Kauffman-Harary Conjecture 1) *Let D be a reduced alternating knot diagram with a prime determinant p. Then every nontrivial Fox's p-coloring of D assigns different colors to different arcs of D.*

The conjecture was first posed in [KH], and proved for torus knots $T(2, n)$ in the same paper. In a later paper [KL], it was shown that the Kauffman-Harary conjecture holds for any 2-bridge knot (rational knots) without restrictions on the knot determinant. Later, in [AM], the conjecture was proved for all Montesinos links and a more general conjecture was made involving the homology of the double cover of the 3-sphere S^3 branched along a link. Finally, in [AS], a conjecture associated with the Alexander quandle was introduced, which generalizes the original conjecture for Fox colorings.

This paper deals with an analogue of the original Kauffman-Harary conjecture for virtual knots. To accomplish this, we introduce a move called the k-swap move. It is instructive to note that Kauffman studied 1-swap moves (he only described what they did, and he did not have a name for them) in [KVKT]. He showed that the 1-swap does not change the Jones polynomial and the involutory quandle but may change the fundamental quandle. This fact had a corollary which was used to easily construct a virtual knot that had a nontrivial fundamental group but a trivial Jones polynomial. To put this result into perspective, the question of whether a nontrivial classical link could have a trivial Jones polynomial was only answered in 2000 ([ES]), but for knots it is still an open question. In virtual knot theory, however, the question was answered quickly. So Kauffman's approach to what we call a 1-swap move was to show that we can build nontrivial virtual knots from trivial classical knots, keeping some invariants unchanged. Our approach is to simplify virtual knot diagrams and apply results from classical knot theory to virtual knot theory.

The paper is organized in the following way. Section 1 organizes the background needed for the main results. It explains virtual knots, quandles, knot colorings, and the Kauffman-Harary conjecture. In Section 2.1, the k-swap move for virtual knots is defined and shown to be a coloring invariant over virtual knots. Also in Section 2.1 is the result that the k-swap move does not change the Kauffman bracket, and therefore the Jones polynomial is unaffected as well. Section 2.2 and 2.3 show that alternating closed 2-string virtual braid diagrams, alternating virtual pretzel knot diagrams, and alternating virtual 2-bridge knot diagrams each satisfy the Kauffman-Harary conjecture, using the k-swap move defined in Section 2.1.

Finally, we would like to thank Dr. Gregory McColm for his numerous contributions to this thesis. Without his help, it would not have been accomplished.

1.2 Preliminaries

1.2.1 Classical Knots

The most fundamental of all questions in knot theory is examining whether two knots are the same knot. This is a subtle problem, and many invariants and techniques have been found to help. To procede further, we need to know what a knot is. A *knot* is the image of a differentiable embedding $f : S^1 \to \mathbb{R}^3$ from a circle S^1 to the 3-dimensional Euclidiean space $\mathbb{R}^3$. Two knots are equivalent if there is a diffeomorphism of $\mathbb{R}^3$ that takes one knot to the other. This definition excludes knots with limit points, and polygonal knots, both of which we do not need. If interested, check [CP] for more information.

Since knots are subsets of $\mathbb{R}^3$, we can move the strands of a knot in $\mathbb{R}^3$ as long as two strands do not pass through each other. Projecting a knot down into the standard xy-plane gives us a *knot diagram*. Since we can move strands in $\mathbb{R}^3$ without cutting them, and since we can rotate the knot at will, there are infinitely many knot diagrams of any given knot. This makes recognizing that two knots are equivalent difficult. For an example of two knot diagrams whose knot is equivalent, see Figure 1.1. To see that these knot diagrams are

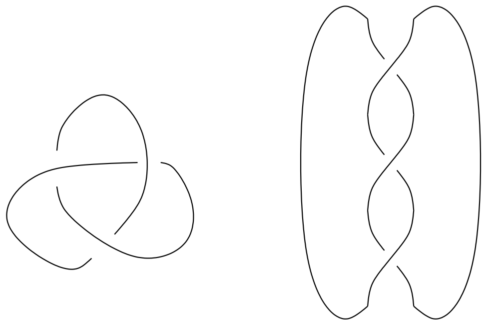


Figure 1.1: Two diagrams of an equivalent knot: a trefoil

equivalent, we can employ the Reidemeister moves to transform one diagram into the other. These moves are pictured in Figure 1.2.

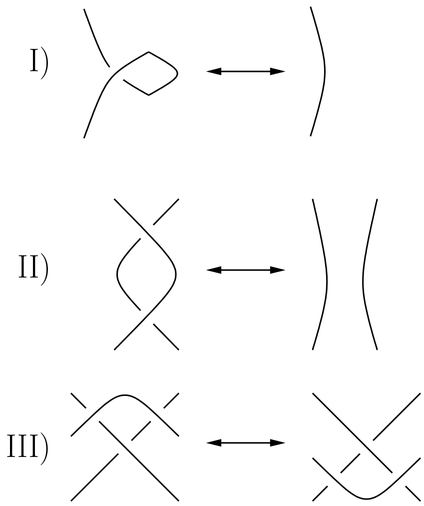


Figure 1.2: The Reidemeister moves

It is known that two knot diagrams are equivalent if, and only if, one diagram can be transformed into the other using a finite sequence of Reidemeister moves. It is usually impractical to try and find a sequence of Reidemeister moves to prove that two knot diagrams are diagrams of an equivalent knot. Thus, in this thesis, we study equivalence classes of knots by diagrams and their moves, instead of by differential mappings, and we turn our attention to knot invariants which tell us if two knot diagrams are diagrams of different knots. A few knot invariants are discussed later in this paper so we do not talk about any specific invariants yet.

When we talk of knots, we mean that we have only one component (or strand). Cutting the knot once would leave us with one line segment. There is no reason to restrict ourselves to one component, so we call any differentiably embedded circles with more than one component a *link*. We adopt the convention that when we mean a link with one component, we will specifically say knot. This is helpful because many invariants of knots are also invariants of links. For an example of a link, see Figure 1.3. Note that Reidemeister moves apply to links as well as knots.

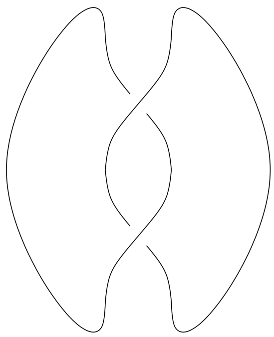


Figure 1.3: The Hopf link

Now we present some basic definitions of the anatomy of a link. A *crossing* is a 4-valent vertex in a link diagram which has over and under information retained. The arc that is an unbroken line is an overarc and the two broken arcs are the underarcs. See Figure 1.4.

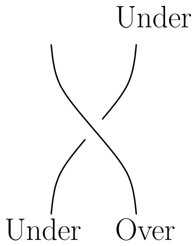


Figure 1.4: Over and under arcs of a crossing

A link diagram is said to have a *nugatory crossing*, or removable crossing, if there exists a circle in the plane of the diagram that intersects the diagram at a single crossing. This situation is shown in Figure 1.5. This crossing is nugatory because it can be removed simply by rotating a portion of the link. In the case of Figure 1.5, the F portion was flipped horizontally from bottom to top. A diagram with no nugatory crossings is said to be *reduced*.

Definition 1.2.1 A diagram D is *alternating* if we can travel along the link and pass over and under alternately. Otherwise, it is called *non-alternating*. A link K is *alternating* if there exists at least one link diagram D of K such that the link diagram D is alternating.

See Figure 1.6 for an alternating and a non-alternating link diagram, respectively. The

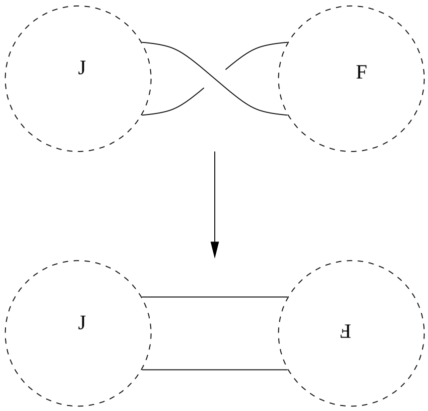


Figure 1.5: A nugatory crossing

link corresponding to the link diagram on the right of Figure 1.6 is the same as the link on the left. We can use a Reidemeister II move on the right-hand link to transform it into the link on the left.

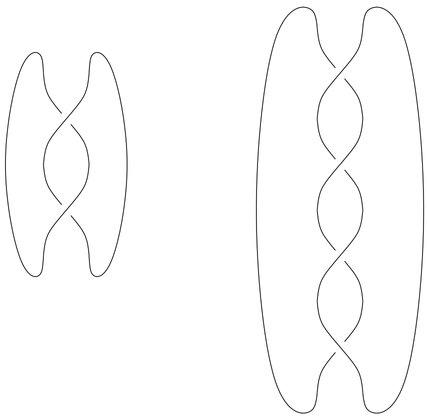


Figure 1.6: An alternating and a non-alternating diagram

Sometimes it is useful to assign on orientation to link diagrams. Then it makes sense to talk about positive or negative crossings in a diagram. For this convention, see Figure 1.7.

1.2.2 Virtual Knots

In classical knot theory, we consider projections of links to some plane to define the link diagrams. For the motivations of virtual knot theory, we encourage the reader to consult [KVKT]. Virtual knot theory is similar in using diagrams except that there is an extra type of crossing called a virtual crossing. Thus, there are three types of crossings for an oriented virtual knot diagram: positive or negative classical crossings and virtual crossings (see Figure 1.8). Furthermore, a virtual link diagram is alternating if we can travel along

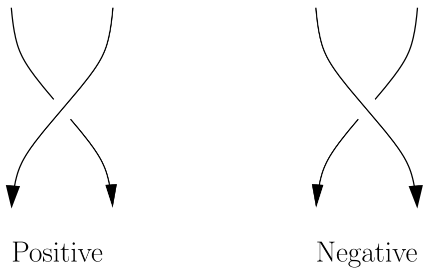


Figure 1.7: Positive and negative crossings

the link passing over and under alternately, just as in the classical case. Note that virtual crossings are not considered "over" or "under", so we just travel to the next classical crossing.

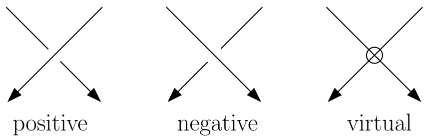


Figure 1.8: Types of crossings

Two virtual diagrams are equivalent if one can be transformed into the other by a finite sequence of extended Reidemeister moves (shown in Figure 1.9) combined with orientation preserving homeomorphisms of the plane to itself, as in the classical case. Moves I, II, and III are just the classical Reidemeister moves. A variation of the type III move with a single virtual crossing and two classical crossings is prohibited.

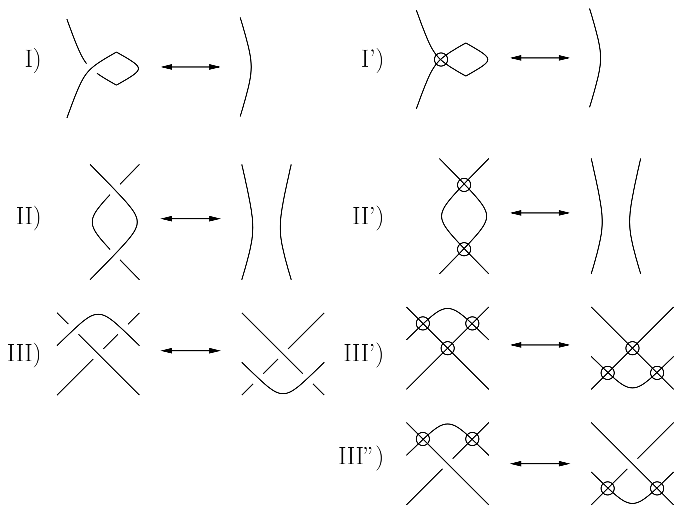


Figure 1.9: Classical and virtual Reidemeister moves

1.2.3 Quandles

We present a brief review of quandles and quandle colorings.

Definition 1.2.2 ([JD]) A *quandle*, X, is a set with a binary operation $* : X \times X \to X$ satisfying the following conditions:

(Q1) For any x in X, $x * x = x$,

(Q2) For any x, y in X, there exists a unique z in X such that $x = z * y$,

(Q3) For any x, y, z in X, $(x * y) * z = (x * z) * (y * z)$.

