2011

Automorphism Groups of Quandles

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by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
Department of Mathematics and Statistics
College of Arts and Sciences
University of South Florida

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Date of Approval:
October 20, 2011

Keywords: inner automorphism group, knot theory, dihedral group, Reidmeister moves, conjugation

Dedication

I dedicate this to my patient and supportive husband and to our adorable baby Logan.

Acknowledgments

I would like to thank my advisor, Mohamed Elhamdadi, for all his help and support throughout my time here at USF. And to thank him for introducing me to knot theory and more specifically to quandles. I am grateful to my thesis committee, Brian Curtin, and Masahiko Saito, for their time and input. I would also like to thank Edwin Clark for all his help and fruitful suggestions.

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Abstract

This thesis arose from a desire to better understand the structures of automorphism groups and inner automorphism groups of quandles. We compute and give the structure of the automorphism groups of all dihedral quandles. In their paper *Matrices and Finite Quandles*, Ho and Nelson found all quandles (up to isomorphism) of orders 3, 4, and 5 and determined their automorphism groups. Here we find the automorphism groups of all quandles of orders 6 and 7. There are, up to isomoprhism, 73 quandles of order 6 and 289 quandles of order 7.

Chapter 1

Review of Quandles and Quandle Colorings

1.1 Introduction

Quandles and racks are algebraic structures whose axiomatization comes from Reidemeister moves in knot theory. The earliest known work on racks is contained within 1959 correspondence between John Conway and Gavin Wraith who studied racks in the context of the conjugation operation in a group. Around 1982, the notion of a quandle was introduced independently by Joyce [15] and Matveev [17]. In Joyce's doctoral dissertation he used the name quandle, as quandle was a word that didn't mean anything else. They used quandles to construct representations of the braid groups. Joyce and Matveev associated to each knot a quandle that determines the knot up to isotopy and mirror image. Since then quandles have been investigated by topologists in order to construct knot and link invariants and their higher analogues (see for example [5] and references therein).

We compute and give the structure of the automorphism groups of all dihedral quandles. In [13], Ho and Nelson gave the list of quandles (up to isomorphism) of orders 3, 4 and 5 and determined their automorphism groups. In this paper, we extend their results to the case of quandles of orders 6 and 7.

This thesis is organized in the following way. Chapter 1 reviews the background information needed for our main results. It explains quandles and knot colorings. Then Chapter 2, computes and gives the structure of automorphism and inner automorphism groups of the dihedral quandles. Chapter 2 also includes a list of the automorphism and inner automorphism groups of all quandles of order 6 and 7. Finally, Chapter 3 is a conclusion to provide a description of how all automorphism and inner automorphism groups of all quandles of order 6 and 7 were found.

1.2 Background

The fundamental question in knot theory is determining whether two knots are equivalent or not. Many invariants have been found to help answer this question. A *knot* is an embedding of the circle S^1 into the 3-dimensional Euclidean space $\mathbb{R}^3$. In order to manipulate knots and work with knot invariants, it is convenient to work with knot diagrams. A *knot diagram* is a projection of a knot into the 2-dimensional Euclidean space $\mathbb{R}^2$ with transverse double points where one of the two arcs is broken to indicate that it is an under-arc. Since we can rotate the knot and move strands at will, there are infinitely many knot diagrams that represent the same knot. This makes distinguishing one knot from another a difficult task. Figure 1 shows two different knot diagrams of the trefoil knot.

Figure 1.: Knot Diagrams of the Trefoil Knot

The Reidemeister theorem states that

THEOREM 1.1 *[3] Two knot diagrams, D_1 and D_2, represent the same knot if and only if D_1 can be transformed into D_2 by a sequence of Reidmeister moves I, II, III and planar isotopy.*

See Figure 2 for the Reidmeister moves.

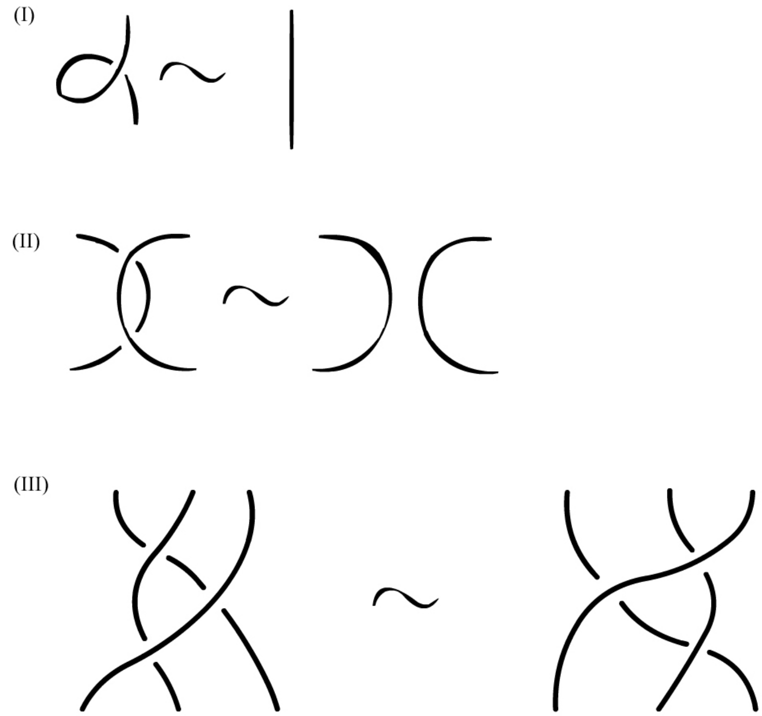

Figure 2.: Reidmeister Moves

It is easy to check that the two knot diagrams in Figure 1 are equivalent. The first diagram of the trefoil can be turned into the other by a finite number of Reidmeister moves.

1.3 Definitions

DEFINITION 1.3.1 *[15] A* quandle *is a set X with a binary operation $(a,b) \mapsto a * b$ such that*

1. *For any $a \in X$, $a * a = a$.*

2. *For any $a, b \in X$, there is a unique $x \in X$ such that $a = x * b$.*

3. *For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.*

The axioms for a quandle correspond respectively to the Reidemeister moves of type I, II, and III. Figure 3 shows the relation of the quandle axioms to the Reidmeister moves.

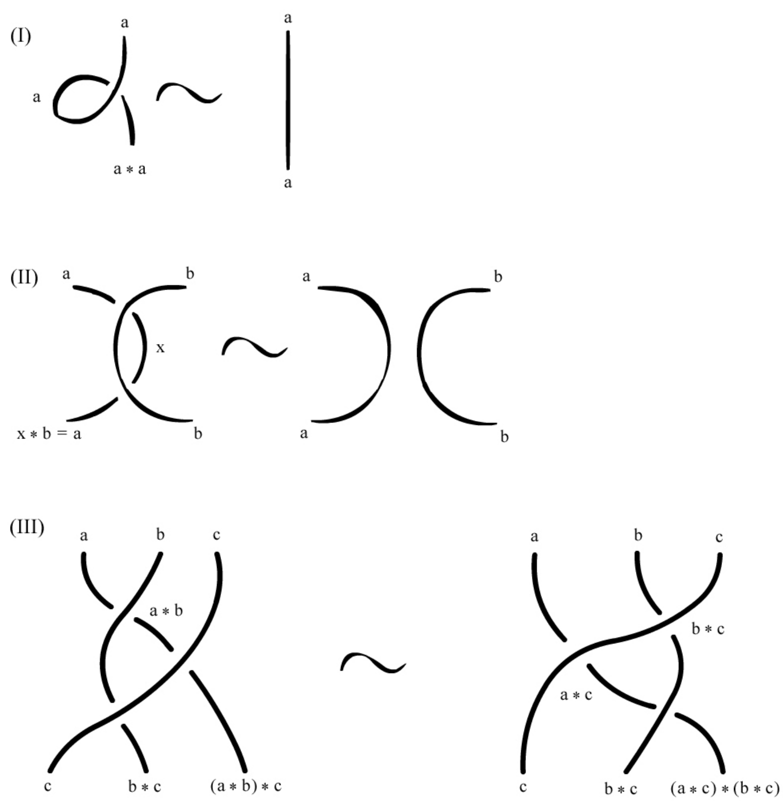

Figure 3.: Correspondence Between Quandle Axioms and Reidmeister Moves

Axiom (2) states that for each $u \in X$, the map $S_u : X \to X$ with $S_u(x) := x * u$ is a bijection. Its inverse will be denoted by the mapping $\bar{S}_u : X \to X$ with $\bar{S}_u(x) = x \bar{*} u$, so that $(x * u) \bar{*} u = x = (x \bar{*} u) * u$.

