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Radiographic Imagery of a Variable Density 3D Object

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Radiographic Imagery of a Variable Density 3D Object

Abstract
The purpose of this project is to develop a mathematical model to study 4D (three spatial dimensions plus density) shapes using 3D projections. In the model, the projection is represented as a function that can be applied to data produced by a radiation detector. The projection is visualized as a three-dimensional graph where x and y coordinates represent position and the z coordinate corresponds to the object's density and thickness. Contour plots of such 3D graphs can be used to construct traditional 2D radiographic images.

Keywords
Radiographic Imagery, Integral Projection, Mathematical Model

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PROBLEM STATEMENT

Radiographic images are created by measuring the amount of radiation absorbed by a material as the radiation travels through it. The amount of the absorbed radiation depends on the distance that the radiation has to travel as well as the material's optical density denoted by the function $f(x, y)$. Furthermore, the radiographic image of a thin slice of a material can be interpreted as a 2D projection (see Figure 1).

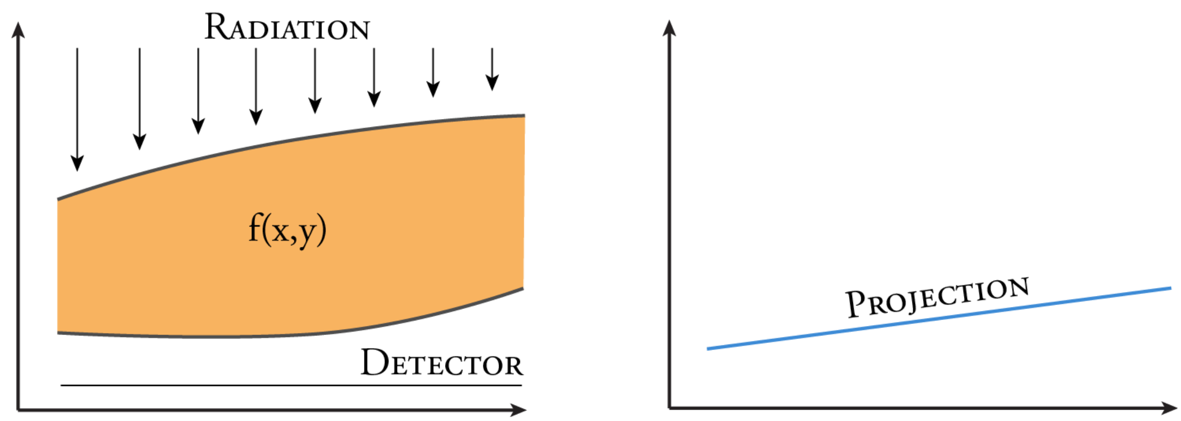

Figure 1: (left) Imaging of a thin slice of a material; (right) corresponding projection.

The projection $P(x)$ of a material is given by the equation (see (Zhou and Geng) for a description of a similar approach used for eye detection in digital images)

$$P(x) = \int f(x, y)dy.$$

If loss in intensity outside of the material is negligible and the density is constant (denoted by c), the above equation becomes

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$$P(x) = c\int_{A(x)}^{B(x)} dy \tag{1}$$

where functions $A(x)$ and $B(x)$ represent the object's boundary.

The goal of this work is to model a bone containing a tumor as a three-dimensional object separated into a collection of two-dimensional slices and use the above expression for $P(x)$ to describe each slice.

MOTIVATION

The purpose of this work is to model projections acquired from radiographic images of variable density objects. Such models could be useful for analysis of the object's density and size as well as other physical properties. Such theoretical estimates are useful because they can decrease the amount of exposure to radiation of patients and doctors as well as reduce the cost.

MATHEMATICAL DESCRIPTION AND SOLUTION APPROACH

For simplicity, we model the bone as a cylinder and the tumor as a sphere contained inside this cylinder (see Figure 2). We assume that the bone and the tumor have different densities (Choi, Choi et al.). This allows representing the projection of this system as a function of two variables whose values at any given point are proportional to the density and the height of the bone and the tumor. Furthermore, a contour map of the graph of this function can be interpreted as a radiographic image of this system.

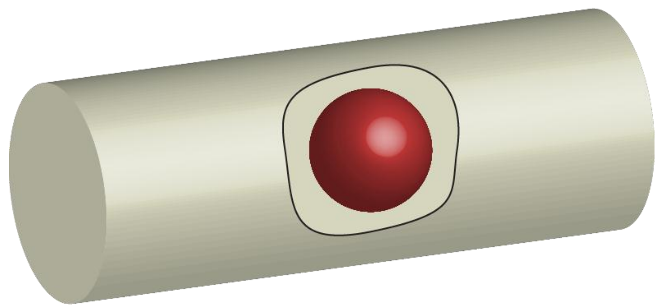

Figure 2: 3D model of a bone containing a tumor.