The condition (Q2) is also equivalent to the following condition:

(Q2') There is a binary operation $*^{-1} : X \times X \to X$ such that $(x*y)*^{-1}y = x = (x*^{-1}y)*y$, for any x, y in X.

The relation of quandles to links is shown in Figure 1.10. Each quandle axiom corresponds to one of the Reidemeister moves. See [BE, JD, MS] for more detailed aspects of quandles. The following is a couple of some commonly used examples of quandles.

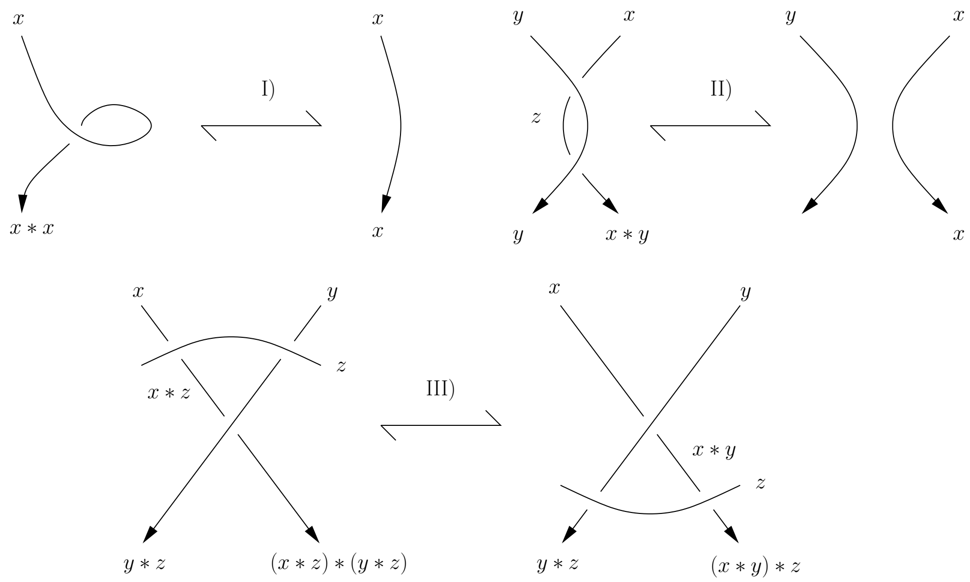


Figure 1.10: Relation of quandles to the Reidemeister moves

Example 1.2.3 Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be a Laurent Polynomial ring over $\mathbb{Z}$, and let $J \subseteq \Lambda$ be an ideal of Λ. Then the quotient ring Λ/J with the binary operation defined by $x * y = tx + (1-t)y$ for any $x, y \in \Lambda/J$ is a quandle, called an *Alexander quandle*. The operation $*^{-1}$ is given by $x *^{-1} y = t^{-1}x + (1-t^{-1})y$.

Example 1.2.4 Let m be a positive integer. For elements $i, j \in \mathbb{Z}_m$, define $i * j = 2j - i \pmod{m}$. Then $*$ defines a quandle called a *dihedral quandle* R_m *of order* m. Also note that the dihedral quandle R_m is isomorphic to $\Lambda/(m, t+1)$. Furthermore, $*^{-1}$ is the same as $*$.

Occassionally, we will need another type of quandle which does not depend upon the orientations of the knot itself.

Definition 1.2.5 [TM] An *involutory quandle*, X, is a quandle such that $(x * y) * y = x$ for all x and $y \in X$.

The notion of involutory quandles first appeared as early as 1942 ([TM]). Also, notice that the dihedral quandle is an involutory quandle, but that, in general, the Alexander quandle is not.

1.2.4 Colorings of Knots

Let X be a quandle, D be an oriented virtual knot diagram, and $\mathcal{A}$ be the set of (over)-arcs. The normal vectors are given in such a way that the ordered pair (tangent, normal) agrees with the orientation of the standard orientation of the xy-plane. See Figure 1.11.

Definition 1.2.6 ([FR]) A *coloring* $\mathcal{C}$ is a map $\mathcal{C} : \mathcal{A} \longrightarrow X$ such that at every classical crossing, the relation $\mathcal{C}(\alpha) * \mathcal{C}(\beta) = \mathcal{C}(\gamma)$ holds, where the normal to the overarc β points from the arc α to the arc γ (see Figure 1.11). At every virtual crossing, the coloring $\mathcal{C}(\alpha) = a$ holds for one arc α, and $\mathcal{C}(\beta) = b$ holds for the other arc β. The image $\mathcal{C}(\alpha)$ is called a *color* of the arc α. The colors in the ordered pair $\langle a, b \rangle$ are called the *source colors.*

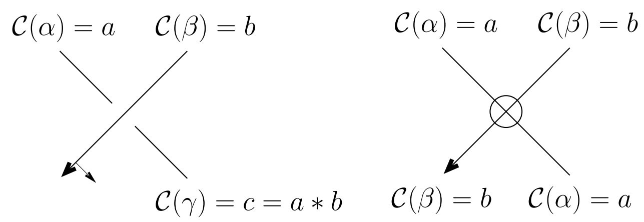


Figure 1.11: Coloring relations

Let $\mathrm{Col}_{\mathrm{X}}(D)$ denote the set of colorings of a knot diagram D of a knot K by a quandle X. Then there is a one-to-one correspondence between the sets of colorings of two diagrams of the same knot. In particular, the number of elements $|\mathrm{Col}_{\mathrm{X}}(D)|$ of $\mathrm{Col}_{\mathrm{X}}(D)$ for a finite quandle X is a knot invariant. Any knot diagram D has at least one coloring for a given quandle X, the *trivial coloring* obtained by letting every arc have the same color. Also, if a knot diagram D can be non-trivially colored by the dihedral quandle R_n, then K is said to be n-colorable.

Example 1.2.7 Let X be the dihedral quandle R_3, and let K be the trefoil as shown in Figure 1.12. Now, let the source colors be given by $\langle 0, 1\rangle$ at the top of the twist in the knot. Then the trefoil is colored by R_3. Notice that this is just one possible coloring of the trefoil from the set $\mathrm{Col}_{\mathrm{X}}(D)$.

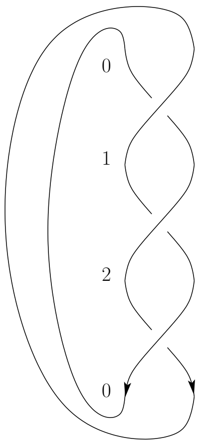


Figure 1.12: A trefoil colored by R_3

The example above is also called a Fox 3-coloring of the trefoil. In general, a Fox n-coloring is a coloring by the dihedral quandle R_n.

1.2.5 The Fundamental Quandle

Definition 1.2.8 ([SJ]) Let $\{x_1, x_2, \ldots, x_k\}$ be variables assigned to arcs of a virtual knot diagram K, let $x_l = x_i * x_j = 2x_j - x_i$, where x_j is the variable assigned to the overarc and x_i is the variable assigned to one of the underarcs. Then x_l is assigned to the other underarc. Now let A be the matrix of relations associated with the set of these equations such that the equations correspond to the rows in the matrix, and the ith variable corresponds to the ith column of the matrix. If $M_{ij}(K)$ is the (i, j)-minor of A, then the determinant of the knot K is defined by

$$\mathrm{Det}(K) = \gcd(\{|\mathrm{Det}(M_{ij}(K))| : 1 \leq i \leq j \leq k\}). \tag{1.2.1}$$

The natural question now is what would be a suitable n to color any given diagram with a dihedral quandle R_n? The most natural choice is the determinant of the knot $\mathrm{Det}(K)$.

Theorem 1.2.9 *[F] Let K be a classical knot. For a prime p, K is p-colorable if* $p|\mathrm{Det}(K)$.

Example 1.2.10 Let K be the trefoil shown in Figure 1.13. Then the matrix A is given by

$$\begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 2 & -1 & -1 \end{bmatrix},$$

where the rows correspond to the relations and the columns correspond to the variables. The determinant of each minor is ± 3, so the determinant of K will be 3. Thus, this knot can be colored non-trivially using the dihedral quandle R_3. Note that in the classical case, any minor of A will have the same determinant so the gcd of all the minors is equivalent to finding one of the minors. See the figure.

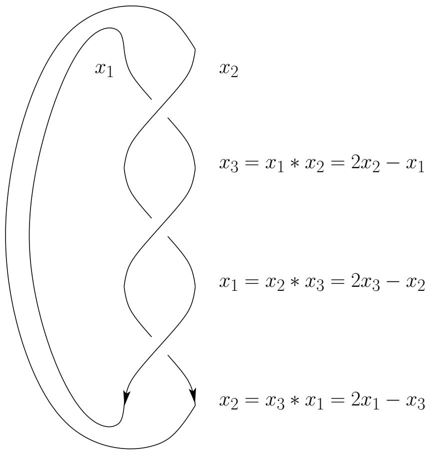


Figure 1.13: Trefoil and its relations

Remark 1.2.11 Consider the figure-eight knot diagram shown below in Figure 1.14. The determinant of this knot is 5, but it has only 4 arcs. This implies that the knot is colored by R_5, but not all elements of R_5 are used for colors.

Definition 1.2.12 [JD, KVKT, MS] Let $\{x_1, x_2, \ldots, x_k\}$ be variables assigned to arcs of a virtual knot diagram K, let $x_l = x_i * x_j$ be assigned at each crossing, where x_j is the variable assigned to the overarc and x_i is the variable assigned to one of the underarcs from which the orientation of the normal vector of the over-arc points. Then x_l is assigned to the other underarc. The quandle $Q(K)$ determined by the set of generators $\{x_1, x_2, \ldots, x_k\}$ and the set of relations $\{x_l = x_i * x_j\}$ over all crossings is called the *fundamental quandle* of K. In

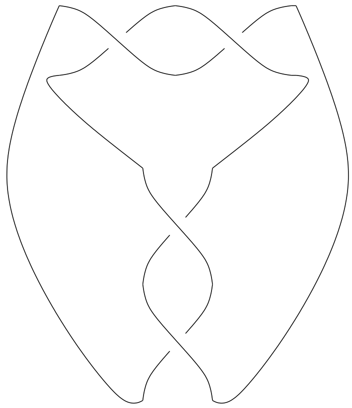


Figure 1.14: Figure 8 knot

addition, when the relations $(x * y) * y = x$ are imposed for all elements of $Q(K)$, then the quandle thus determined is called the *fundamental involutory quandle*.

A quandle defined by the set of generators $\{x_1, x_2, \ldots, x_k\}$ and relations $\{r_1, r_2, \ldots, r_m\}$ as above is denoted by $\langle x_1, x_2, \ldots, x_k \mid r_1, r_2, \ldots, r_m \rangle$, and this notation is called a *presentation* of the quandle. The fundamental involutory quandle is denoted by $IQ(K)$.

1.2.6 Kauffman-Harary Conjecture

In this paper we consider the following conjecture by Kauffman and Harary ([KH]).

Conjecture 1.2.13 (Kauffman-Harary Conjecture 1) *Let D be a reduced alternating knot diagram with a prime determinant p. Then every nontrivial Fox's p-coloring of D assigns different colors to different arcs of D.*

Instead of restricting the conjecture to only classical knots, we consider the virtualization of the Kauffman-Harary Conjecture.

Conjecture 1.2.14 (Kauffman-Harary Conjecture 2) *Let D be a reduced alternating virtual knot diagram with a prime determinant p. Then every nontrivial Fox's p-coloring of D assigns different colors to different arcs of D.*

Kauffman and Harary first proved Conjecture 1.2.13 for the family of rational (or 2-bridge) classical knots in [KL], and [PL]. Asaeda, et al., [AM], proved the conjecture to be true for Montesinos links (which include pretzel knots). Our aim is to prove Kauffman-Harary Conjecture 2 to be true for certain families of virtual knots.

1.2.7 Bracket Polynomial and the Jones Polynomial

The following material is standard, see [KKP, CP] for more information. Even though the letter A was used earlier for a matrix, we now follow the standard notation and use A as a variable in this section. Let D be an unoriented diagram of a knot K. The *bracket polynomial* $\langle D\rangle$ of D is a Laurent polynomial of a variable A defined by the following axioms:

1. $\langle D_+\rangle = A\langle D_0\rangle + A^{-1}\langle D_\infty\rangle$,

2. $\langle \bigcirc D\rangle = d\langle D\rangle$,

3. $\langle \bigcirc\rangle = d$,

where $d = -A^2 - A^{-2}$ and D_+, D_0, and D_∞ are identical outside of a small ball neighborhood, inside which they look like as depicted in Figure 1.15. The axioms give a recursive computation of $\langle D\rangle$. By repeated application of axiom 1, a diagram becomes a set of diagrams with no crossings. The value of each of these other diagrams is calculated by using axioms 2 and 3.