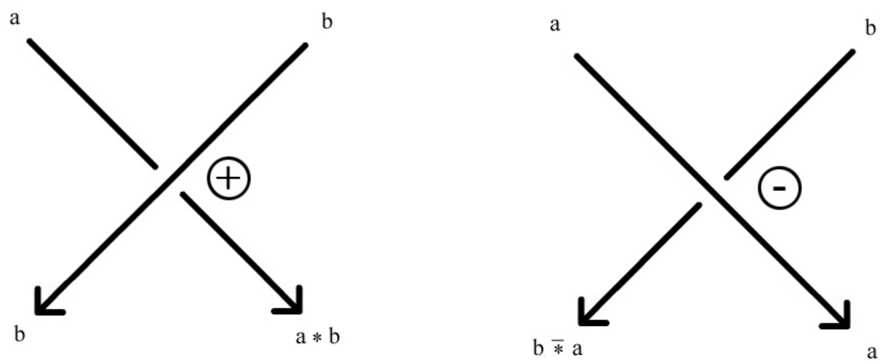

Figure 4.: Positive and Negative Crossings

DEFINITION 1.3.2 *A* rack *is a set with a binary operation that satisfies (2) and (3).*

Racks and quandles have been studied in, for example, [9, 15, 17].

DEFINITION 1.3.3 *[19] Given two quandles $(X, *)$ and $(Y, \triangleleft)$, a map $f : (X, *) \longrightarrow (Y, \triangleleft)$ is a* quandle homomorphism *if $f(a * b) = f(a) \triangleleft f(b)$ for any $a, b \in X$. If f is a bijection, then f is called an* isomorphism*, and we say $(X, *)$ and $(Y, \triangleleft)$ are* isomorphic quandles*.*

REMARK 1 Unlike the group case, there is no identity in a quandle. So there is no notion of a kernel for quandle homomorphisms, but there is a notion of an equalizer. Let $f : X \longrightarrow Y$ be a quandle homomorphism. For $y \in Y$, let $E_y = \{x \in X; f(x) = y\}$. Then E_y is a subquandle of X.

DEFINITION 1.3.4 *[19] The* automorphism group *of a quandle $(X, *)$, denoted $\mathrm{Aut}(X)$, is the group of all isomorphisms $\rho : X \longrightarrow X$. The elements of $\mathrm{Aut}(X)$ act on those of X by right action.*

DEFINITION 1.3.5 *[19] The* inner automorphism group *of a quandle $(X, *)$, denoted $\mathrm{Inn}(X)$, is the subgroup of $\mathrm{Aut}(X)$ generated by all S_x, where $S_x(y) := y * x$, for any $x, y \in X$. The map S_x: $X \longrightarrow X$ that maps u to $u * x$ defines a right action of X on X, so that we obtain a map $X \longrightarrow Inn(X)$.*

DEFINITION 1.3.6 *[16] The* transvection group *of a quandle* $(X, *)$*, denoted* Trans(X)*, is the subgroup of* Inn(X) *generated by the elements of the form* $S_x S_y^{-1}$ *for* $x \in X$ *and* $y \in X$*.*

It is important to note that the Trans(X) is a normal subgroup of Inn(X) which itself is a normal subgroup of Aut(X). The quotient group Inn(X)/Trans(X) is a cyclic group since any two generators S_x and S_y are congruent modulo Trans(X).

DEFINITION 1.3.7 *A quandle* $(X, *)$ *is* Latin *when for any* $x \in X$ *and for any* $z \in X$*, there exists exactly one* y *such that* $x * y = z$*. In other words, we can define left action, for any* $x \in X$ *define a mapping* L_x*:* $X \longrightarrow X$ *that maps* u *to* $x * u$*. That is, for finite Latin quandles, the mappings* L_x *are bijections for all* x*.*

DEFINITION 1.3.8 *Consider the map* $S : X \longrightarrow \mathrm{Inn(X)}$ *sending* u *to* S_u*, where* S_u *is the map* $S_u : X \longrightarrow X$ *is a bijection and quandle homomorphism. The quandle* $(X, *)$ *is called* faithful *when the map* S *is injective. If (X,*) is faithful, then the center of* $\mathrm{Inn(X)}$ *is trivial.*

DEFINITION 1.3.9 *[19] The* orbit *of* $x \in X$ *is the subset of elements* $y \in X$ *such that there exists some* $f \in \mathrm{Inn(X)}$ *satisfying* $f(x) = y$*.*

DEFINITION 1.3.10 *[19] A quandle* $(X, *)$ *is* connected *when there exists exactly one orbit in* X*. That is, for any* $\in X$*, the orbit of* x *is all of* X*.*

DEFINITION 1.3.11 *A quandle* $(X, *)$ *is* abelian *if it satisfies the identity* $(w * x) * (y * z) = (w * y) * (x * z)$*, for any* $w, x, y, z \in X$*, not to be confused with a communative quandle.*

DEFINITION 1.3.12 *A quandle* $(X, *)$ *is* commutative *if it satisfies the identity* $x * y = y * x$*, for any* $x, y \in X$*.*

1.4 Examples of Quandles

EXAMPLE 1 Any set X with the operation $x * y = x$ for any $x, y \in X$ is a quandle called the *trivial* quandle. The trivial quandle of n elements is denoted by T_n.

EXAMPLE 2 Let G be a group. The binary operation (conjugation) $a * b = b^{-1}ab$ for any $a, b \in G$ gives a quandle structure on G. This quandle is called a conjugation quandle, denoted Conj(G). An n-fold conjugation quandle is the group G with n-fold conjugation as the operation $a * b = b^{-n}ab^n$.

This is interesting when G is non-abelian. It is easy to check that the following three properties hold:

1. $a * a = a$

2. $x * a = b \Longleftrightarrow a^{-n}xa^n = b \Longleftrightarrow x = a^nba^{-n} \Longrightarrow x$ is unique.

3.

$$\begin{aligned}(a * b) * c &= (b^{-n}ab^n) * c \\ &= c^{-n}b^{-n}ab^nc^n\end{aligned}$$

$$\begin{aligned}(a * c) * (b * c) &= (c^{-n}ac^n) * (c^{-n}bc^n) \\ &= (c^{-n}bc^n)^{-n}(c^{-n}ac^n)(c^{-n}bc^n)^n \\ &= c^{-n}b^{-n}c^nc^{-n}ac^nc^{-n}b^nc^n \\ &= c^{-n}b^{-n}ab^nc^n\end{aligned}$$

EXAMPLE 3 Let n be a positive integer. For elements $i, j \in \mathbb{Z}_n$, define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n. This set can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation.

EXAMPLE 4 Let Λ be the ring of Laurent polynomials in the one variable T. Any Λ -module M is a quandle with $a * b = Ta + (1 - T)b$; $a, b \in M$. This quandle is called an *Alexander quandle*. Furthermore for a positive integer n, a *mod-n Alexander quandle* $\mathbb{Z}_n[T, T^{-1}]/(h(T))$ is a quandle for a Laurent polynomial $h(T)$. The mod-n Alexander quandle is finite if the coefficients of the highest and lowest degree terms of h are units in $\mathbb{Z}_n$.

1.5 Fundamental Quandle

This is the most important quandle of all and is the *raison d'etre* of the whole theory.

DEFINITION 1.5.1 *[15, 17] Let* $\{x_1, x_2, ..., x_k\}$ *be variables assigned to the arcs of a knot diagram* K. *Let* $x_l = x_i * x_j$ *be assigned at each crossing, where* x_j *is the variable assigned to the over-arc and* x_i *is the variable assigned to the under-arc from which the orientation of the normal vector of the over-arc points. Then* x_l *is assigned to the other under-arc. The quandle* $Q(K)$ *determined by the set of generators* $\{x_1, x_2, ..., x_k\}$ *and the set of relations* $\{x_l = x_i * x_j\}$ *over all crossings is called the* fundamental quandle *of* K.

Figure 5 shows over and under crossings.

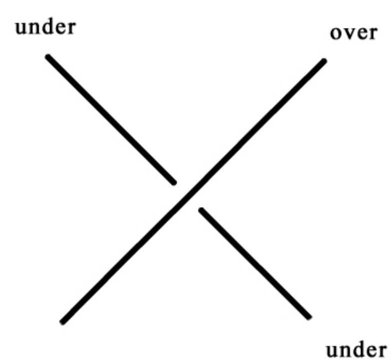

Figure 5.: Over-Arcs and Under-Arcs

A quandle defined by the set of generators $\{x_1, x_2, ..., x_k\}$ and relations $\{r_1, r_2, ..., r_m\}$, as in Definition 1.5.1, is denoted by $< x_1, x_2, ..., x_k \mid r_1, r_2, ..., r_m >$. This is called a *presentation* of the quandle [9].

EXAMPLE 5 Consider the trefoil knot. Let $\{x_1, x_2, x_3\}$ be the variables assigned to the arcs of the trefoil knot. See Figure 6.