The bone is modeled as a circle in the xz-plane, centered at the origin, with radius R_c that is extruded along the y-axis. That is

$$x^2 + z^2 = {R_c}^2 \text{ from } y_i \text{ to } y_f.$$

Solving the above equation for z gives

$$z = \pm\sqrt{{R_c}^2 - x^2} \text{ from } y_i \text{ to } y_f. \tag{2}$$

Next, the tumor is modeled as a sphere with the center point (X, Y, Z) and radius R_s

$$(x - X)^2 + (y - Y)^2 + (z - Z)^2 = {R_s}^2.$$

Solving for z gives

$$z = Z \pm \sqrt{{R_s}^2 - (x - X)^2 - (y - Y)^2} \tag{3}$$

Now that both equations in the bone/tumor system have been determined (equations 2 and 3), we turn to calculating the projection function. First, we add an extra dimension to the "slices" described in the problem statement. It, therefore, follows that the projection $P(x, y)$ is a function of two variables rather than one. As mentioned previously, the density of the bone and

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the tumor are assumed to be constant. They are denoted by f_c and f_s respectively. We use these observations to first determine the projection function of a cylinder.

$$P_c(x,y) = \int_{-\sqrt{{R_c}^2-x^2}}^{\sqrt{{R_c}^2-x^2}} f_c\,dz - \int_{-\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}}^{\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}} f_c\,dz$$

$$= f_c\left[\sqrt{{R_c}^2-x^2}+\sqrt{{R_c}^2-x^2}\right] - f_c\left[\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}+\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}\right]$$

$$= 2f_c\sqrt{{R_c}^2-x^2} - 2f_c\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}$$

$$= 2f_c\sqrt{{R_c}^2-x^2} - 2f_c\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2} \qquad (4)$$

In (4), the projection is calculated by subtracting the projection of the part of the cylinder bounded by the sphere from the projection of the entire cylinder.

Next we calculate the projection of the sphere.

$$P_s(x,y) = \int_{-\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}}^{\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}} f_s\,dz$$

$$= f_s\left[\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}+\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2}\right]$$

$$= 2f_s\sqrt{{R_s}^2-(x-X)^2-(y-Y)^2} \qquad (5)$$

Finally, the two projections are added together to get the final equation for the projection of the entire system (6).

$$
\begin{aligned}
P(x,y) &= P_c(x,y) + P_s(x,y) \\
&= \left(2f_c\sqrt{{R_c}^2 - x^2} - 2f_c\sqrt{{R_s}^2 - (x-X)^2 - (y-Y)^2}\right) + \left(2f_s\sqrt{{R_s}^2 - (x-X)^2 - (y-Y)^2}\right) \\
&= 2f_c\sqrt{{R_c}^2 - x^2} + (2f_s - 2f_c)\sqrt{{R_s}^2 - (x-X)^2 - (y-Y)^2} \qquad (6)
\end{aligned}
$$

DISCUSSION

Now that a mathematical model for the projection of the image has been obtained, we apply it to the system consisting of a bone with length of 10, radius of 2, and density of 1; and a tumor whose density is 1.4 and a radius is 1 centered at $(0,7,0)$. Substituting these values into the equation for the projection $P(x,y)$ and simplifying the result gives

$$P(x,y) = 2\sqrt{4 - x^2} + (0.8)\sqrt{1 - (x)^2 - (y-7)^2} \qquad (7)$$

The equation (7) has the following graph

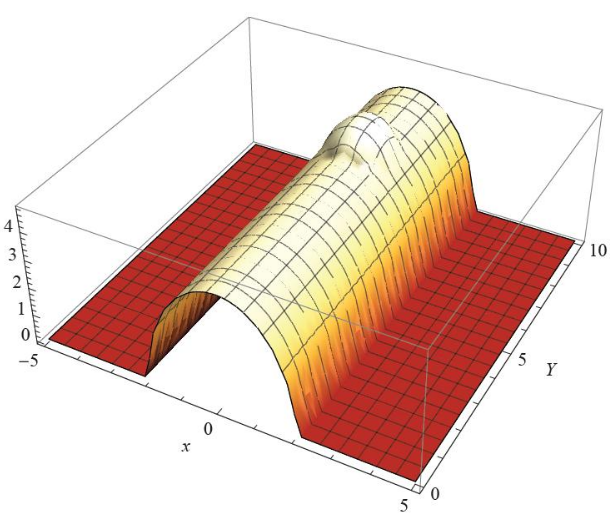

Figure 4 (generated by wolframalpha.com): Three-dimensional plot of the projection of the bone/tumor system. Notice that in addition to the height, the ***z***-axis also characterizes density of the objects in the original model.