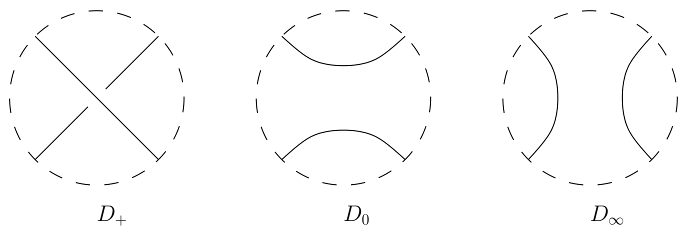


Figure 1.15: Bracket crossing computations

Theorem 1.2.15 *[KJ] The bracket polynomial $\langle D\rangle$ is invariant under Reidemeister II and III moves.*

In order for the bracket polynomial to be invariant under the Reidemeister I move, we need the writhe of a knot ([KKP], for example). The *writhe* $w(D)$ of a knot is the number of positive crossings minus the number of negative crossings.

Definition 1.2.16 Let D be an oriented diagram, and let $|D|$ denote the diagram D without orientation. Then the *normalized bracket polynomial* is given by

$$\tilde{V}_D(A) = (-A^{-3})^{w(D)}\langle |D|\rangle(A).$$

This polynomial is invariant under all the Reidemeister moves.

Theorem 1.2.17 *[KJ] The normalized bracket polynomial is equivalent to the Jones polynomial under a change of variable:*

$$V_L(A^{-4}) = \tilde{V}_L(A),$$

where $V_L(t)$ is the Jones polynomial.

2 Swap Moves and Some Families of Virtual Knot Diagrams

2.1 The k-Swap Moves

In this section, we define k-swap moves and investigate their properties.

2.1.1 Definition of k-Swap Moves

Let $\mathcal{A}(K_i)$ be the set of arcs of the virtual knot diagrams K_i, for $i = 1, 2$, and let $\mathcal{C}_i \in \mathrm{Col}_n(K_i)$ such that $\mathcal{C}_i : \mathcal{A}(K_i) \to R_n$, where n is a positive integer.

Definition 2.1.1 Let K_1 be a twist with k positive crossings and one virtual crossing at the bottom as depicted in the left of Figure 2.1. Then a *k-swap move* is a move which changes K_1 into the twist K_2 such that K_2 has a virtual crossing at the top followed by k negative crossings as in the right of Figure 2.1. In Figure 2.1, K_1 has positive crossings, by convention. The change from K_2 to K_1, conversely, is also called a *k-swap move*. Note that if the two arcs have parallel orientation, then the crossings of K_1 are positive, but if the arcs have anti-parallel orientation, then the crossings are negative. Thus our convention for the positive or negative sign of a twist with a virtual crossing is the same as the sign when the arcs have parallel orientation.

2.1.2 Colorings and the Determinant under k-Swap Moves

Definition 2.1.2 Let D_2 be a diagram obtained from D_1 by applying a k-swap move once. Then define a map

$$\mathrm{Sp} : \mathcal{A}(D_1) \to \mathcal{A}(D_2)$$

as follows. If $\alpha \in \mathcal{A}(D_1)$ is not contained in the twist where the k-swap move is performed, then define $\mathrm{Sp}(\alpha) = \alpha \in \mathcal{A}(D_2)$. If $\alpha \in \mathcal{A}(D_1)$ is contained in the twist, then α is one of the arcs α_i or β_j in the twist such that $0 \leq i, j \leq l$, where $l = k/2$ if k is even, and $l = (k-1)/2$

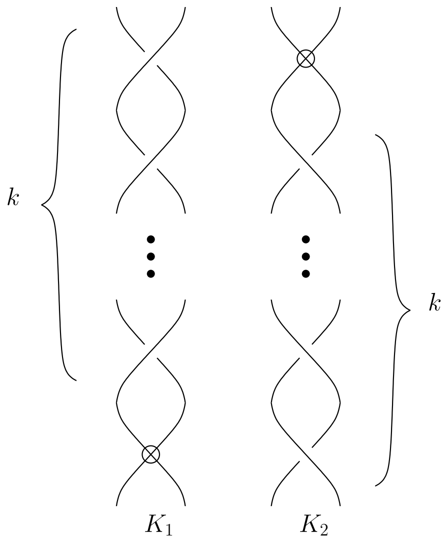


Figure 2.1: A k-swap move

if k is odd, as depicted in the left of Figure 2.2. The arcs after the move are labeled as in the right of Figure 2.2. Then define $\mathrm{Sp}(\alpha_i) = \alpha'_i$ and $\mathrm{Sp}(\beta_j) = \beta'_j$ for all i and j.

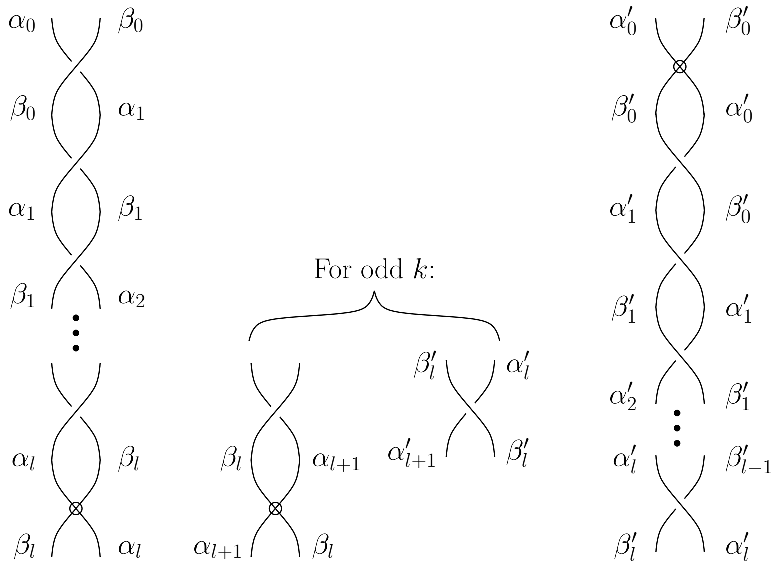


Figure 2.2: Arcs under a k-swap

Lemma 2.1.3 *The function*

$$\mathrm{Sp} : \mathcal{A}(D_1) \to \mathcal{A}(D_2)$$

is a bijection.

Proof. This follows from the definition. ■

Theorem 2.1.4 *Again, let D_2 be the diagram obtained from D_1 by applying a k-swap move once. Let*

$$\mathrm{Sp} : \mathcal{A}(D_1) \to \mathcal{A}(D_2)$$

be the bijection defined in Definition 2.1.2. Let $\mathcal{C}_1 \in \mathrm{Col}_n(D_1)$. Then the map defined by $\mathcal{C}_2 = \mathcal{C}_1 \circ \mathrm{Sp}^{-1}$ is an n-coloring of D_2: $\mathcal{C}_2 \in \mathrm{Col}_n(D_2)$.

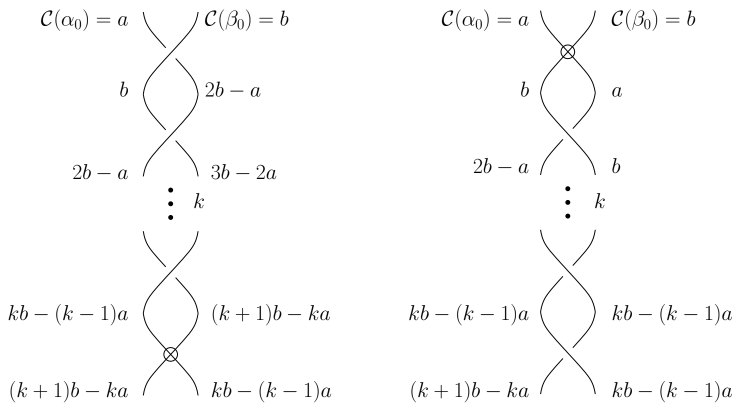


Figure 2.3: Dihedral coloring proof

Proof. Let D_1 be the twist such that D_1 has k positive crossings with a virtual crossing at the bottom. Now let $\langle a, b\rangle$ be the vector of source colors at the top of the twist, where $a = \mathcal{C}_1(\alpha_0)$ and $b = \mathcal{C}_1(\beta_0)$. Then, using the equation $x * y = 2y - x$, we end up with

$$\langle (k+1)b - ka, kb - (k-1)a\rangle$$

at the bottom of D_1 (see the left of Figure 2.3). Similarly, D_2 is the twist with a virtual crossing at the top followed by k negative crossings. If $\langle a, b\rangle$ is the vector of source colors at the top of the twist, then

$$\langle (k+1)b - ka, kb - (k-1)a\rangle$$

is again found at the bottom (see the right of Figure 2.3).

Thus, $\mathcal{C}_1(\alpha_i) = \mathcal{C}_1(\mathrm{Sp}^{-1}(\alpha_i')) = \mathcal{C}_2(\alpha_i')$, for some $\alpha_i \in \mathcal{A}(D_1)$.

■

Corollary 2.1.5 *Under the assumptions of Theorem 2.1.4, we have the equality $\mathcal{C}_1(\mathcal{A}(D_1)) = \mathcal{C}_2(\mathcal{A}(D_2))$.*

Proof. Let $m \in \mathcal{C}_1(\mathcal{A}(D_1)) \subseteq R_n$. Then there is an arc $\gamma \in \mathcal{A}(D_1)$ such that $\mathcal{C}_1(\gamma) = m$. Then $\mathcal{C}_2(\mathrm{Sp}(\gamma)) = \mathcal{C}_1(\mathrm{Sp}^{-1}(\mathrm{Sp}(\gamma))) = m$. Hence, $m \in \mathcal{C}_2(\mathcal{A}(D_2))$.

■

Corollary 2.1.6 *Again using the assumptions of Theorem 2.1.4, we see that*

$$\mathrm{Det}(D_1) = \mathrm{Det}(D_2).$$

Proof. From Theorem 2.1.4, we know that the arcs do not change so the matrix defined in the definition of determinant does not change.

■

Corollary 2.1.7 *If the assumptions of Theorem 2.1.4 are true, then* $|\mathrm{Col}_p(D_1)| = |\mathrm{Col}_p(D_2)|$.

Proof. Using Corollary 2.1.5, each coloring of D_1 has a corresponding coloring of D_2, and $\mathcal{C}_1(\mathcal{A}(D_1)) = \mathcal{C}_2(\mathcal{A}(D_2))$. The result follows.

■

Corollary 2.1.8 *If D_2 is the diagram obtained from D_1 by applying a k-swap move once, and if D_1 is alternating, then D_2 is alternating as well.*

Proof. Again using Theorem 2.1.4, the arcs do not change between D_1 and D_2 so the result follows.

■

2.1.3 Quandles and the Jones Polynomial under k-Swap Moves

Next we investigate the fundamental involutory quandle and the Jones polynomial.

Theorem 2.1.9 *Let X be the fundamental involutory quandle of a virtual knot K_1 with a diagram D_1. If D_2 is obtained from D_1 by a k-swap move, then X is the fundamental involutory quandle of the virtual knot K_2, where D_2 is a diagram of K_2.*

Proof. This follows from Definition 2.1.1 and Figure 2.4. Also, the $k = 1$ case was proved in [KVKT] by Kauffman.

■

Theorem 2.1.10 *For two virtual knots K_1 and K_2, if K_2 is obtained from K_1 by a k-swap move, then their Jones polynomial coincide: $V_{K_1}(t) = V_{K_2}(t)$.*

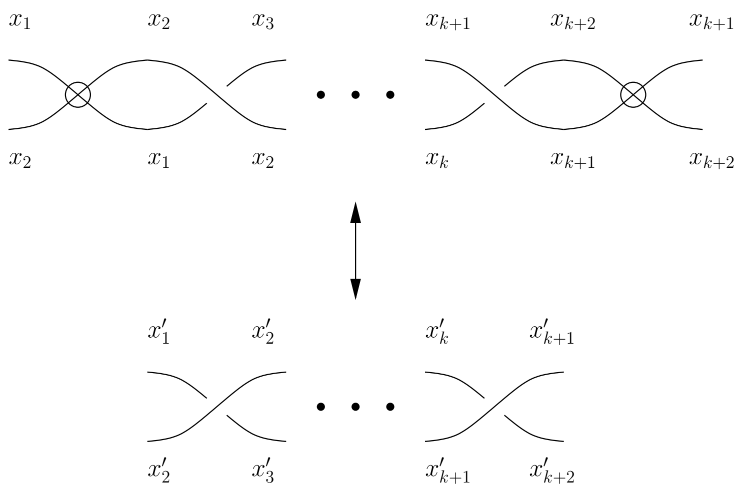


Figure 2.4: A k-swap on the fundamental involutory quandle

Proof. Given a twist $\mathcal{T}$ in a classical knot K, use the virtual Reidemeister II' move to make a knot K'. This knot K' is still an unreduced classical knot, so perform a k-swap to obtain a virtual knot K''. By Lemma 2.1.11, this knot K'' has the same Jones polynomial as K. ■

This follows from the next lemma.