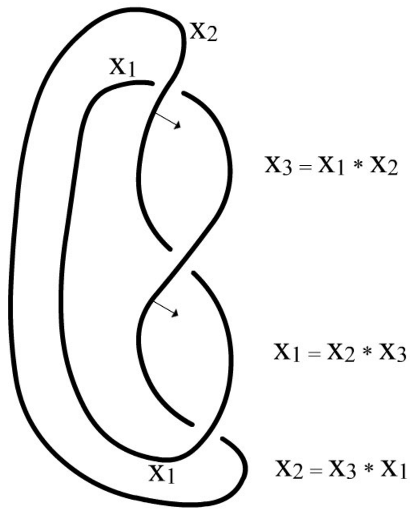

Figure 6.: Trefoil and Its Relations

The presentation of the fundamental quandle of the trefoil knot is

$$< x_1, x_2, x_3 \mid x_1 * x_2 = x_3, x_2 * x_3 = x_1, x_3 * x_1 = x_2 > .$$

1.6 Coloring of Knots

DEFINITION 1.6.1 *[9] Let X be a quandle, let D be an oriented knot diagram and A be the set of over-arcs. A coloring f is a map $f : A \longrightarrow X$ such that at every crossing where the normal to the over-arc β points from the arc α to the arc γ, the relation $f(\alpha) * f(\beta) = f(\gamma)$ holds, see Figure 7. That is, a coloring is a quandle homorphism that maps the generators of the fundamental quandle to a fixed quandle. The image $f(\alpha)$ is called a color of the arc α. The colors in the ordered pair (a, b) are called source colors.*

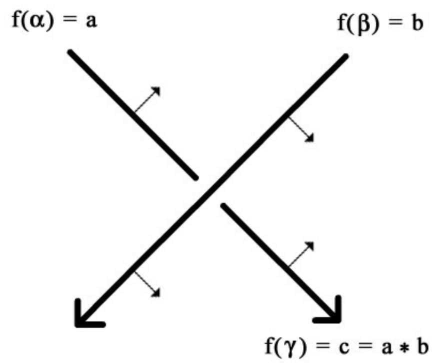

Figure 7.: Coloring Relations

Let $Col_X(D)$ be the set of colorings of a knot diagram D by a quandle X. There is a one-to-one correspondence between the set of colorings of two diagrams of the same knot. The set of all colorings of D by quandle X is the set of quandle homomorphisms from the fundamental quandle to the quandle X. So the number of colorings by X is the cardinality of this set. It is proved in [9] that the number of colorings is a *knot invariant*.

Any knot diagram K has at least one coloring for a given quandle X, the *trivial coloring*. The trivial coloring is obtained by letting every arc have the same color. Clearly the reason knot coloring works so well is that the axioms of quandles correspond to the Reidmeister moves.

DEFINITION 1.6.2 *[10] If a knot diagram K can be non-trivially colored by the dihedral quandle R_n, then K is said to be* n-colorable.

EXAMPLE 6 Coloring the trefoil by R_3:

Let X be the dihedral quandle R_3 and let D be the trefoil knot diagram as shown in Figure 8. Let the source colors be given by $(0, 1)$ at the top of the knot. Note that this is just one possible coloring of the trefoil by R_3. Any pair $(a, b) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ colors the trefoil. Then there are a total of 9 colorings of the trefoil by R_3. Of those 9 colorings, 3 are trivial. This example is also called a Fox *3-coloring* of the trefoil. In general a Fox *-coloring* is a coloring of a knot diagram by the dihedral quandle R_n.

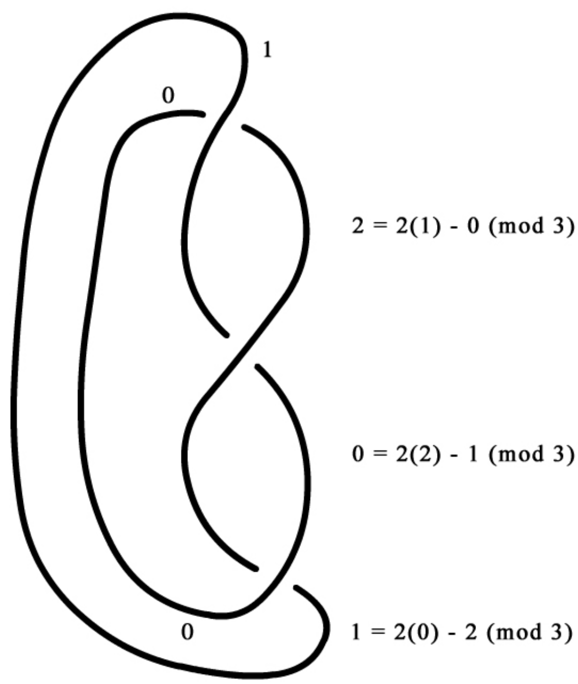

Figure 8.: Coloring the Trefoil by R_3

Chapter 2

Automorphism Groups of Quandles

2.1 Review of Automorphism Groups of Quandles of Orders 3, 4 and 5

Ho and Nelson's *Matrices and Finite Quandles* [13] classify finite quandles with up to five elements and compute the automorphism group for each of those quandles. The results obtained by Ho and Nelsen were confirmed by the algorithm used by Elhamdadi and myself. The algorithm we used to obatin some results in [8] was written with the software *Maple* with the help of Edwin Clark. This algorithm finds all quandles of order 9 or less up to isomorphism and finds the order and order sequence of their automorphism and inner automorphism groups. A brief description of the algorithm is included in [8].

DEFINITION 2.1.1 *The* symmetric group*, on a finite set of n elements, is the group whose elements are all the permutations of the n elements. The symmetric group is denoted by* Σ_n*. The cardinality of the symmetric group is* $n!$ *(factorial* n*). For example,* Σ_3 *is the symmetric group on a set of 3 elements.*

DEFINITION 2.1.2 *The* alternating group*, denoted by* A_n*, is the group of even permutations of a finite set of n elements. The cardinality of the alternating group is* $n!/2$*. For example,* A_4 *is the alternating group on a set of 4 elements.*

DEFINITION 2.1.3 *The* dihedral group*, denoted by* D_n*, is the group of symmetries of a regular* n*-gon (including both rotations and reflections). The cardinality of the dihedral group is* $2n$*. For example,* D_5 *is the group of rotations and reflections of a regular pentagon.*

Table 1: Quandles of order 3 as disjoint cycles of columns with automorphism groups

Quandle X	Disjoint Cycle Notation	Inn(X)	Aut(X)
Q_1	$(1),(1),(1)$	$\{1\}$	Σ_3
Q_2	$(1),(1),(12)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Q_3	$(23),(13),(12)$	Σ_3	Σ_3

Table 2: Quandles of order 4 as disjoint cycles of columns with automorphism groups

Quandle X	Disjoint Cycle Notation	Inn(X)	Aut(X)
Q_1	$(1),(1),(1),(1)$	$\{1\}$	Σ_4
Q_2	$(1),(1),(1),(23)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Q_3	$(1),(1),(1),(123)$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
Q_4	$(1),(1),(12),(12)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Q_5	$(1),(34),(24),(23)$	Σ_3	Σ_3
Q_6	$(34),(34),(12),(12)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4
Q_7	$(234),(143),(124),(132)$	A_4	A_4

Since every permutation of a finite set can be written as a product of disjoint cycles we will adopt the following notation: the identity permutation $(x \mapsto x)$ of a finite set of n elements will be denoted by (1). The permutation σ of the set $\{1, 2, 3, 4, 5, 6\}$ sending 1 to 1, 2 to 5, 3 to 4, 4 to 3, 5 to 2, and 6 to 6 will be denoted by $(25)(34)$.

First determined by Nelsen and Ho in [13], there are exactly three quandles of order 3 up to isomorphism, there are exactly seven quandles of order 4 up to isomorphism, and there are 22 quandles of order 5 up to isomorphism. See Table 1 and Table 2 for the list of quandles and their automorphism groups of orders 3 and 4. For a list of quandles of order 5 and their automorphism groups please refer to Ho and Nelsen's paper. For any quandle of order 5 the inner automorphism group is one of the following groups: $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, Σ_3, $\mathbb{Z}_2 \times \mathbb{Z}_2$, A_4, D_5, or $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$. Section 2.3 provides a complete list of the quandles of orders 6 and 7 with their automorphism and inner automorphism groups. The *Maple* algorithm generates quandles up to order 9. The 1581 quandles of order 8 and

11079 quandles of order 9 are too numerous to list here. We do have a list of all automorphism and inner automorphism groups of quandles of order 8.

2.2 Quandles and Their Automorphism and Inner Automorphism Groups

We will denote the group of automorphisms of a quandle X by $\mathrm{Aut}(X)$. Axioms (2) and (3) respectively state that for each $u \in X$, the map $S_u : X \to X$ with $S_u(x) = x * u$ is respectively a bijection and a quandle homomorphism. We call the subgroup of $\mathrm{Aut}(X)$ generated by the *symmetries* S_x, the group of $inner$ automorphism of X denoted by $\mathrm{Inn}(X)$. By axiom (3), the map $S : X \longrightarrow \mathrm{Inn}(X)$ sending u to S_u satisfies the equation $S_z\ S_y = S_{y*z}\ S_z$, for any $y, z \in X$, which can be written as $S_z\ S_y\ {S_z}^{-1} = S_{y*z}$. Thus, if the group $\mathrm{Inn}(X)$ is considered as a quandle with conjugation, then the map S becomes a quandle homomorphism.

