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Motion of a Pendulum

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Motion of a Pendulum

Abstract
The objective of this project is to derive and solve the equation of motion for a pendulum swinging at small angles in one dimension. The pendulum may be either a simple pendulum like a ball hanging from a string or a physical pendulum like a pendulum on a clock. For simplicity, we only considered small rotational angles so that the equation of motion becomes a harmonic oscillator.

Keywords
Pendulum, Harmonic Oscillator, Periodic Motion

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PROBLEM STATEMENT

The goal of this project is to accurately describe the motion of pendulums in terms of calculus and physics. There are two types of pendulums. A simple pendulum is a mass dangling at the bottom of a string. A physical pendulum is a rigid hanging object which oscillates about a stationary point that is not located at its center of mass. We shall derive and solve the differential equation of motion of pendulums. In order to simplify the differential equation for the motion of a pendulum, we shall assume that the pendulum is swinging at small angles and only swinging back and forth in one dimension rather than swinging roundly. Throughout this project, we shall ignore the effects of air resistance and friction.

MOTIVATION

People use pendulums in some mechanical systems such as grandfather clocks, car alarm systems, and seismographic instruments to measure the magnitude of physical shock. Many simple devices are also pendulums. Such everyday devices include swings on a swing set and rope swings suspended from a tree branch. The best example of a physical pendulum is the grandfather clock arm because it is rigid and pivots from a stationary point. It is therefore worthwhile to characterize a pendulum's motion as it swings back and forth.

MATHEMATICAL DESCRIPTION AND SOLUTION APPROACH

Newton's second law of motion, namely $F = ma$, can be used to mathematically describe the motion of a pendulum. In terms of rotation, Newton's second law becomes the equation

$$\tau = I\,\alpha \tag{1}$$

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where I is the moment of inertia of the pendulum, α is the angular acceleration, and τ is the torque acting on the pendulum. Using Newton's second law in angular form, we shall derive the equation of motion for a pendulum.

Since torque is directly proportional to the angle of rotation θ, we get

$$\tau = -c \sin\theta \tag{2}$$

where c is the torque of the pendulum at a $90°$ angle. If the pendulum is a simple pendulum or a physical pendulum, we get

$$c = mgd \tag{3}$$

where mg is the normal force, and d is the center of mass. Since the angular acceleration α is the change in angular velocity $\frac{d\theta(t)}{dt}$, we get

$$\alpha(t) = \frac{d^2\theta(t)}{dt^2}, \tag{4}$$

so

$$-c \sin\theta(t) = I\,\alpha(t) = I\,\frac{d^2\theta(t)}{dt^2}. \tag{5}$$

When the angle θ is small, the function $\sin\theta$ is approximately θ. Therefore, for small angles, the pendulum satisfies the equation

$$\left(-\frac{c}{I}\right)\theta(t) = \frac{d^2\theta(t)}{dt^2}. \tag{6}$$

Assuming the initial velocity is zero, $\theta'(0) = 0$, the general solution to (6) is

$$\theta(t) = A\cos\left(\sqrt{\frac{c}{I}}\,t\right) = A\cos\left(\sqrt{\frac{mgd}{I}}\,t\right) \tag{7}$$

where A is the amplitude or maximum angular position of the object. Since the sine function is 2π periodic, the amount of time needed for the pendulum to swing one full period (back and forth) is

$$T = 2\,\pi\sqrt{I/c} = 2\,\pi\sqrt{I/mgd} \tag{8}$$

and the frequency of the pendulum is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{mgd/I}. \tag{9}$$

DISCUSSION

Note that equation (1) is essentially

$$-\omega^2\theta = \frac{d^2\theta}{dt^2}, \tag{10}$$

the equation for a harmonic oscillator. Also take note that as θ grows larger,equation (6) becomes less accurate since θ does not approximate $\sin\theta$ very well.Instead, one has to deal with the equation $-\frac{c}{I}\sin(\theta) = \frac{d^2\theta}{dt^2}$ which cannot be solved in terms of elementary functions. Therefore one must use special functions or numerical methods to solve the equation of motion of a pendulum swinging at larger angles.

CONCLUSION AND RECOMMENDATIONS

We recommend further studies on the equation of motion for a pendulum. For example, one could do a study on the equation of motion of a pendulum for pendulums that swing at larger angles. Furthermore, one could also incorporate air resistance and friction in the equation of motion of the pendulum. One could even study the motion of a pendulum moving in two dimensions instead of a pendulum just moving back and forth.

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NOMENCLATURE

Symbol	Description	Units
F	Force	N
m	Mass	kg
a	Acceleration	$\frac{m}{s^2}$
c	Torque at 90°	Nm
α	Angular Acceleration	$\frac{rad}{s^2}$
I	Moment of Inertia	$\frac{kg}{m^2}$
θ	Angle	deg
t	Time	s
D	Distance	m
A	Amplitude	deg
T	Period	s
f	Frequency	$\frac{rad}{s}$

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