Then, converting this 3D plot into a contour plot provides the final radiographic image.

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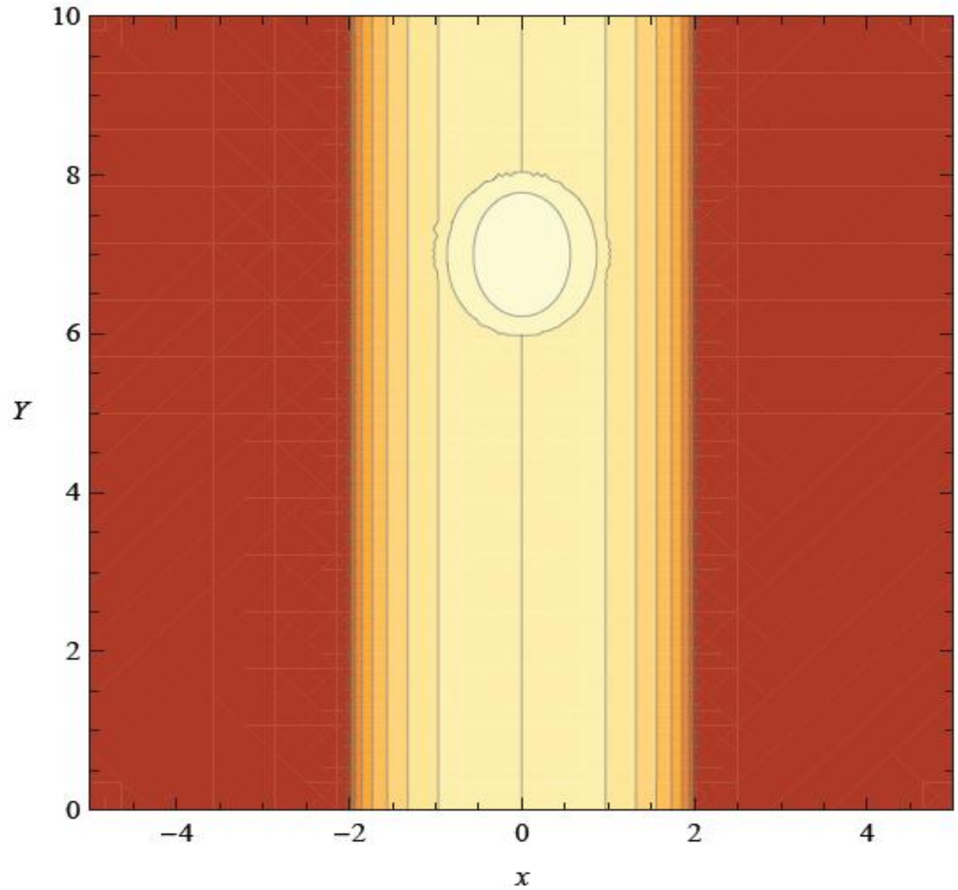

As can be seen from the contour plot, the radiograph of the bone appears uniform everywhere except for in the region containing the tumor. Here, the radiograph shows a brighter spot, indicating a localized area of different density.

The bone/tumor system has been modeled by subdividing the problem into simpler parts (slices). These parts can then be combined together to describe the complete model. In this way, the presented technique can be applied to create a wide class of models for describing and studying radiographic images.

CONCLUSION AND RECOMMENDATIONS

This project describes a mathematical model of a radiographic image of a bone/tumor system. By interpreting the density as the height in a 3D graph, a contour map could be plotted in a way that provides a representation of the radiograph.

This work can be extended to develop a technique that could predict the vertical location of a tumor based on the radiographic output.

NOMENCLATURE

Symbol	Definition
$\boldsymbol{R_c}$	radius of the cylinder
$\boldsymbol{R_s}$	radius of the sphere
$\boldsymbol{f_c}$	density of the bone (cylinder)
$\boldsymbol{f_s}$	density of the tumor (sphere)
$\boldsymbol{P_c(x,y)}$	projection of the cylinder
$\boldsymbol{P_s(x,y)}$	projection of the sphere
$\boldsymbol{P(x,y)}$	projection of the entire system

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