Now we will characterize the automorphisms of the dihedral quandles. Recall that the affine group of $\mathbb{Z}_n$ is the group of all invertible affine transformations of $\mathbb{Z}_n$,

$$\mathrm{Aff}(\mathbb{Z}_n) := \{f_{a,b} : \mathbb{Z}_n \to \mathbb{Z}_n,\ \ f_{a,b}(x) = ax + b,\ a\ \in \mathbb{Z}_n^{\times},\ \ b \in \mathbb{Z}\}.$$

The element $f_{a,b}$ is identified with the pair (a, b) and the group multiplication is given by $(a, b)(c, d) = (ac, ad+b)$. The identity is $(1, 0)$ and the inverse is given by $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$. Usually the element (a, b) is represented in a matrix notation as $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ so group multiplication corresponds to multiplication of matrices.

THEOREM 2.1 *Let* $R_n = \mathbb{Z}_n$ *be the dihedral quandle with the operation* $i * j = 2j - i(mod\ n)$. *Then* $\mathrm{Aut}(R_n)$ *is isomorphic to the affine group* $\mathrm{Aff}(\mathbb{Z}_n)$ *for any positive integer* n.

Proof. It is clear that for $a \neq 0$, the map $f_{a,b}$ (with $f_{a,b}(x) = ax + b$) is a quandle homomorphism. It is bijective if and only if a is a unit in ${\mathbb{Z}_n}^{\times}$. Now we show that any quandle automorphism of $\mathbb{Z}_n$ (with the operation $x * y = 2y - x$) is an affine transformation $f_{a,b}$ for some invertible a. Let $f \in Aut(\mathbb{Z}_n)$, then for all $x, y \in \mathbb{Z}_n, f(2y - x) = 2f(y) - f(x)$. Now consider the mapping $g : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $g(x) = f(x) - f(0)$. It is clear that $g(0) = 0$ and $g(-a) = -g(a)$. We now prove that $g(\lambda x) = \lambda g(x)$ for all $\lambda \in \mathbb{Z}_n$. We have $g(2b - a) = 2g(b) - g(a)$, thus

$g(2b) = 2g(b)$ and by induction on even integers $g(2ka) = 2kg(a)$. Now we do induction on odd integers: $g[(2k+1)a] = g[2ka - (-a)] = 2kg(a) - g(-a) = 2kg(a) + g(a) = (2k+1)g(a)$. Now g is a bijection if and only if λ is a unit in $\mathbb{Z}_n^{\times}$. □

COROLLARY 2.0.1 *The cardinality of* $\mathrm{Aut}(\mathbb{Z}_n)$ *is* $n\phi(n)$*, where* ϕ *denotes the Euler function.*

THEOREM 2.2 *The inner automorphism group* $\mathrm{Inn}(R_n)$ *of the dihedral quandle* R_n *is isomorphic to the dihedral group* $D_{\frac{m}{2}}$ *of order* m*, where* m *is the least common multiple of* n *and* 2.

Proof. By definition, the Inn(R_n) is the subgroup of Aut(R_n) generated by all S_x, where $S_x(y) := y * x = 2x - y \,(mod\, n), \forall x, y \in R_n$. We claim that the Inn$(R_n)$ of the dihedral quandle R_n is generated by two elements of order 2. This is a direct consequence of the relation $S_{i+2} = S_{i+1}S_iS_{i+1}^{-1}$. It is easy to check that this relation holds. First observe that $S_iS_j = S_{j*i}S_i$. Then $S_{i+1}S_i = S_{i*(i+1)}S_{i+1} = S_{2i+2-i}S_{i+1} = S_{i+2}S_{i+1}$. That is, $S_{i+2} = S_{i+1}S_iS_{i+1}^{-1}$. By induction, S_i can be obtained from S_0 and S_1, each of order 2. It is easy to check that S_i is of order 2. $S_i(S_i(j)) = S_i(j*i) = S_i(2i-j) = (2i-j)*i = 2i-(2i-j) = j$. We have $S_0S_1 \neq S_1S_0$, thus Inn(R_n) is non-abelian. Then since Inn(R_n) is a finite non-abelian group generated by two elements of order 2, Inn(R_n) is isomorphic to a dihedral group. Since the cardinality of Inn(R_{2n}) equals $2n$, the Inn(R_{2n}) is the dihedral group D_n. Since the cardinality of Inn(R_{2n+1}) equals $2(2n+1)$, the Inn(R_{2n+1}) is the dihedral group D_{2n+1}. Hence the Inn(R_n) isomorphic to the dihedral group $D_{\frac{m}{2}}$ of order m, where m is the least common multiple of n and 2. □

THEOREM 2.3 *[1] Let* X *be the quandle on the set of* G *with operation* $x * y = y^{-1}xy$*. Then the Inner automorphism group of* X *is isomorphic (as a group) to the quotient of* G *by the center* $Z(G)$*. The quandle* X *is usually denoted by* $\mathrm{Conj}(G)$*.*

Proof. The proof is straightforward from the fact that in this case the surjective map $S : X \longrightarrow$ Inn(X) sending $a \in X$ to S_a is a quandle homomorphism with kernel equal to the center $Z(G)$ of G. □

EXAMPLE 7 The symmetric group Σ_3 is the smallest group with trivial center then Inn(Conj$(\Sigma_3)) \cong \Sigma_3$.

The converse of Theorem 2.3 is also true, namely if $(X, *)$ is a quandle for which the map $S : X \longrightarrow \text{Inn}(X)$ is one-to-one and onto then $(X, *) \cong$ Conj(Inn(X)) with Z(Inn(X)) being trivial group.

An interesting project would be to calculate the automorphism groups Aut(Conj(G)). Obviously for the symmetric group Σ_3, we have Aut(Conj$(\Sigma_3)) \cong$ Inn(Conj$(\Sigma_3)) \cong \Sigma_3$.

2.3 Automorphism Groups of Quandles of Orders 6 and 7

In this section, the automorphism and inner automorphism groups of all 73 quandles of order 6 and all 298 quandles of order 7 are computed. This computation is accomplished with the help of the software *Maple* [20], which also allows for the computation of the quandles of order 8. Since there are 1581 they will not be included in this paper.

We describe each quandle Q_j of order 6 for $1 \leq j \leq 73$ (there are 73 isomorphism classes of quandles of order 6) by explicitly giving each symmetry S_k $(1 \leq k \leq 6)$, in terms of products of disjoint cycles. The symmetries are the columns in the Cayley table. For example, the quandle Q_{46} of Table 4 has the following Cayley table (a Cayley table for quandle X is the square matrix such that the ij entry correspond to $i * j$).

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 5 & 2 & 5 \\ 3 & 4 & 3 & 3 & 4 & 4 \\ 4 & 3 & 4 & 4 & 3 & 3 \\ 5 & 5 & 2 & 2 & 5 & 2 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$

The quandle Q_{46} is described by the symmetries $S_1 = (1)$, $S_2 = (34)$, $S_3 = (25)$, $S_4 = (25)$, $S_5 = (34)$, $S_6 = (25)(34)$. Here and through the rest of the paper, every permutation is written as a product of transpositions. For example, $S_1 = (1)$ means that S_1 is the identity permutation. The permutation $S_4 = (25)$ stands for the transposition sending 2 to 5 and $S_6 = (25)(34)$ stands for the product of the two transpositions (25) and (34).

In this example, $Aut(Q_{46}) = D_4$ is the dihedral group of 8 elements and $Inn(Q_{48}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ is the direct product of two copies of $\mathbb{Z}_2$. Another example in Table 6, the $Aut(Q_{49}) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ is the semi-direct product of the cyclic group $\mathbb{Z}_5$ by $\mathbb{Z}_4$ and the $Inn(Q_{49}) = D_5$ is the dihedral group of 10 elements. The semi-direct product of groups is given by the following:

DEFINITION 2.3.1 *Let G, H be groups and let $f : H \longrightarrow Aut(G)$ be the map sending u to f_u a group homomorphism. As a set, the* semi-direct product *is $G \times H$. The group operation is defined as follows. For any $(g_1, h_1), (g_2, h_2) \in G \times H$ has the formula $(g_1 \cdot f_{h_1}(g_2), h_1 h_2)$.*

With the help of Edwin Clark, we wrote a program using the software *Maple* to find as much information about the automorphism and inner automorphism as possible. The following is a description of the *Maple* algorithm we used:

1. A representation of each quandle using the symmetries S_x, recall these are permutations written as disjoint cycles. The command *permgroup* from the *Maple* group package was used.