Lemma 2.1.11 *Let $\langle D_1 \rangle$ be the bracket polynomial of a virtual knot diagram D_1. If D_2 is obtained from D_1 by a k-swap move, then $\langle D \rangle = \langle D' \rangle$.*

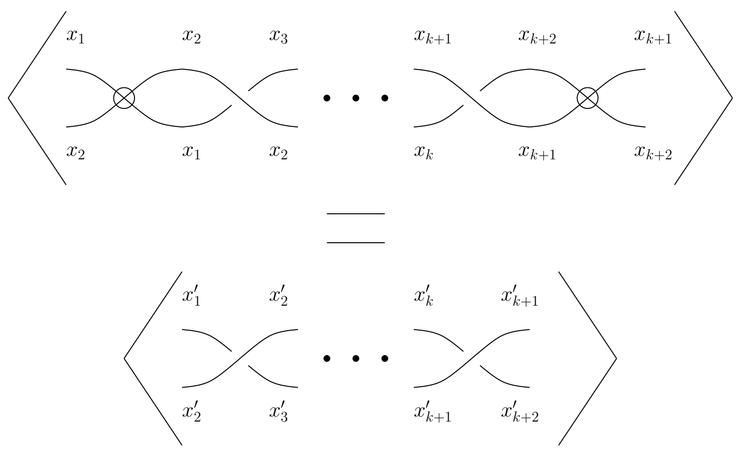


Figure 2.5: A k-swap's effect on the bracket polynomial

Proof. Use Definition 2.1.1 and look at Figure 2.5. More specifically, apply the axiom in Fig. 1.15 at a crossing in between virtual crossings in the top of Fig. 2.5 and a crossing of the bottom of the figure, then after Reidemeister moves, we obtain the same invariant values. Also, note that Kauffman proved the $k = 1$ case in [KVKT], and this proof is a direct analogue.

■

The following sections will give many examples of the above statements, so we will not delve into any examples here.

2.2 Some Families of Alternating Virtual Knot Diagrams

2.2.1 Closed 2-String Virtual Braid Diagrams

In order to show what the colorings of closed 2-string virtual braids are, we need some background.

Definition 2.2.1 For a postive integer n, let $k_i \in \mathbb{Z}$ for $i = 1, ..., n$. A *virtual twist* $[x_1, k_1, k_2, ..., k_n, x_2]$ is a sequence of classical twists and virtual crossings such that the following holds true. First, each k_i corresponds to a classical twist with $|k_i|$ classical crossings, positive crossings if $k_i > 0$ and negative if $k_i < 0$. Between the i-th and $(i+1)$-th twists is a virtual crossing for $i = 1, 2, ..., n$, and each x_ℓ, where $\ell = 1, 2$, represents either a virtual crossing (denoted by $x_\ell = v$) or no virtual crossing (denoted by $x_\ell = \emptyset$). An *alternating virtual twist* $[x_1, k_1, k_2, ..., k_n, x_2]$, where x_ℓ is either v or $\emptyset$, is a virtual twist such that $k_i > 0$ for odd i and $k_j < 0$ for even j, or $k_i < 0$ for odd i and $k_j > 0$ for even j. We also write $[k_1, k_2, ..., k_n, x_\ell]$ for $[\emptyset, k_1, k_2, ..., k_n, x_\ell]$. For examples of this, see Figure 2.6.

The above definition agrees with our intuition and use of the word alternating because either strand in an alternating virtual twist is seen to alternate in overarcs and underarcs.

Definition 2.2.2 If $[k_1, k_2, ..., k_n, x_1]$ is a virtual twist, then the closed 2-string virtual braid diagram is denoted by $T(2, [k_1, k_2, ..., k_n, x_1])$. Also, $T(2, [k_1])$ is the classical $(2, k_1)$-torus knot ([CP]), also denoted by $T(2, k_1)$.

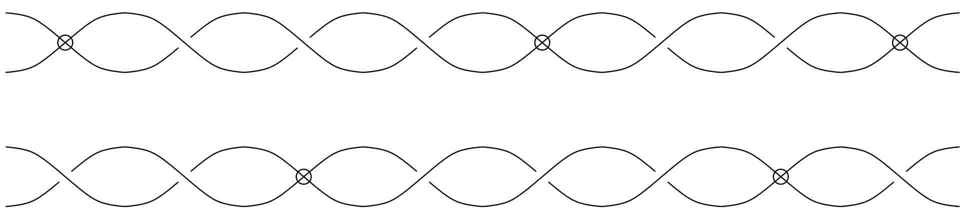


Figure 2.6: Examples of virtual twists: the top is $[v, 3, -2, v]$ and the bottom is $[2, -3, 1]$

Note that if $D = T(2, [k_1, k_2, ..., k_n])$, then there are $n-1$ virtual crossings in D. Also, if $D = T(2, [k_1, k_2, ..., k_n, v])$, then there are n virtual crossings in D.

The proof of the following lemma are straightforward from the definition, so they are omitted.

Lemma 2.2.3 *The virtual knot diagram $T(2, [k_1, k_2, ..., k_n])$ is alternating iff n is odd and k_i is positive for odd i and negative for even i, or positive for even i and negative for odd i. Similarly, the virtual knot diagram $T(2, [k_1, k_2, ..., k_n, v])$ is alternating iff n is even and the same condition is satisfied. In Figure 2.6, the torus knot closure of the top virtual twist is not alternating but the bottom one is.*

Note that Lemma 2.2.3 implies that if $D = T(2, [k_1, k_2, ..., k_n, x])$ is alternating then it has an even number of virtual crossings, where x is either v or $\emptyset$. Here $x = \emptyset$ means that $D = T(2, [k_1, ..., k_n])$.

Lemma 2.2.4 (Reducing Lemma) *If $\mathcal{T} = [x_1, k_1, k_2, ..., k_n, x_2]$ is an alternating virtual twist, where x_ℓ is either v or $\emptyset$, then the virtual twist becomes*

$$\mathcal{T}' = \left[\ x(\mathcal{T}), \chi(\mathcal{T})\epsilon(x_2)\sum_{i=1}^{n}(-1)^i k_i\ \right],$$

where

$$x(\mathcal{T}) = \begin{cases} v & \text{if } \chi(\mathcal{T})\epsilon(x_1)\epsilon(x_2) = -1 \\ \emptyset & \text{if } \chi(\mathcal{T})\epsilon(x_1)\epsilon(x_2) = 1 \end{cases},$$

$$\chi(\mathcal{T}) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases},$$

$$\epsilon(x_i) = \begin{cases} 1 & \text{if } x_i = v \\ -1 & \text{if } x_i = \emptyset \end{cases},$$

after a sequence of k-swap moves and Reidemeister II$'$ *moves.*

Proof. Suppose $\mathcal{T} = [x_1, k_1, k_2, ..., k_n, x_2]$ is an alternating virtual twist. We prove the lemma by induction on n.

For $n = 1$, we have $\mathcal{T} = [\ x_1, k_1, x_2\ ]$. First, we prove the case where $\mathcal{T} = [\ v, k_1, v\ ]$. We perform a k_1-swap and we obtain $[\ v, v, k_1\ ]$. Since two virtual crossings are adjacent, we can perform a Reidemeister type II$'$ move to get $\mathcal{T}' = [\ -k_1\ ]$. Using the formula given in the statement of the lemma, we notice that $\chi(\mathcal{T}) = 1$, since $n = 1$. Also, $\epsilon(x_i) = 1$ for both $i = 1, 2$. See Figure 2.7. Thus, by the formula, $\mathcal{T}' = [\ \emptyset, (1)(1)(1)((-1)^1 k_1)\ ] = [\ -k_1\ ]$. The other cases are similar and we obtain

$$\begin{aligned}[\ \emptyset, k_1, \emptyset\ ] &= [\ k_1\ ],\\ [\ \emptyset, k_1, v\ ] &= [\ v, -k_1\ ],\\ [\ v, k_1, \emptyset\ ] &= [\ v, k_1\ ].\end{aligned}$$

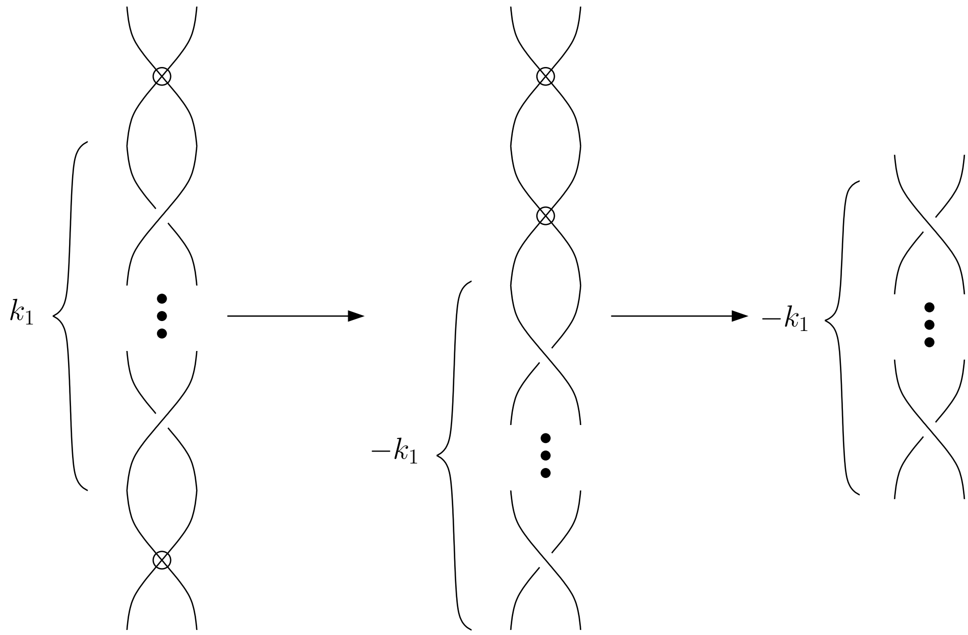


Figure 2.7: The case $n = 1$ of the Reducing Lemma

For the next part of the induction, there are 8 cases total, but two sets of four can be checked in a similar manner. For the first part, we prove the cases where $x_2 = \emptyset$. Thus, the cases we check first are:

$$\begin{aligned}n \text{ is even, } x_1 = v,\ x_2 = \emptyset &:\ [\ v, k_1, ..., k_{n-2}, k_{n-1}, k_n, \emptyset\ ],\\ n \text{ is even, } x_1 = \emptyset,\ x_2 = \emptyset &:\ [\ \emptyset, k_1, ..., k_{n-2}, k_{n-1}, k_n, \emptyset\ ],\\ n \text{ is odd, } x_1 = v,\ x_2 = \emptyset &:\ [\ v, k_1, ..., k_{n-2}, k_{n-1}, k_n, \emptyset\ ],\\ n \text{ is odd, } x_1 = \emptyset,\ x_2 = \emptyset &:\ [\ \emptyset, k_1, ..., k_{n-2}, k_{n-1}, k_n, \emptyset\ ].\end{aligned}$$

For an illustration of these cases, see Figure 2.8.

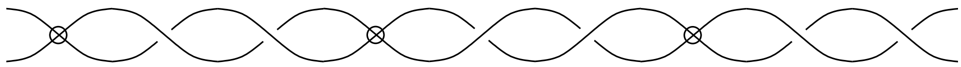


Figure 2.8: The case when n is odd, $x_1 = v$ and $x_2 = \emptyset$.