2. For both inner automorphism and automorphism groups, first calculate the group order using *group order* command.

3. Identify abelian or non-abelian using the *isabelian* command in the *Maple* group package.

4. Calculate the order sequence. The order sequence is a list of the number of elements of each order. That is, $[1, 1], [2, 1], [4, 2]$ means there is one element of order 1, 1 element of order 2 and 2 elements of order 4.

Using the *Maple* output we were left to find all inner automorphism and automorphism groups. Using the classification of small groups we were able to find most all of them. It is known that two abelian groups, say G and H, are isomorphic if they have the same order and the same order sequence [12]. But this is not true for non-abelian groups. That is, there exists two non-isomorphic non-abelian groups with the same order and the same number of elements of each order. This implies that the number of elements of each order is not enough to determine the structure of non-abelian groups. Fortunately, most of the non-abelian groups for the automorphism and inner automorphism groups of the quandles in this paper were of small enough order that is was possible to find the groups. A list was made of all the non-abelian groups of the given order with the respective order sequences. From there we had only one non-abelian group with the desired order sequence. There were a few cases of non-ableian groups of orders large enough in which listing all possibilities was not an efficient way to obtain the desired group. Knowing that for quandles, the inner automorphism group is a normal subgroup of the automorphism group and with the software *Maple* we were able to find our non-abelian groups of large orders. For example, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $(D_3 \times D_3) \rtimes \mathbb{Z}_2$, $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$, $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ and $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

The following is an example of the *Maple* output a quandle of order 6, specifically Q_{70}.

(70) matrix columns in disjcyc notation:
$[[[2,3],[4,5]],[[1,5],[3,6]],[[1,4],[2,6]],[[1,5],[3,6]],[[1,4],[2,6]],[[2,3],[4,5]]]$
(70) inner automorphism group order = 6, not abelian, not cyclic, order seq = $\{[1,1],[2,3],[3,2]\}$
(70) automorphism group order = 12, not abelian, order sequence = $\{[1,1],[2,7],[3,2],[6,2]\}$

Recall from earlier in this chapter that the quandles are represented as Cayley tables. The output for the quandle Q_{70} gives the matrix columns in disjoint cycle notation. The following is the Cayley table for the quandle Q_{70}.

$$\begin{bmatrix} 1 & 5 & 4 & 5 & 4 & 1 \\ 3 & 2 & 6 & 2 & 6 & 3 \\ 2 & 6 & 3 & 6 & 3 & 2 \\ 5 & 4 & 1 & 4 & 1 & 5 \\ 4 & 1 & 5 & 1 & 5 & 4 \\ 6 & 3 & 2 & 3 & 2 & 6 \end{bmatrix}$$

Using the *Maple* output the following steps were taken to find the automorphism and inner automorphism groups of each quandle.

1. For the quandle Q_{70} the *Maple* output shows the group order to be 6 and the group to be non-abelian. The only non-abelian groups of order 6 are the dihedral group D_3 and the symmetric group Σ_3. Since D_3 is isomorphic to Σ_3 it is easy to identify the desired group. It is also important to note that the order sequence for D_3 and Σ_3 is the same as the order sequence of the *Maple* output.

2. For the automorphism group, the *Maple* output shows the group order to be 12 and the group to be non-abelian. A list is then made of the non-ableian groups of order 12. The non-ableian groups of order 12 are $D_6 \cong D_3 \times \mathbb{Z}_2$, A_4, and $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$. It is easy to check that the order sequence of D_6 is the same as the order sequence of the given output. However since two non-abelian groups can have the same order sequence but have different structure, the rest of the groups listed need to be checked. Looking at the order sequence of A_4 we see that there are 3 elements of order 2. Since the group we are looking for has 7 elements of order 2 we know A_4 is not the desired group. Finally looking at the order sequence of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ we see that this group contains elements of order 3 and order 4. Since our desired group does not contain any elements of order 4 we know this is not the group. It is now clear that the automorphism group for Q_{70} is D_6. This should make sense because we know that the inner automorphism group is a subgroup, in fact a normal subgroup, of the automorphism group of a quandle.

Table 3: Quandles of order 6 as disjoint cycles of columns: 1–28 of 73

Quandle X	Disjoint Cycle Notation
Q_1	$(1),(1),(1),(1),(1),(1)$
Q_2	$(1),(1),(1),(1),(1),(12)$
Q_3	$(1),(1),(1),(1),(1),(132)$
Q_4	$(1),(1),(1),(1),(1),(1243)$
Q_5	$(1),(1),(1),(1),(1),(12)(34)$
Q_6	$(1),(1),(1),(1),(1),(15234)$
Q_7	$(1),(1),(1),(1),(1),(134)(25)$
Q_8	$(1),(1),(1),(1),(12),(12)$
Q_9	$(1),(1),(1),(1),(12),(12)(34)$
Q_{10}	$(1),(1),(1),(1),(12),(34)$
Q_{11}	$(1),(1),(1),(1),(132),(132)$
Q_{12}	$(1),(1),(1),(1),(132),(123)$
Q_{13}	$(1),(1),(1),(1),(1243),(1243)$
Q_{14}	$(1),(1),(1),(1),(1243),(1342)$
Q_{15}	$(1),(1),(1),(1),(1243),(14)(23)$
Q_{16}	$(1),(1),(1),(1),(12)(34),(12)(34)$
Q_{17}	$(1),(1),(1),(1),(12)(34),(13)(24)$
Q_{18}	$(1),(1),(1),(12),(12),(12)$
Q_{19}	$(1),(1),(1),(12),(12),(12)(45)$
Q_{20}	$(1),(1),(1),(12),(12),(45)$
Q_{21}	$(1),(1),(1),(132),(132),(132)$
Q_{22}	$(1),(1),(1),(132),(132),(123)$
Q_{23}	$(1),(1),(1),(132),(132),(45)$
Q_{24}	$(1),(1),(1),(132),(132),(123)(45)$
Q_{25}	$(1),(1),(1),(132),(132),(132)(45)$
Q_{26}	$(1),(1),(12)(56),(12)(46),(12)(45)$
Q_{27}	$(1),(1),(1),(12)(56),(13)(46),(23)(45)$
Q_{28}	$(1),(1),(1),(56),(46),(45)$

Table 4: Quandles of order 6 as disjoint cycles of columns: 29–56 of 73

Quandle X	Disjoint Cycle Notation
Q_{29}	$(1), (1), (1), (123)(56), (123)(46), (123)(45)$
Q_{30}	$(1), (1), (12), (12), (12), (12)$
Q_{31}	$(1), (1), (12), (12), (12), (12)(34)$
Q_{32}	$(1), (1), (12), (12), (12), (34)$
Q_{33}	$(1), (1), (12), (12), (12), (345)$
Q_{34}	$(1), (1), (12), (12), (12), (12)(345)$
Q_{35}	$(1), (1), (12), (12), (12)(34), (12)(34)$
Q_{36}	$(1), (1), (12), (12), (12)(34), (34)$
Q_{37}	$(1), (1), (12), (12), (34), (34)$
Q_{38}	$(1), (1), (12), (12)(56), (12)(46), (12)(45)$
Q_{39}	$(1), (1), (12), (56), (46), (45)$
Q_{40}	$(1), (1), (12)(45), (12)(36), (12)(36), (12)(45)$
Q_{41}	$(1), (1), (12)(45), (36), (36), (12)(45)$
Q_{42}	$(1), (1), (45), (36), (36), (45)$
Q_{43}	$(1), (1), (456), (365), (346), (354)$
Q_{44}	$(1), (1), (12)(456), (12)(365), (12)(346), (12)(354)$
Q_{45}	$(1), (34), (25), (25), (34), (34)$
Q_{46}	$(1), (34), (25), (25), (34), (25)(34)$
Q_{47}	$(1), (34), (256), (256), (34), (34)$
Q_{48}	$(1), (354), (26)(45), (26)(35), (26)(34), (345)$
Q_{49}	$(1), (36)(45), (25)(46), (23)(56), (26)(34), (24)(35)$
Q_{50}	$(1), (3546), (2456), (2365), (2643), (2534)$
Q_{51}	$(1), (3546), (2564), (2653), (2436), (2345)$
Q_{52}	$(23), (13), (12), (56), (46), (45)$
Q_{53}	$(23), (14), (14), (23), (23), (23)$
Q_{54}	$(23), (14), (14), (23), (23), (14)(23)$
Q_{55}	$(23), (14), (14), (23), (23), (14)$
Q_{56}	$(23), (14), (14), (23), (14)(23), (14)(23)$

Table 5: Quandles of order 6 as disjoint cycles of columns: 57–73

Quandle X	Disjoint Cycle Notation
Q_{57}	$(23),(154),(154),(23),(23),(23)$
Q_{58}	$(23),(154),(154),(23),(23),(154)(23)$
Q_{59}	$(23),(154),(154),(23),(23),(154)$
Q_{60}	$(23),(154),(154),(23),(23),(145)$
Q_{61}	$(23),(154),(154),(23),(23),(145)(23)$
Q_{62}	$(23),(45),(45),(16)(23),(16)(23),(23)$
Q_{63}	$(23),(45),(45),(16),(16),(23)$
Q_{64}	$(23),(1564),(1564),(23),(23),(23)$
Q_{65}	$(23),(15)(46),(15)(46),(23),(23),(23)$
Q_{66}	$(23),(15)(46),(15)(46),(15)(23),(23),(15)(23)$
Q_{67}	$(243),(165),(165),(165),(243),(243)$
Q_{68}	$(2354),(1463),(1265),(1562),(1364),(2453)$
Q_{69}	$(2354),(16)(34),(16)(25),(16)(25),(16)(34),(2453)$
Q_{70}	$(23)(45),(15)(36),(14)(26),(15)(36),(14)(26),(23)(45)$
Q_{71}	$(23)(45),(15)(46),(14)(56),(16)(23),(16)(23),(23)(45)$
Q_{72}	$(23)(45),(13)(46),(12)(56),(15)(26),(14)(36),(24)(35)$
Q_{73}	$(23)(45),(16)(45),(16)(45),(16)(23),(16)(23),(23)(45)$

Table 6: Quandles of order 6 with their automorphism groups 1–56 of 73

Quandle X	Inn(X)	Aut(X)	Quandle X	Inn(X)	Aut(X)
Q_1	$\{1\}$	S_6	Q_{29}	$D_3 \times \mathbb{Z}_3$	$D_3 \times \mathbb{Z}_3$
Q_2	$\mathbb{Z}_2$	$D_3 \times \mathbb{Z}_2$	Q_{30}	$\mathbb{Z}_2$	$S_4 \times \mathbb{Z}_2$
Q_3	$\mathbb{Z}_3$	Z_6	Q_{31}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Q_4	$\mathbb{Z}_4$	$\mathbb{Z}_4$	Q_{32}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Q_5	$\mathbb{Z}_2$	D_4	Q_{33}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_6	$\mathbb{Z}_5$	Z_5	Q_{34}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_7	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{35}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_8	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{36}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Q_9	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{37}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{10}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4	Q_{38}	$D_3 \times \mathbb{Z}_2$	$D_3 \times \mathbb{Z}_2$
Q_{11}	$\mathbb{Z}_3$	$\mathbb{Z}_6$	Q_{39}	$D_3 \times \mathbb{Z}_2$	$D_3 \times \mathbb{Z}_2$
Q_{12}	$\mathbb{Z}_3$	D_3	Q_{40}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$
Q_{13}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{41}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{14}	$\mathbb{Z}_4$	D_4	Q_{42}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$
Q_{15}	$\mathbb{Z}_4$	$\mathbb{Z}_4$	Q_{43}	A_4	$A_4 \times \mathbb{Z}_2$
Q_{16}	$\mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$	Q_{44}	$A_4 \times \mathbb{Z}_2$	$A_4 \times \mathbb{Z}_2$
Q_{17}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4	Q_{45}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{18}	$\mathbb{Z}_2$	$D_3 \times \mathbb{Z}_2$	Q_{46}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4
Q_{19}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$Z_2 \times \mathbb{Z}_2$	Q_{47}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{20}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$Z_2 \times \mathbb{Z}_2$	Q_{48}	D_3	D_3
Q_{21}	$\mathbb{Z}_3$	$D_3 \times \mathbb{Z}_3$	Q_{49}	D_5	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
Q_{22}	$\mathbb{Z}_3$	$\mathbb{Z}_6$	Q_{50}	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
Q_{23}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{51}	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
Q_{24}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{52}	$D_3 \times D_3$	$(D_3 \times D_3) \rtimes \mathbb{Z}_2$
Q_{25}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{53}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{26}	D_3	$D_3 \times \mathbb{Z}_2$	Q_{54}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{27}	D_3	D_3	Q_{55}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4
Q_{28}	D_3	$D_3 \times D_3$	Q_{56}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$

Table 7: Quandles of order 6 with their automorphism groups 57–73

Quandle X	Inn(X)	Aut(X)	Quandle X	Inn(X)	Aut(X)
Q_{57}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{66}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$Z_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{58}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{67}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$D_3 \times \mathbb{Z}_3$
Q_{59}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{68}	S_4	S_4
Q_{60}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{69}	D_4	D_4
Q_{61}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{70}	D_3	$D_3 \times \mathbb{Z}_2$
Q_{62}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{71}	D_4	D_4
Q_{63}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$A_4 \times \mathbb{Z}_2$	Q_{72}	S_4	S_4
Q_{64}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{73}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$S_4 \times \mathbb{Z}_2$
Q_{65}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \mathbb{Z}_2$			

Table 8: Quandles of order 7 as disjoint cycles of columns: 1–28 of 298

Quandle X	Disjoint Cycle Notation
Q_1	$(1),(1),(1),(1),(1),(1),(1)$
Q_2	$(1),(1),(1),(1),(1),(1),(12)$
Q_3	$(1),(1),(1),(1),(1),(1),(132)$
Q_4	$(1),(1),(1),(1),(1),(1),(1243)$
Q_5	$(1),(1),(1),(1),(1),(1),(12)(34)$
Q_6	$(1),(1),(1),(1),(1),(1),(15234)$
Q_7	$(1),(1),(1),(1),(1),(1),(134)(25)$
Q_8	$(1),(1),(1),(1),(1),(1),(124365)$
Q_9	$(1),(1),(1),(1),(1),(1),(165)(243)$
Q_{10}	$(1),(1),(1),(1),(1),(1),(1265)(34)$
Q_{11}	$(1),(1),(1),(1),(1),(1),(12)(34)(56)$
Q_{12}	$(1),(1),(1),(1),(1),(12),(12)$
Q_{13}	$(1),(1),(1),(1),(1),(12),(12)(34)$
Q_{14}	$(1),(1),(1),(1),(1),(12),(34)$
Q_{15}	$(1),(1),(1),(1),(1),(12),(345)$
Q_{16}	$(1),(1),(1),(1),(1),(12),(12)(345)$
Q_{17}	$(1),(1),(1),(1),(1),(132),(132)$
Q_{18}	$(1),(1),(1),(1),(1),(132),(123)$
Q_{19}	$(1),(1),(1),(1),(1),(132),(123)(45)$
Q_{20}	$(1),(1),(1),(1),(1),(132),(132)(45)$
Q_{21}	$(1),(1),(1),(1),(1),(1243),(1243)$
Q_{22}	$(1),(1),(1),(1),(1),(1243),(1342)$
Q_{23}	$(1),(1),(1),(1),(1),(1243),(14)(23)$
Q_{24}	$(1),(1),(1),(1),(1),(12)(34),(12)(34)$
Q_{25}	$(1),(1),(1),(1),(1),(12)(34),(13)(24)$
Q_{26}	$(1),(1),(1),(1),(1),(15234),(15234)$
Q_{27}	$(1),(1),(1),(1),(1),(15234),(12453)$
Q_{28}	$(1),(1),(1),(1),(1),(15234),(14325)$

Table 9: Quandles of order 7 as disjoint cycles of columns: 29–56 of 298

Quandle X	Disjoint Cycle Notation
Q_{29}	$(1),(1),(1),(1),(1),(134)(25),(134)(25)$
Q_{30}	$(1),(1),(1),(1),(1),(134)(25),(143)(25)$
Q_{31}	$(1),(1),(1),(1),(12),(12),(12)$
Q_{32}	$(1),(1),(1),(1),(12),(12),(12)(34)$
Q_{33}	$(1),(1),(1),(1),(12),(12),(34)$
Q_{34}	$(1),(1),(1),(1),(12),(12),(34)(56)$
Q_{35}	$(1),(1),(1),(1),(12),(12),(12)(34)(56)$
Q_{36}	$(1),(1),(1),(1),(12),(12),(12)(56)$
Q_{37}	$(1),(1),(1),(1),(12),(12),(56)$
Q_{38}	$(1),(1),(1),(1),(12),(12)(34),(12)(34)$
Q_{39}	$(1),(1),(1),(1),(12),(12)(34),(34)$
Q_{40}	$(1),(1),(1),(1),(132),(132),(132)$
Q_{41}	$(1),(1),(1),(1),(132),(132),(123)$
Q_{42}	$(1),(1),(1),(1),(132),(132),(123)(56)$
Q_{43}	$(1),(1),(1),(1),(132),(132),(132)(56)$
Q_{44}	$(1),(1),(1),(1),(132),(132),(56)$
Q_{45}	$(1),(1),(1),(1),(1243),(1243),(1243)$
Q_{46}	$(1),(1),(1),(1),(1243),(1243),(1342)$
Q_{47}	$(1),(1),(1),(1),(1243),(1243),(14)(23)$
Q_{48}	$(1),(1),(1),(1),(1243),(1243),(1243)(56)$
Q_{49}	$(1),(1),(1),(1),(1243),(1243),(1342)(56)$
Q_{50}	$(1),(1),(1),(1),(1243),(1243),(14)(23)(56)$
Q_{51}	$(1),(1),(1),(1),(1243),(1243),(56)$
Q_{52}	$(1),(1),(1),(1),(1243),(1342),(14)(23)$
Q_{53}	$(1),(1),(1),(1),(1243),(14)(23),(14)(23)$
Q_{54}	$(1),(1),(1),(1),(12)(34),(12)(34),(12)(34)$
Q_{55}	$(1),(1),(1),(1),(12)(34),(12)(34),(13)(24)$
Q_{56}	$(1),(1),(1),(1),(12)(34),(12)(34),(34)(56)$