So, assume the result holds for $n-2$ classical twists. We use the notation

$$\mathcal{T} = [\ v, k_1, ..., k_{n-2}, \emptyset\ ] \xrightarrow{ks} \left[\ x(\mathcal{T}), \chi(\mathcal{T})(-1)\sum_{i=1}^{n-2}(-1)^i k_i\ \right],$$

to express the condition that T is changed to $\left[\ x(\mathcal{T}), \chi(\mathcal{T})(-1)\sum_{i=1}^{n-2}(-1)^i k_i\ \right]$ by a k-swap move. Starting with an alternating virtual twist

$$\mathcal{S} = [\ v, k_1, ..., k_{n-2}, k_{n-1}, k_n, \emptyset\ ],$$

we perform a (k_{n-1})-swap and a Reidemeister II$'$ move to yield

$$\mathcal{S}' = [\ v, k_1, ..., k_{n-2} - k_{n-1} + k_n\ ].$$

If n is even, then $\mathcal{S}'$ still has an even number of classical twists (similarily if n is odd). This means that $\chi(\mathcal{S}) = \chi(\mathcal{S}')$. Because $\mathcal{S}$ is alternating, $\mathcal{S}'$ is as well, and $-k_{n-1}$ is the same parity as k_{n-2} and k_n. Now let $k_m = k_{n-2} - k_{n-1} + k_n$. Notice that the k_m twist is actually the $(n-2)$-th twist. This means that

$$\mathcal{S}' = \left[\ x(\mathcal{T}), \chi(\mathcal{S})(-1)\sum_{i=1}^{m}(-1)^i k_i\ \right],$$

by the induction hypothesis.

Now for the other four cases:

$$\begin{aligned}
n \text{ is even, } x_1 = v,\ x_2 = v &\ :\ [\ v, k_1, ..., k_{n-2}, k_{n-1}, k_n, v\ ],\\
n \text{ is even, } x_1 = \emptyset,\ x_2 = v &\ :\ [\ \emptyset, k_1, ..., k_{n-2}, k_{n-1}, k_n, v\ ],\\
n \text{ is odd, } x_1 = v,\ x_2 = v &\ :\ [\ v, k_1, ..., k_{n-2}, k_{n-1}, k_n, v\ ],\\
n \text{ is odd, } x_1 = \emptyset,\ x_2 = v &\ :\ [\ \emptyset, k_1, ..., k_{n-2}, k_{n-1}, k_n, v\ ].
\end{aligned}$$

For an illustration of these cases, see Figure 2.9.

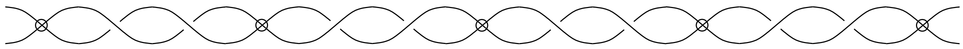


Figure 2.9: The case when n is even and $x_i = v$.

We start with

$$\mathcal{T} = [\ x(\mathcal{T}), k_1, ..., k_{n-2}, k_{n-1}, k_n, v\ ],$$

and we perform a k_n-swap and a Reidemeister type II′ move to obtain

$$\mathcal{T}' = [\ x(\mathcal{T}), k_1, ..., k_{n-2}, k_{n-1} - k_n\ ].$$

Using the previous result from the induction step,

$$\begin{aligned} \mathcal{T}' \xrightarrow{ks} \mathcal{T}'' &= \left[\ x(\mathcal{T}), \chi(\mathcal{T}')(-1)\sum_{i=1}^{n}(-1)^i k_i\ \right] \\ &= \left[\ x(\mathcal{T}), \chi(\mathcal{T})\sum_{i=1}^{n}(-1)^i k_i\ \right], \end{aligned}$$

because $\chi(\mathcal{T}') = (-1)\chi(\mathcal{T})$. This is true because if $\mathcal{T}$ has an odd number of classical twists, then $\mathcal{T}'$ has an even number of classical twists. This is what we would have obtained if we used

$$\left[\ x(\mathcal{T}), \chi(\mathcal{T})\epsilon(x_2)\sum_{i=1}^{n}(-1)^i k_i\ \right]$$

on the original $\mathcal{T}$, where $\epsilon(x_2) = 1$. Hence, the result follows.

■

Lemma 2.2.5 *If $D = T(2, [k_1, k_2, ..., k_n, x_2])$ is alternating, then one of the following holds:*
a) n is even and $x_2 = v$,
b) n is odd and $x_2 = \emptyset$.

Proof. Since D is alternating, the twist $[k_1, k_2, ..., k_n, x_2]$ must be alternating too, and its strands must connect in such a way that D is still alternating. See Figure 2.10. Now since D is a torus knot, $x_1 = \emptyset$ because if $x_1 = v$, it can be moved to the bottom of the torus knot. By Lemma 2.2.5, D is alternating if it has an even number of virtual crossings. So if n is even, then there are $n - 1$ virtual crossings between each classical twist so $x_2 = v$. A similar argument holds for odd n.

■

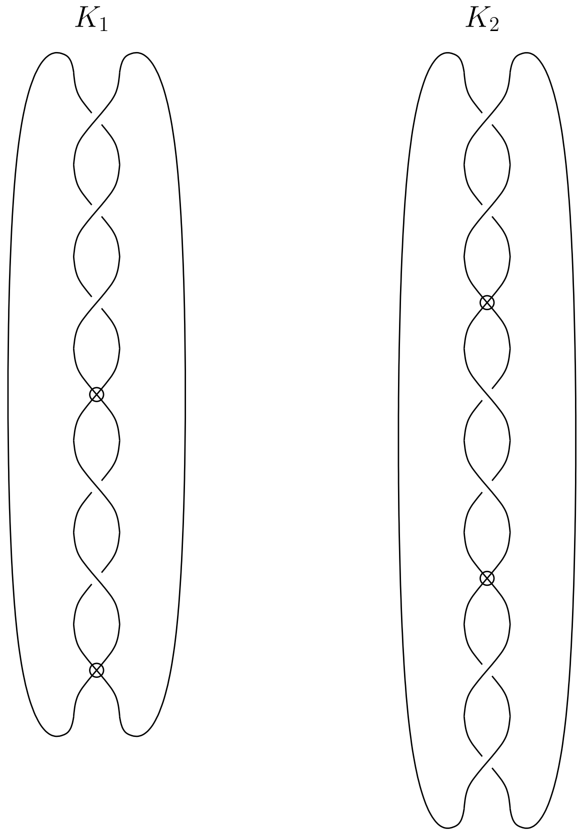


Figure 2.10: The two cases from Lemma 2.2.5: D_1 has even n and $x_2 = v$, while D_2 has odd n and $x_2 = \emptyset$.

Theorem 2.2.6 *Let $T(2, [k_1, k_2, ..., k_n, x_2])$ be alternating, p be prime, $\mathcal{T}$ be the virtual twist in K, and x_2 be either v or $\emptyset$ (according to whether n is even or odd, respectively). Then*

$$\text{Col}_p(T(2, [k_1, k_2, ..., k_n, x_2])) = \text{Col}_p\left(T\left(2, \left[\ \chi(\mathcal{T})\epsilon(x_2)\sum_{i=1}^{n}(-1)^i k_i\ \right]\right)\right),$$

where $\chi(\mathcal{T})$ and $\epsilon(x_2)$ are as in the Reducing Lemma.

Proof. Let $\mathcal{T}$ be the virtual twist in D. Applying the formula from the Reducing Lemma

$$D' = \left[x(\mathcal{T}), \chi(\mathcal{T})\epsilon(x_2)\sum_{i=1}^{n}(-1)^i k_i\ \right]$$

to D, we see that $x_1 = \emptyset \Rightarrow \epsilon(x_1) = -1$. It suffices to check what $x(\mathcal{T})$ is. Using Lemma 2.2.5, we have two cases to check. If n is even, then $\chi(\mathcal{T}) = -1$ and $x_2 = v \Rightarrow \epsilon(x_2) = 1$. Thus, $x(\mathcal{T}) = \emptyset$ because $\chi(\mathcal{T})\epsilon(x_1)\epsilon(x_2) = 1$. Now if n is odd, then $\chi(\mathcal{T}) = 1$ and $x_2 = \emptyset \Rightarrow \epsilon(x_2) = -1$. Thus, $x(\mathcal{T}) = \emptyset$ because $\chi(\mathcal{T})\epsilon(x_1)\epsilon(x_2) = 1$. So by Corollary 2.1.7, the result is obtained.

■

Theorem 2.2.7 *The determinant of an alternating knot $D = T(2, [k_1, k_2, ..., k_n, x_2])$, where x_2 is either v or $\emptyset$, is given by*

$$\text{Det}(D) = \left| \sum_{i=1}^{n} (-1)^i k_i \right|.$$

Proof. Using the Reducing Lemma and Theorem 2.2.6, we obtain the knot

$$D \xrightarrow{ks} D_1 = T\left(2, \left[\chi(\mathcal{T})\epsilon(x_2) \sum_{i=1}^{n} (-1)^i k_i \right]\right).$$

By Corollary 2.1.6, we know that $\text{Det}(D) = \text{Det}(D_1)$. Hence, using the fact that $\text{Det}(T(2, n)) = n$ (see [CP], for example), we see that

$$\text{Det}(T(2, [k_1, ..., k_n, x_2])) = \left| \sum_{i=1}^{n} (-1)^i k_i \right|,$$

where x_2 is either v or $\emptyset$.

■

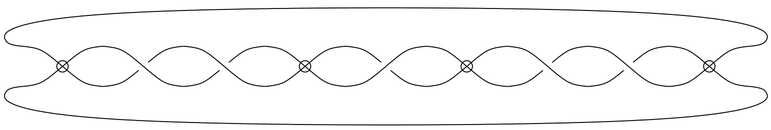


Figure 2.11: The 2-string virtual braid diagram $D = T(2, [v, 2, -1, 2, v])$

Example 2.2.8 Consider the 2-string virtual braid diagram $D = T(2, [v, 2, -1, 2, v])$. By inspection of Figure 2.11 and Lemma 2.2.5, D is alternating. So, performing two 2-swap moves on D and then using a Reidemeister II′ move gives us $D' = T(2, [-5])$. Using the Reducing Lemma gives us the same D'. We start with n being odd, $x_1 = v, x_2 = v$, so then

$$\epsilon(x_1) = 1, \epsilon(x_2) = 1, \chi(\mathcal{T}) = 1, x(\mathcal{T}) = \emptyset,$$

where $\mathcal{T} = [v, 2, -1, 2, v]$. By the Reducing Lemma, we get

$$D' = T\left(2, \left[\emptyset, (1)(1) \sum_{i=1}^{3} (-1)^i k_i \right]\right) = T(2, [(-2 - 1 - 2)]) = T(2, [-5]).$$

This shows that the Reducing Lemma agrees with the figure. By Theorem 2.2.6, we know that

$$\mathrm{Col}_p(D) = \mathrm{Col}_p(D') = \mathrm{Col}_p(T(2, [-5])).$$

Furthermore, using Theorem 2.2.7, we also know that

$$\mathrm{Det}(D) = \mathrm{Det}(D') = 5.$$

2.2.2 Virtual Pretzel Knot Diagrams

Definition 2.2.9 A virtual pretzel knot diagram is denoted by

$$P(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n),$$

where a given twist $\mathcal{T}_i = [x_{i,1}, k_{i,1}, k_{i,2}, ..., k_{i,j}, x_{i,2}]$ is inserted into the i-th box in Figure 2.12, where $n \geq 3$. See Figure 2.13 for an example.

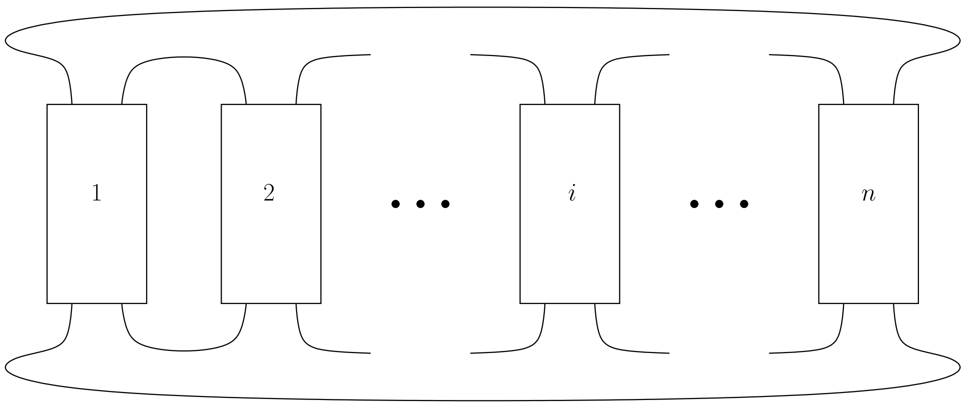


Figure 2.12: The general setup of a pretzel knot.