Table 10: Quandles of order 7 as disjoint cycles of columns: 57–84 of 298

Quandle X	Disjoint Cycle Notation
Q_{57}	$(1),(1),(1),(1),(12)(34),(12)(34),(12)(34)(56)$
Q_{58}	$(1),(1),(1),(1),(12)(34),(12)(34),(1423)(56)$
Q_{59}	$(1),(1),(1),(1),(12)(34),(12)(34),(13)(24)(56)$
Q_{60}	$(1),(1),(1),(1),(12)(34),(12)(34),(56)$
Q_{61}	$(1),(1),(1),(1),(12)(34),(13)(24),(14)(23)$
Q_{62}	$(1),(1),(1),(1),(1432)(67),(1432)(57),(1432)(56)$
Q_{63}	$(1),(1),(1),(1),(243)(67),(243)(57),(243)(56)$
Q_{64}	$(1),(1),(1),(1),(34)(67),(34)(57),(34)(56)$
Q_{65}	$(1),(1),(1),(1),(34)(67),(24)(57),(23)(56)$
Q_{66}	$(1),(1),(1),(1),(12)(34)(67),(12)(34)(57),(12)(34)(56)$
Q_{67}	$(1),(1),(1),(1),(67),(57),(56)$
Q_{68}	$(1),(1),(1),(12),(12),(12),(12)$
Q_{69}	$(1),(1),(1),(12),(12),(12),(12)(45)$
Q_{70}	$(1),(1),(1),(12),(12),(12),(45)$
Q_{71}	$(1),(1),(1),(12),(12),(12),(465)$
Q_{72}	$(1),(1),(1),(12),(12),(12),(12)(465)$
Q_{73}	$(1),(1),(1),(12),(12),(12)(45),(12)(45)$
Q_{74}	$(1),(1),(1),(12),(12),(12)(45),(45)$
Q_{75}	$(1),(1),(1),(12),(12),(45),(45)$
Q_{76}	$(1),(1),(1),(12),(12)(67),(12)(57),(12)(56)$
Q_{77}	$(1),(1),(1),(12),(67),(57),(56)$
Q_{78}	$(1),(1),(1),(132),(132),(132),(132)$
Q_{79}	$(1),(1),(1),(132),(132),(132),(123)$
Q_{80}	$(1),(1),(1),(132),(132),(132),(45)$
Q_{81}	$(1),(1),(1),(132),(132),(132),(123)(45)$
Q_{82}	$(1),(1),(1),(132),(132),(132),(132)(45)$
Q_{83}	$(1),(1),(1),(132),(132),(132),(132)(465)$
Q_{84}	$(1),(1),(1),(132),(132),(132),(123)(465)$

Table 11: Quandles of order 7 as disjoint cycles of columns: 85–112 of 298

Quandle X	Disjoint Cycle Notation
Q_{85}	$(1),(1),(1),(132),(132),(132),(465)$
Q_{86}	$(1),(1),(1),(132),(132),(123),(123)$
Q_{87}	$(1),(1),(1),(132),(132),(123),(45)$
Q_{88}	$(1),(1),(1),(132),(132),(123),(123)(45)$
Q_{89}	$(1),(1),(1),(132),(132),(123),(132)(45)$
Q_{90}	$(1),(1),(1),(132),(132),(45),(45)$
Q_{91}	$(1),(1),(1),(132),(132),(45),(123)(45)$
Q_{92}	$(1),(1),(1),(132),(132),(45),(132)(45)$
Q_{93}	$(1),(1),(1),(132),(132),(123)(45),(123)(45)$
Q_{94}	$(1),(1),(1),(132),(132),(123)(45),(132)(45)$
Q_{95}	$(1),(1),(1),(132),(132),(132)(45),(132)(45)$
Q_{96}	$(1),(1),(1),(132),(123)(67),(123)(57),(123)(56)$
Q_{97}	$(1),(1),(1),(132),(132)(67),(132)(57),(132)(56)$
Q_{98}	$(1),(1),(1),(132),(67),(57),(56)$
Q_{99}	$(1),(1),(1),(12)(56),(12)(47),(12)(47),(12)(56)$
Q_{100}	$(1),(1),(1),(12)(56),(47),(47),(12)(56)$
Q_{101}	$(1),(1),(1),(56),(132)(47),(132)(47),(56)$
Q_{102}	$(1),(1),(1),(56),(47),(47),(56)$
Q_{103}	$(1),(1),(1),(123)(56),(132)(47),(132)(47),(123)(56)$
Q_{104}	$(1),(1),(1),(123)(56),(123)(47),(123)(47),(123)(56)$
Q_{105}	$(1),(1),(1),(132)(576),(132)(467),(132)(475),(132)(456)$
Q_{106}	$(1),(1),(1),(23)(576),(23)(467),(23)(475),(23)(456)$
Q_{107}	$(1),(1),(1),(576),(467),(475),(456)$
Q_{108}	$(1),(1),(12),(12),(12),(12),(12)$
Q_{109}	$(1),(1),(12),(12),(12),(12),(12)(34)$
Q_{110}	$(1),(1),(12),(12),(12),(12),(34)$
Q_{111}	$(1),(1),(12),(12),(12),(12),(345)$
Q_{112}	$(1),(1),(12),(12),(12),(12),(12)(345)$

Table 12: Quandles of order 7 as disjoint cycles of columns: 113–140 of 298

Quandle X	Disjoint Cycle Notation
Q_{113}	$(1), (1), (12), (12), (12), (12), (12)(3465)$
Q_{114}	$(1), (1), (12), (12), (12), (12), (3465)$
Q_{115}	$(1), (1), (12), (12), (12), (12), (34)(56)$
Q_{116}	$(1), (1), (12), (12), (12), (12), (12)(34)(56)$
Q_{117}	$(1), (1), (12), (12), (12), (12)(34), (12)(34)$
Q_{118}	$(1), (1), (12), (12), (12), (12)(34), (34)$
Q_{119}	$(1), (1), (12), (12), (12), (34), (34)$
Q_{120}	$(1), (1), (12), (12), (12), (345), (345)$
Q_{121}	$(1), (1), (12), (12), (12), (345), (12)(345)$
Q_{122}	$(1), (1), (12), (12), (12), (345), (354)$
Q_{123}	$(1), (1), (12), (12), (12), (345), (12)(354)$
Q_{124}	$(1), (1), (12), (12), (12), (12)(345), (12)(345)$
Q_{125}	$(1), (1), (12), (12), (12), (12)(345), (12)(354)$
Q_{126}	$(1), (1), (12), (12), (12)(34), (12)(34), (12)(34)$
Q_{127}	$(1), (1), (12), (12), (12)(34), (12)(34), (34)$
Q_{128}	$(1), (1), (12), (12), (12)(34), (12)(34), (34)(56)$
Q_{129}	$(1), (1), (12), (12), (12)(34), (12)(34), (12)(34)(56)$
Q_{130}	$(1), (1), (12), (12), (12)(34), (12)(34), (12)(56)$
Q_{131}	$(1), (1), (12), (12), (12)(34), (12)(34), (56)$
Q_{132}	$(1), (1), (12), (12), (12)(34), (34), (34)$
Q_{133}	$(1), (1), (12), (12), (34), (34), (34)$
Q_{134}	$(1), (1), (12), (12), (34), (34), (34)(56)$
Q_{135}	$(1), (1), (12), (12), (34), (34), (12)(34)(56)$
Q_{136}	$(1), (1), (12), (12), (34), (34), (12)(56)$
Q_{137}	$(1), (1), (12), (12), (34), (34), (56)$
Q_{138}	$(1), (1), (12), (12), (34)(67), (34)(57), (34)(56)$
Q_{139}	$(1), (1), (12), (12), (12)(34)(67), (12)(34)(57), (12)(34)(56)$
Q_{140}	$(1), (1), (12), (12), (12)(67), (12)(57), (12)(56)$