Remark 2.2.10 If $n = 1$ or $n = 2$, we have a $T(2, \mathcal{T})$ torus knot diagram, where $\mathcal{T}$ is some twist.

Definition 2.2.11 A virtual twist $\mathcal{T}$ is *even* if it has an even number of virtual crossings or no virtual crossings, and it is *odd* if it has an odd number of virtual crossings.

Lemma 2.2.12 *If* $D = P(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$ *is an alternating pretzel link diagram such that* $\mathcal{T}_r = [x_{r,1}, k_{r,1}, k_{r,2}, ..., k_{r,m}, x_{r,2}]$*, where* $r = 1, \ldots, n$*, then one of the following properties will hold:*

a) $\mathcal{T}_r = [x_{r,1}, k_{r,1}, k_{r,2}, ..., k_{r,m}, x_{r,2}]$ *for all* $r = 1, 2, ..., n$ *is an alternating even virtual twist*

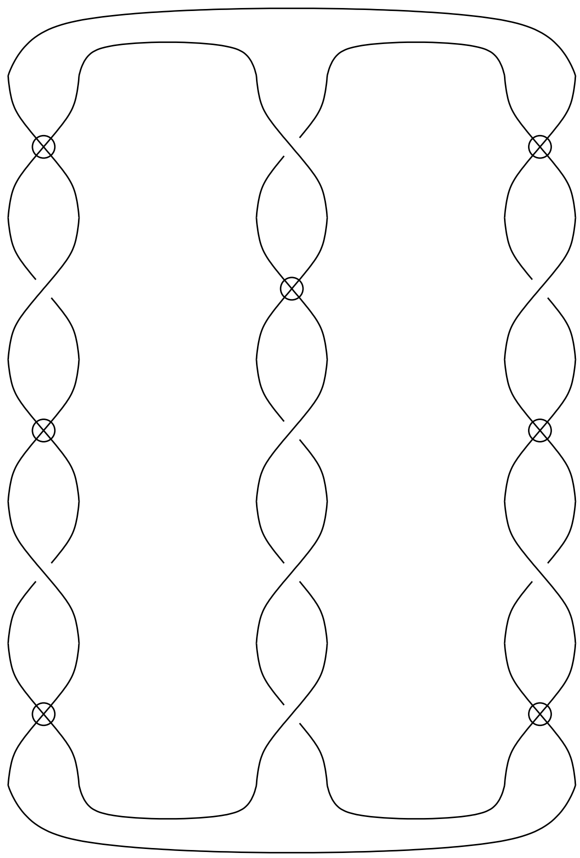


Figure 2.13: The alternating virtual pretzel knot $P([v, 1, -1, v], [-1, 3], [v, 1, -1, v])$.

in D (see Figure 2.12), or

b) $\mathcal{T}_r$ *is an alternating odd virtual twist in D for all* $r = 1, 2, ..., n$ *(see Figure 2.12).*

Conversely, for any sequence $(m_1, m_2, ..., m_n)$ *of all even or all odd non-negative integers, there is an alternating diagram* $P(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$ *such that the number of virtual crossings of* $(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$ *are* $(m_1, m_2, ..., m_n)$*.*

Proof. Assume $\mathcal{T}_r$ is an even alternating twist in D, $\mathcal{T}_{r+1}$ is the twist to the right of $\mathcal{T}_r$ and connected to $\mathcal{T}_r$. By the Reducing Lemma, let $\mathcal{T}_r'$ and $\mathcal{T}_{r+1}'$ be obtained from $\mathcal{T}_r$ and $\mathcal{T}_{r+1}$, respectively. Since $\mathcal{T}_r$ is even, $\mathcal{T}_r'$ is a classical twist, so that if $\mathcal{T}_{r+1}$ is even, then $\mathcal{T}_{r+1}'$ is classical as well. Now we show that $\mathcal{T}_{r+1}$ cannot be odd. If it was, then $\mathcal{T}_{r+1}'$ would have one virtual crossing at the top. Without loss of generality, assume that $\mathcal{T}_r'$ has all positive classical crossings. Then the upper right strand of $\mathcal{T}_r'$ is an overarc and it forces $\mathcal{T}_{r+1}'$ to have all negative classical crossings in order to be alternating. But looking at the bottom crossings of $\mathcal{T}_r'$ and $\mathcal{T}_{r+1}'$, we find a contradiction. Indeed, the bottom strand between $\mathcal{T}_r'$ and $\mathcal{T}_{r+1}'$ will force $\mathcal{T}_{r+1}'$ to have all positive classical crossings. The situtation is depicted in Figure 2.14. Conversely, if all virtual twists $\mathcal{T}_r$ are even, then there is an alternating diagram by Corollary 2.1.8.

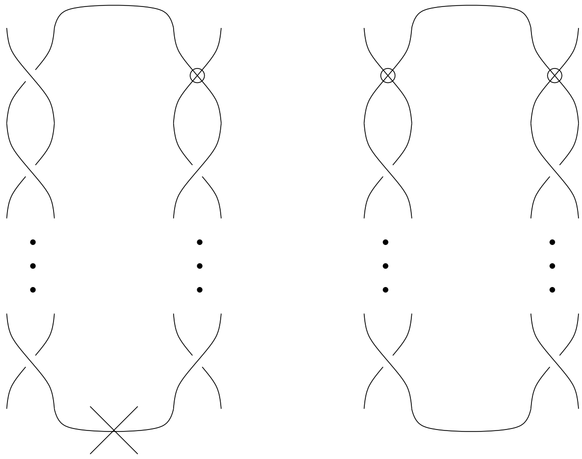


Figure 2.14: One non-alternating pretzel knot diagram construction, and the two alternating pretzel knot diagrams

Now assume $\mathcal{T}_r$ is an odd alternating twist in D. We claim that $\mathcal{T}_{r+1}$ is odd as well. We know that $\mathcal{T}_{r+1}$ cannot be even by the previous argument. It needs to be shown that there are alternating diagrams if all of $\mathcal{T}_r$ are odd. By the Reducing Lemma, $\mathcal{T}_r'$ can have one virtual crossing at the top. Without loss of generality, assume that $\mathcal{T}_r'$'s classical crossings are all positive. Then the top right strand connects to the top left strand of $\mathcal{T}_{r+1}'$, where $\mathcal{T}_{r+1}'$ is obtained from $\mathcal{T}_{r+1}$ by the Reducing Lemma. Then that strand must be an overarc, so the classical crossings in $\mathcal{T}_{r+1}'$ are all positive as well. Now, the bottom right strand of $\mathcal{T}_r'$ is an underarc and in $\mathcal{T}_{r+1}'$, it is an overarc, as expected. See Figure 2.14.

∎

Theorem 2.2.13 *If* $D = P(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$ *is an alternating virtual pretzel knot diagram, where* $\mathcal{T}_\ell = [x_{\ell,1}, k_{\ell,1}, k_{\ell,2}, ..., k_{\ell,m}, x_{\ell,2}]$ *is an alternating even virtual twist for* $\ell = 1, 2, ..., n$, *p is prime, and* $x_{i,j}$ *is either* v *or* $\emptyset$, *then*

$$\mathrm{Col}_p(D) = \mathrm{Col}_p(P([a_1], [a_2], ..., [a_n])),$$

where $[a_\ell] = \left[\ \chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n} (-1)^i k_{\ell,i}\ \right]$, *for* $\ell = 1, 2, ..., n$, *and where* χ *and* ϵ *are defined as in the Reducing Lemma.*

Proof. By Lemma 2.2.12, we know that each twist $\mathcal{T}_\ell$ from D has either an even number of virtual crossings, or no virtual crossings. In either case, we use the Reducing Lemma to see that

$$\mathcal{T}_\ell \xrightarrow{ks} \left[\chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n} (-1)^i k_{\ell,i} \right].$$

Letting $[a_\ell] = \left[\chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n} (-1)^i k_{\ell,i} \right]$ gives us the desired result by Corollary 2.1.5. ■

Corollary 2.2.14 *The determinant of an alternating pretzel knot diagram* $D = P(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$, *where* $\mathcal{T}_\ell = [x_{\ell,1}, k_{\ell,1}, k_{\ell,2}, ..., k_{\ell,m}, x_{\ell,2}]$ *is an alternating even virtual twist for* $\ell = 1, 2, ..., n$, *and* $x_{i,j}$ *is either* v *or* $\emptyset$, *is given by*

$$\text{Det}(D) = \sum_{j=1}^{n} a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_n,$$

where $a_\ell = \left[\chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n} (-1)^i k_{\ell,i} \right]$, *for* $\ell = 1, 2, ..., n$.

Proof. From the previous theorem,

$$D \xrightarrow{ks} P([a_1], [a_2], ..., [a_n]),$$

where $a_\ell = \left[\chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n} (-1)^i k_{\ell,i} \right]$, for $\ell = 1, 2, ..., n$. By Conway's formula [CJ], any pretzel knot has the determinant given in the statement of the theorem, therefore D does as well.

■

Example 2.2.15 For this example, we shall investigate the virtual even pretzel knot diagram $D = P([v, 2, -2], [v, 1, -3], [-6])$. By inspection of Figure 2.15 and Lemma 2.2.12, the diagram D is alternating. Using a 2-swap and a 1-swap on D, then using two Reidemeister II′ moves, we get

$$D' = P([-4], [-4], [-6]).$$

Our goal is to show that the Reducing Lemma gives the same result for $D = P(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$, where $\mathcal{T}_1 = [v, 2, -2], \mathcal{T}_2 = [v, 1, -3]$, and $\mathcal{T}_3 = [-6]$. We calculate only $\mathcal{T}_1$ and leave $\mathcal{T}_2$ to the reader. For $\mathcal{T}_1 = [v, 2, -2]$, we have the following data: n is even, $x_1 = v$, and $x_2 = \emptyset$. Then we obtain

$$\epsilon(x_1) = 1, \epsilon(x_2) = -1, \chi(\mathcal{T}_1) = -1, x(\mathcal{T}_1) = \emptyset.$$

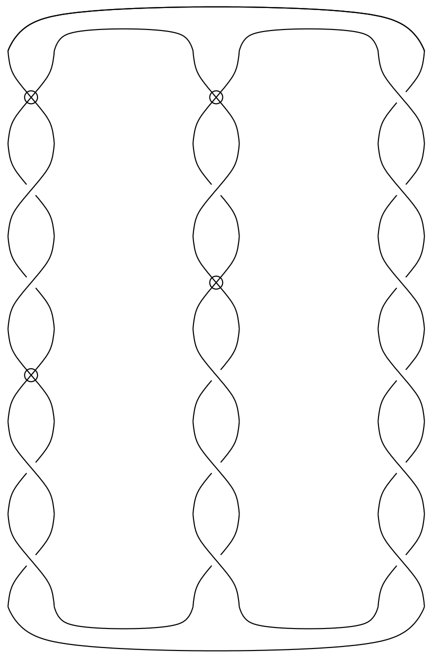


Figure 2.15: The virtual even pretzel knot diagram $D = P([v,2,-2],[v,1,-3],[-6])$

Thus,

$$\mathcal{T}_1' = \Big[\ \emptyset, (-1)(-1)(-2-2)\ \Big] = [-4].$$

Also, $\mathcal{T}_2 = [-4]$ by a similar calculation. Thus,

$$D_1 \xrightarrow{ks} D_1' = P([-4],[-4],[-6]),$$

which was exactly what was obtained by analyzing the diagram. So, by Theorem 2.2.13,

$$\mathrm{Col}_p(D) = \mathrm{Col}_p(D') = \mathrm{Col}_p(P([-4],[-4],[-6])).$$

A quick application of Corllary 2.2.14 shows that

$$\mathrm{Det}(D) = \mathrm{Det}(D') = (-4)(-6) + (-4)(-6) + (-4)(-4) = 64.$$

Example 2.2.16 For the next example, we consider the odd virtual pretzel knot diagram $D = P(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$, where $\mathcal{T}_1 = [v,5], \mathcal{T}_2 = [v,1,-1,1]$, and $\mathcal{T}_3 = [v,5]$. Notice that by inspection of Figure 2.16 and Lemma 2.2.12, the diagram D is alternating. Performing a (-1)-swap in the middle of $\mathcal{T}_2$ and then using a Reidemeister II$'$ move, we get

$$D' = P([v,4],[v,3],[v,4]).$$

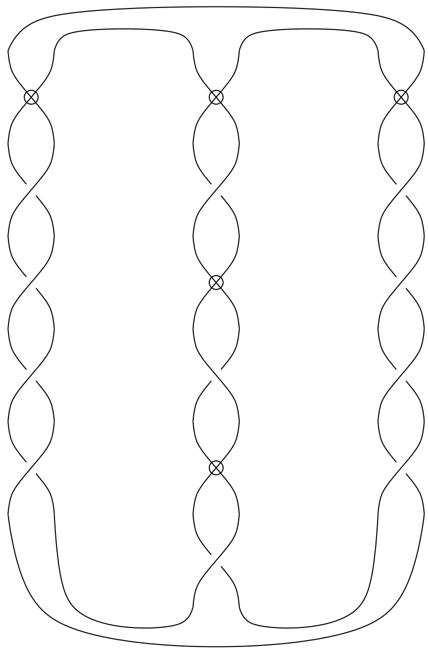


Figure 2.16: The odd virtual pretzel knot diagram $D = P([v,5],[v,1,-1,1],[v,5])$

Now we show the same result using the formula given in the Reducing Lemma. Since $\mathcal{T}_1$ and $\mathcal{T}_3$ are already reduced, we only need to look at $\mathcal{T}_2 = [v,1,-1,1]$. The data we have then is n is odd, $x_1 = v$, and $x_2 = \emptyset$. Thus,

$$\epsilon(x_1) = 1, \epsilon(x_2) = -1, \chi(\mathcal{T}_2) = 1, x(\mathcal{T}_2) = v,$$

so finally we have

$$\mathcal{T}_2 \xrightarrow{ks} \mathcal{T}_2' = [v,(1)(-1)(-1-1-1)] = [v,3],$$

which is what we obtained before. Now, we come upon a situation that we cannot handle as of yet and needs more research: what is $\mathrm{Col}_p(P([v,4],[v,3],[v,4]))$? In general, what is $\mathrm{Col}_p(E)$, where E is any odd virtual alternating pretzel diagram? Similarly, what is the determinant of these knots? Obviously, these areas need more research.

2.2.3 Virtual 2-Bridge Knot Diagrams

For our final family of alternating virtual knot diagrams, we turn our attention to virtualized 2-bridge knot diagrams.

Definition 2.2.17 A virtual 2-bridge link diagram is denoted by

$$B(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n),$$

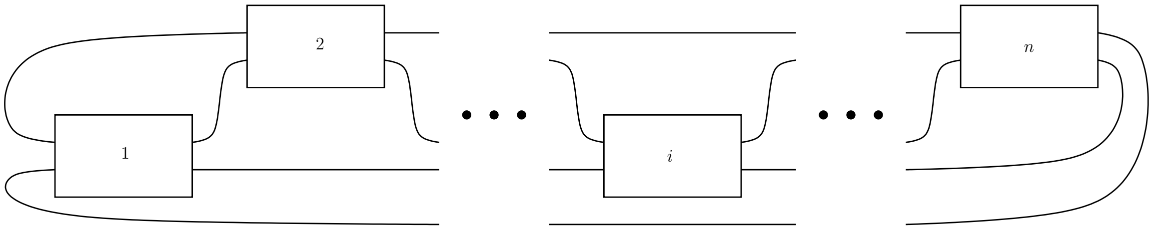


Figure 2.17: The general 2-bridge virtual knot diagram

where a given twist $\mathcal{T}_i = [x_{i,1}, k_{i,1}, k_{i,2}, ..., k_{i,j}, x_{i,2}]$ is inserted into the i-th box in Figure 2.17. See Figure 2.18 for an example. In the case there are no virtual crossings in any $\mathcal{T}_i$, then the classical 2-bridge knot will still be denoted by $B(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$, regardless of the conventional notations.

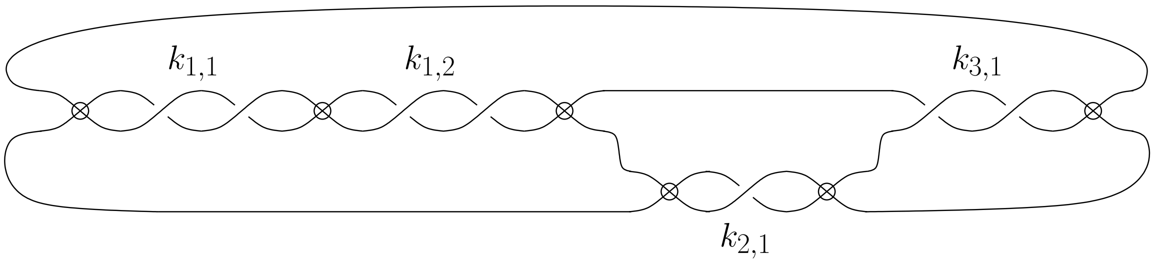


Figure 2.18: The 2-bridge virtual knot diagram $B([v, -2, -2, v], [v, -1, v], [-1, v])$

Lemma 2.2.18 *If a virtual 2-bridge diagram D is alternating, then all virtual twists in D are even and alternating. Conversely, for any sequence $(m_1, m_2, ..., m_n)$ of all even non-negative integers, there is an alternating diagram $B(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$ such that the number of virtual crossings of $(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$ are $(m_1, m_2, ..., m_n)$.*

Proof. Assume $D = B(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$ is alternating. Using the Reducing Lemma, we reduce all twists $\mathcal{T}_r$ to $\mathcal{T}_r'$ so that, for each twist $\mathcal{T}_r'$, there is either one virtual crossing or no virtual crossings. Note that we can place the lone virtual crossing anywhere in a given twist that we choose by using a k-swap move. We claim that there are only even alternating twists in D. This would mean that after reducing, there would be no virtual crossings in any twists. There are three cases to check for the middle twists, as depicted in Figure 2.19. Furthermore, there are two cases each for the case when $r = n$ and for the case when $r = 1$, as in Figure 2.20. Note that Figure 2.20 is a refinement of Figure 2.17.

For the first case, assume that $\mathcal{T}_{r+1}'$ and $\mathcal{T}_{r-1}'$ are both classical twists and $\mathcal{T}_r'$ is positive with one virtual crossing to the right of its classical crossings. Then the strand between $\mathcal{T}_r'$

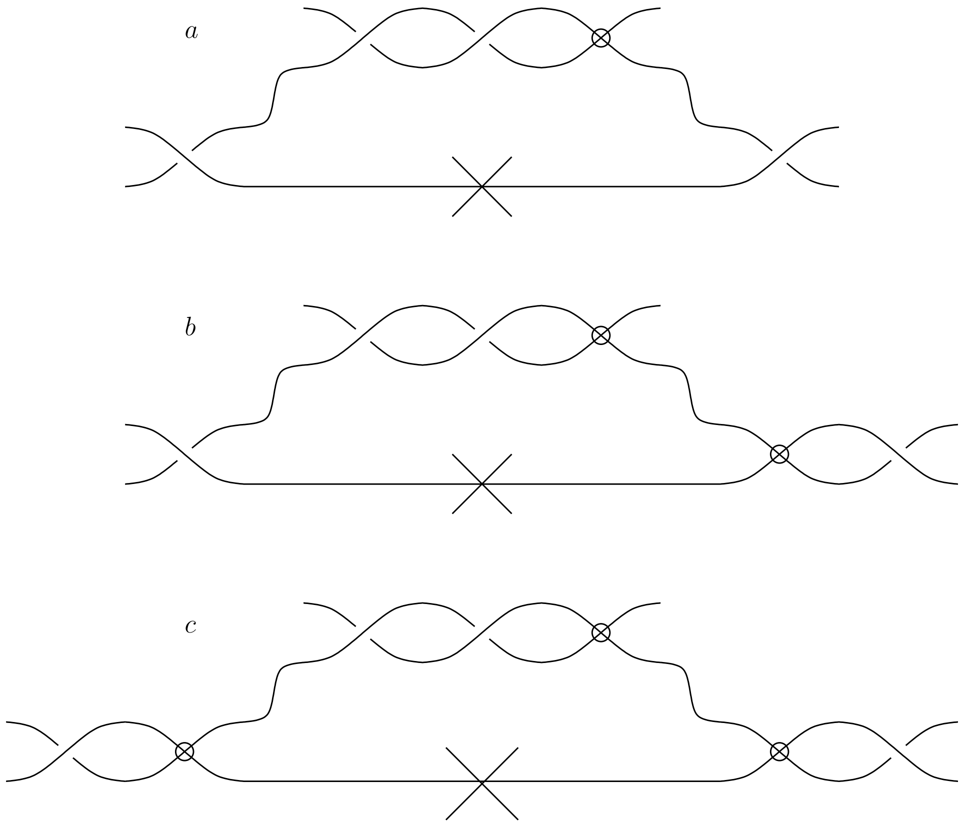


Figure 2.19: The three cases to check

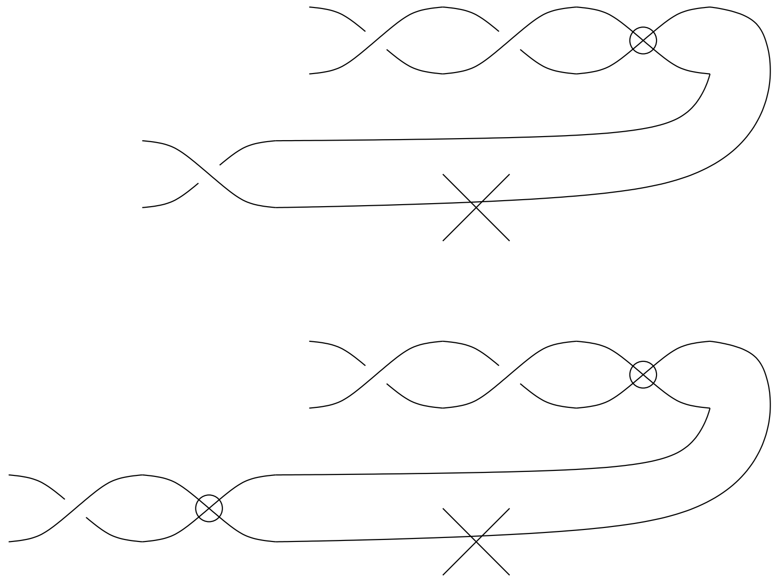
and $\mathcal{T}'_{r+1}$ forces $\mathcal{T}'_{r+1}$ to be a positive twist. The strand between $\mathcal{T}'_r$ and $\mathcal{T}'_{r-1}$ makes $\mathcal{T}'_{r-1}$ a negative twist, so the strand between $\mathcal{T}'_{r-1}$ and $\mathcal{T}'_{r+1}$ is non-alternating. See Figure 2.19, part *a*. The other two cases are similar and are depicted in Figure 2.19, parts *b* and *c*.

The case where $r = n$ is similar to the middle cases. We examine the subcase of $\mathcal{T}_{n-1}$ being classical, and $\mathcal{T}_n$ being positive. Since $\mathcal{T}_n$ is positive, the strand between $\mathcal{T}_{n-1}$ and $\mathcal{T}_n$ forces $\mathcal{T}_{n-1}$ to be a negative twist. However, the other strand between them would force $\mathcal{T}_{n-1}$ to be a positive twist. Thus, this setup is non-alternating and the other cases are similar. See Figure 2.20. Hence, the first part of the proof is done.

The converse relies on the fact that we can have an alternating classical 2-bridge knot diagram, add pairs of virtual crossings to any twist, and apply the Reducing Lemma in reverse to get an alternating virtual 2-bridge knot diagram. This is possible because of Corollary 2.1.8.

■

Theorem 2.2.19 *If* $D = B(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$ *is an alternating virtual 2-bridge link diagram, where* $\mathcal{T}_\ell = [x_{\ell,1}, k_{\ell,1}, k_{\ell,2}, ..., k_{\ell,m}, x_{\ell,2}]$ *is an alternating even virtual twist for* $\ell = 1, 2, ..., n$*,*



$r = n$

Figure 2.20: The cases when $r = n$

p is prime, and $x_{i,j}$ is either v or $\emptyset$, then

$$\mathrm{Col}_p(D) = \mathrm{Col}_p(B([a_1], [a_2], ..., [a_n])),$$

where $[a_\ell] = \left[\ \chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n}(-1)^i k_{\ell,i}\ \right]$, *for* $\ell = 1, 2, ..., n$, *and where* χ *and* ϵ *are defined as in the Reducing Lemma.*

Proof. By Lemma 2.2.18, we know that each twist $\mathcal{T}_\ell$ from D has an even number of virtual crossings. Thus, a quick application of the Reducing Lemma shows that

$$\mathcal{T}_\ell \xrightarrow{ks} \left[\ \chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n}(-1)^i k_{\ell,i}\ \right].$$

Letting $[a_\ell] = \left[\ \chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n}(-1)^i k_{\ell,i}\ \right]$ gives us the desired result.

■

Theorem 2.2.20 *The determinant of an alternating 2-bridge link diagram* $D = B(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$, *where* $\mathcal{T}_\ell = [x_{\ell,1}, k_{\ell,1}, k_{\ell,2}, ..., k_{\ell,m}, x_{\ell,2}]$ *is an alternating even twist for* $\ell = 1, 2, ..., n$, *and* $x_{i,j}$ *is either* v *or* $\emptyset$, *is given by*

$$\mathrm{Det}(D) = a_n(\cdots a_4(a_3(a_2a_1 + 1) + 2) + 3) + \cdots) + a_n - 1,$$

where $[a_\ell] = \chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2})\sum_{i=1}^n(-1)^i k_{\ell,i}$.

Proof. From the previous theorem,

$$D \xrightarrow{ks} B([a_1], [a_2], ..., [a_n]),$$

where $a_\ell = \left[\ \chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2})\sum_{i=1}^n(-1)^i k_{\ell,i}\ \right]$, for $\ell = 1, 2, ..., n$. By Conway's formula [CJ], any 2-bridge knot has the determinant given in the statement of the theorem, therefore D does as well.

■

Example 2.2.21 In this last example, we consider the virtual 2-bridge knot diagram $D = B(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$, where $\mathcal{T}_1 = [v, 2, -2], \mathcal{T}_2 = [2, -2, v]$, and $\mathcal{T}_3 = [-3]$. By Lemma 2.2.18, the diagram D is alternating. Again performing two k-swaps on D and then using two Reidemeister II′ moves yields

$$D' = B([-4], [4], [-3]).$$

We want to show that the Reducing Lemma gives the same D' from $D = B(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$. We calculate only $\mathcal{T}_2$ and leave $\mathcal{T}_1$ to the reader. Starting with even n, $x_1 = \emptyset$, and $x_2 = v$, we see that

$$\epsilon(x_1) = -1, \epsilon(x_2) = 1, \chi(\mathcal{T}_2) = -1, x(\mathcal{T}_2) = \emptyset.$$

So by the Reducing Lemma,

$$\mathcal{T}_2' = \left[\ \emptyset, (-1)(1)\sum_{i=1}^{2}(-1)^i k_i\ \right] = [-((-1)2 + (-2))] = [4].$$

Also, $\mathcal{T}_1 = [-4]$ by a similar calculation. Thus the Reducing Lemma agrees with our manipulations of the diagram. By Theorem 2.2.19, we know that

$$\mathrm{Col}_p(D) = \mathrm{Col}_p(D') = \mathrm{Col}_p(B([-4], [4], [-3])).$$

Furthermore, using Theorem 2.2.20, we also know that

$$\mathrm{Det}(D) = \mathrm{Det}(D') = (-3)(-16 + 1) + 2 = 47.$$

2.3 Kauffman-Harary Conjecture for Alternating Virtual Knots and Swap Moves

For convenience, we recall the Kauffman-Harary conjectures here.

Conjecture 2.3.1 (Kauffman-Harary Conjecture 1) *Let D be a reduced alternating knot diagram with a prime determinant p. Then every nontrivial Fox's p-coloring of D assigns different colors to different arcs of D.*

Conjecture 2.3.2 (Kauffman-Harary Conjecture 2) *Let D be a reduced alternating virtual knot diagram with a prime determinant p. Then every nontrivial Fox's p-coloring of D assigns different colors to different arcs of D.*

Theorem 2.3.3 *Let D_2 be obtained from D_1 by a finite sequence of k-swap moves. Then the Kauffman-Harary Conjecture (1.2.13) is true for an alternating virtual knot diagram D_1 iff the Kauffman-Harary Conjecture is true for an alternating knot diagram D_2.*

Proof. By Corollary 2.1.6 and Theorem 2.1.4, the arcs in D_1 are only relabeled in D_2 after the k-swap. The result follows.

■

Corollary 2.3.4 *The Kauffman-Harary conjecture is true for an alternating knot diagram $D = T(2, [k_1, k_2, ..., k_n, x_2])$, where x_2 is either v or $\emptyset$.*

Proof. By Theorem 2.2.6,

$$D \xrightarrow{ks} D' = \left(T\left(2, \left[\ \chi(\mathcal{T})\epsilon(x_2)\sum_{i=1}^{n}(-1)^i k_i\ \right]\right)\right),$$

where $\mathcal{T} = [k_1, k_2, ..., k_n, x_2]$. By Corollary 2.1.8 and Theorem 2.2.6, D' is an alternating classical diagram. In [KH], it was proven that alternating $T(2, [n])$ classical knots satisfy the Kauffman-Harary conjecture. Therefore, alternating virtual 2-string braids satisfy the Kauffman-Harary conjecture, by Theorem 2.3.3.

■

Corollary 2.3.5 *The Kauffman-Harary conjecture is true for alternating virtual pretzel knot diagrams $D = P(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$, where $\mathcal{T}_\ell = [x_{\ell,1}, k_{\ell,1}, k_{\ell,2}, ..., k_{\ell,m}, x_{\ell,2}]$ is an alternating even virtual twist for $\ell = 1, 2, ..., n$, and $x_{i,j}$ is either v or $\emptyset$.*

Proof. By Theorem 2.2.13,

$$D \xrightarrow{ks} D' = P([a_1], [a_2], ..., [a_n]),$$

where $[a_\ell] = \left[\chi(\mathcal{T}_\ell)\epsilon(x_{\ell,2}) \sum_{i=1}^{n} (-1)^i k_{\ell,i} \right]$. Since D is an even alternating virtual diagram, D' is a classical alternating diagram (Corollary 2.1.8 and Theorem 2.2.13). In [AM], it was shown that alternating pretzel knots satisfy the Kauffman-Harary conjecture. Thus, Theorem 2.3.3 states that D satisfies the virtual Kauffman-Harary conjecture.

■

Remark 2.3.6 Since there are an odd number of virtual crossings in each virtual twist of the odd alternating pretzel diagrams from Lemma 2.2.12, we can use the Reducing Lemma to get one crossing in each twist. Further results in that area would need more research in order to prove or disprove the Kauffman-Harary conjecture.

Corollary 2.3.7 *The Kauffman-Harary conjecture is true for alternating virtual 2-bridge link diagrams* $D = B(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n)$*, where* $\mathcal{T}_\ell = [x_{\ell,1}, k_{\ell,1}, k_{\ell,2}, ..., k_{\ell,m}, x_{\ell,2}]$ *is an alternating even virtual twist for* $\ell = 1, 2, ..., n$*, and* $x_{i,j}$ *is either* v *or* $\emptyset$*.*

Proof. This is similar to Corollary 2.3.5. Note that 2-bridge knots satisfy the Kauffman-Harary conjecture, as proved in [KL].

■

Remark 2.3.8 It was proved in [KL] that the Kauffman-Harary conjecture holds for any 2-bridge knot without restrictions on the determinant of the knot. However, the statement of the conjecture needs to be changed from "every nontrivial n-coloring" to "there exists a n-coloring," where n is the determinant of a 2-bridge knot K.

3 Conclusion

After defining the k-swap move, we proved that they induce a bijection between colorings before and after the move in order to show that two knots are related by a sequence of k-swap moves, then their colorings are the same. This also allowed us to prove that determinants do not change after k-swap moves are performed. Furthermore, we proved that the k-swap does not change the fundamental involutory quandles, and it does not change Jones polynomials either.

Following the proofs concerning the invariance of the colorings and determinants of the k-swap, we showed what the alternating conditions were for each of the three virtual knot diagrams discussed. Only one of the families can be virtual after reducing via the Reducing Lemma: the virtual pretzel knot diagrams. Next, we showed what each of the alternating virtual knot diagrams' (except for the odd alternating virtual pretzel knot diagrams) colorings and determinants were via the Reducing Lemma. Finally, we proved that the Kauffman-Harary conjecture holds for all of the above alternating virtual knot diagrams, except for the odd alternating virtual pretzel knot diagrams. In that case, it is not clear to us what the colorings are for these knot diagrams or what their determinants are so we cannot conjecture whether or not they satisfy the Kauffman-Harary conjecture.

There are many questions regarding the k-swap and future work which could be done. Does the k-swap hold for Alexander colorings of virtual knots? If not, can the k-swap be generalized to include Alexander colorings? Are there any other colorings in which this move hold? There is also no reason to restrict these moves to colorings or determinants, so are there other invariants that the k-swap could be used to examine? For instance, what happens to the Jones polynomial (or other polynomials) after the k-swap is performed? In Kauffman's work mentioned above, it was shown that the 1-swap is an invariant for Jones polynomials so is this true in general? We would expect that the k-swap does something interesting to the Alexander polynomial or the Conway polynomial, but again, more research is needed.

References

[AC] C. Adams, *The knot book: An elementary introduction to the mathematical theory of knots*, Revised reprint of the 1994 original, American Mathematical Society, Providence, RI, 2004.

[AM] M. Asaeda, J. Przytycki, A. Sikora, Kauffman-Harary conjecture holds for Montesinos knots, *J. Knot Theory Ramifications* 13 (2004), no. 4, 467–477.

[AS] S. Asami, On the Kauffman-Harary conjecture for Alexander quandle colorings, *Rev. Mat. Complut.* 19 (2006), no. 1, 139–143.

[BE] Brieskorn, E., *Automorphic sets and singularities,* Contemporary math., 78 (1988), 45–115.

[CJ] J. H. Conway, An enumeration of knots and links, and some of their algebraic properties, 1970, *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pp. 329–358, Pergamon, Oxford.

[CP] P. Cromwell, *Knots and links*, Cambridge University Press, Cambridge, 2004.

[ES] S. Eliahou, L. Kauffman, M. Thistlethwaite, Infinite families of links with trivial Jones polynomial, Topology 42 (2003), no. 1, 155-169.

[F] R. Fox, A quick trip through knot theory, *1962 Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)*, pp. 120–167, Prentice-Hall, Englewood Cliffs, N.J.

[FR] R. Fenn, C. Rourke, Racks and links in codimension two, *J. Knot Theory Ramifications* 1 (1992), no. 4, 343–406.

[IA] A. Inoue, Quandle homomorphisms of knot quandles to Alexander quandles, *J. Knot Theory Ramifications* 10 (2001), no. 6, 813–821.

[JD] D. Joyce, A classifying invariant of knots, the knot quandle, *J. Pure Appl. Algebra* 23 (1982), no. 1, 37–65.

[KH] F. Harary, L. Kauffman, Knots and graphs I - Arc graphs and colorings, Adv. in Appl. Math. 22 (1999), no. 3, 312–337.

[KJ] L. Kauffman, State models and the Jones polynomial, *Topology* 26 (1987), no. 3, 395–407.

[KKP] L. Kauffman, *Knots and physics*, Third edition, Series on Knots and Everything, 1, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[KL] L. Kauffman, S. Lambropoulou, On the classification of rational tangles, *Adv. in Appl. Math.* 33 (2004), no. 2, 199–237.

[KLC] L. Kauffman, P. Lopes, On the minimum number of colors for knots, Preprint, arxiv:math.GT/0512088.

[KT] T. Kadokami, A classification of closed 2-string virtual braids, Preprint.

[KVKT] L. Kauffman, Virtual knot theory, *European J. Combin.* 20 (1999), no. 7, 663–690.

[MS] Matveev, S., *Distributive groupoids in knot theory,* (Russian) Mat. Sb. (N.S.) 119(161) (1982), no. 1, 78–88, 160.

[PL] L. Person, M. Dunne, J. DeNinno, B. Guntel, L. Smith, Colorings of rational, alternating knots and links, preprint 2002.

[SJ] J. Sawollek, On Alexander-Conway Polynomials for Virtual Knots and Links, Preprint 1999. http://front.math.ucdavis.edu/math.GT/9912173.

[TM] M. Takasaki, Abstraction of Symmetric Transformations (in Japanese), Tohoku Math. J., 49, 1942/43, 145–207.