Table 13: Quandles of order 7 as disjoint cycles of columns: 141–168 of 298

Quandle X	Disjoint Cycle Notation
Q_{141}	$(1),(1),(12),(12),(67),(57),(56)$
Q_{142}	$(1),(1),(12),(12)(56),(12)(47),(12)(47),(12)(56)$
Q_{143}	$(1),(1),(12),(12)(56),(47),(47),(12)(56)$
Q_{144}	$(1),(1),(12),(56),(47),(47),(56)$
Q_{145}	$(1),(1),(12),(576),(467),(475),(456)$
Q_{146}	$(1),(1),(12),(12)(576),(12)(467),(12)(475),(12)(456)$
Q_{147}	$(1),(1),(12)(45),(12)(36),(12)(36),(12)(45),(12)(45)$
Q_{148}	$(1),(1),(12)(45),(12)(36),(12)(36),(12)(45),(45)$
Q_{149}	$(1),(1),(12)(45),(12)(36),(12)(36),(12)(45),(36)(45)$
Q_{150}	$(1),(1),(12)(45),(12)(36),(12)(36),(12)(45),(12)(36)(45)$
Q_{151}	$(1),(1),(12)(45),(36),(36),(12)(45),(12)(45)$
Q_{152}	$(1),(1),(12)(45),(36),(36),(12)(45),(45)$
Q_{153}	$(1),(1),(12)(45),(36),(36),(12)(45),(12)(36)$
Q_{154}	$(1),(1),(12)(45),(36),(36),(12)(45),(36)$
Q_{155}	$(1),(1),(12)(45),(36),(36),(12)(45),(36)(45)$
Q_{156}	$(1),(1),(12)(45),(36),(36),(12)(45),(12)(36)(45)$
Q_{157}	$(1),(1),(12)(45),(376),(376),(12)(45),(12)(45)$
Q_{158}	$(1),(1),(12)(45),(12)(376),(12)(376),(12)(45),(12)(45)$
Q_{159}	$(1),(1),(12)(45),(67),(67),(45),(45)$
Q_{160}	$(1),(1),(45),(36),(36),(45),(45)$
Q_{161}	$(1),(1),(45),(36),(36),(45),(36)(45)$
Q_{162}	$(1),(1),(45),(36),(36),(45),(12)(36)(45)$
Q_{163}	$(1),(1),(45),(376),(376),(45),(45)$
Q_{164}	$(1),(1),(45),(12)(376),(12)(376),(45),(45)$
Q_{165}	$(1),(1),(456),(37)(56),(37)(46),(37)(45),(465)$
Q_{166}	$(1),(1),(456),(12)(37)(56),(12)(37)(46),(12)(37)(45),(465)$
Q_{167}	$(1),(1),(12)(456),(37)(56),(37)(46),(37)(45),(12)(465)$
Q_{168}	$(1),(1),(12)(456),(12)(37)(56),(12)(37)(46),(12)(37)(45),(12)(465)$

Table 14: Quandles of order 7 as disjoint cycles of columns: 169–196 of 298

Quandle X	Disjoint Cycle Notation
Q_{169}	$(1),(1),(12)(4576),(12)(3567),(12)(3746),(12)(3475),(12)(3654)$
Q_{170}	$(1),(1),(12)(4576),(12)(3756),(12)(3674),(12)(3547),(12)(3465)$
Q_{171}	$(1),(1),(4576),(3567),(3746),(3475),(3654)$
Q_{172}	$(1),(1),(4576),(3756),(3674),(3547),(3465)$
Q_{173}	$(1),(1),(45)(67),(37)(56),(36)(47),(34)(57),(35)(46)$
Q_{174}	$(1),(1),(12)(45)(67),(12)(37)(56),(12)(36)(47),(12)(34)(57),(12)(35)(46)$
Q_{175}	$(27),(56),(27),(1),(13),(13),(56)$
Q_{176}	$(27),(56),(27),(27),(14),(14),(56)$
Q_{177}	$(27),(16),(27),(16),(1),(27),(16)$
Q_{178}	$(27),(16),(27),(27),(1),(27),(16)$
Q_{179}	$(27),(16),(27),(27),(34),(27),(16)$
Q_{180}	$(27),(16),(27),(27),(16),(27),(16)$
Q_{181}	$(27),(16),(27),(27),(27),(27),(16)$
Q_{182}	$(27),(17),(56),(1),(36),(35),(12)$
Q_{183}	$(27),(17),(56),(56),(34),(34),(12)$
Q_{184}	$(27)(46),(35),(27),(35),(27),(35),(35)$
Q_{185}	$(27)(46),(15),(1),(15),(27)(46),(15),(15)$
Q_{186}	$(27)(46),(15),(15),(15),(27)(46),(15),(15)$
Q_{187}	$(27)(46),(15),(27),(15),(27)(46),(15),(15)$
Q_{188}	$(27)(46),(15),(27)(46),(15),(27)(46),(15),(15)$
Q_{189}	$(27)(46),(15),(26)(47),(15),(27)(46),(15),(15)$
Q_{190}	$(27)(46),(13)(46),(27)(46),(13)(27),(1),(13)(27),(13)(46)$
Q_{191}	$(27)(46),(13)(46),(27)(46),(13)(27),(27),(13)(27),(13)(46)$
Q_{192}	$(27)(46),(46),(46),(35),(46),(35),(46)$
Q_{193}	$(27)(46),(46),(27),(35),(27),(35),(46)$
Q_{194}	$(27)(46),(46),(27),(27),(1),(27),(46)$
Q_{195}	$(27)(46),(46),(27),(27),(46),(27),(46)$
Q_{196}	$(27)(46),(46),(27),(27),(27),(27),(46)$

Table 15: Quandles of order 7 as disjoint cycles of columns: 197–224 of 298

Quandle X	Disjoint Cycle Notation
Q_{197}	$(27)(46), (46), (27)(46), (13), (1), (13), (46)$
Q_{198}	$(27)(46), (46), (27)(46), (13), (13), (13), (46)$
Q_{199}	$(27)(46), (46), (27)(46), (13), (46), (13), (46)$
Q_{200}	$(27)(46), (46), (27)(46), (13), (27), (13), (46)$
Q_{201}	$(27)(46), (46), (27)(46), (15), (27)(46), (15), (46)$
Q_{202}	$(27)(46), (46), (27)(46), (27), (1), (27), (46)$
Q_{203}	$(27)(46), (46), (27)(46), (27), (13), (27), (46)$
Q_{204}	$(27)(46), (46), (27)(46), (27), (27), (27), (46)$
Q_{205}	$(27)(46), (46), (27)(46), (27), (27)(46), (27), (46)$
Q_{206}	$(27)(46), (46), (27)(46), (13)(27), (1), (13)(27), (46)$
Q_{207}	$(27)(46), (46), (27)(46), (13)(27), (13), (13)(27), (46)$
Q_{208}	$(27)(46), (46), (27)(46), (13)(27), (46), (13)(27), (46)$
Q_{209}	$(27)(46), (46), (27)(46), (13)(27), (27), (13)(27), (46)$
Q_{210}	$(27)(46), (35)(46), (46), (35), (46), (35), (35)(46)$
Q_{211}	$(27)(46), (35)(46), (46), (27), (46), (27), (35)(46)$
Q_{212}	$(27)(46), (35)(46), (27), (35), (27), (35), (35)(46)$
Q_{213}	$(27)(46), (35)(46), (27), (27), (27), (27), (35)(46)$
Q_{214}	$(27)(46), (35)(46), (27), (27)(35), (27), (27)(35), (35)(46)$
Q_{215}	$(27)(46), (35)(46), (27)(46), (35), (27)(46), (35), (35)(46)$
Q_{216}	$(27)(46), (35)(46), (27)(46), (27), (27)(46), (27), (35)(46)$
Q_{217}	$(27)(46), (35)(46), (27)(46), (27)(35), (27)(46), (27)(35), (35)(46)$
Q_{218}	$(27)(46), (15)(47), (1), (15)(26), (24)(67), (15)(47), (15)(26)$
Q_{219}	$(27)(46), (15)(47), (26)(47), (15)(26), (24)(67), (15)(47), (15)(26)$
Q_{220}	$(27)(46), (17)(36), (27)(46), (17)(36), (1), (12)(34), (12)(34)$
Q_{221}	$(27)(46), (17)(36), (26)(47), (16)(37), (1), (14)(23), (12)(34)$
Q_{222}	$(27)(46), (17)(46), (46), (35), (46), (35), (12)(46)$
Q_{223}	$(27)(35)(46), (46), (27), (35), (27), (35), (46)$
Q_{224}	$(27)(35)(46), (46), (27), (27), (27), (27), (46)$

Table 16: Quandles of order 7 as disjoint cycles of columns: 225–251 of 298

Quandle X	Disjoint Cycle Notation
Q_{225}	$(27)(35)(46), (35)(46), (46), (35), (46), (35), (35)(46)$
Q_{226}	$(27)(35)(46), (35)(46), (27), (35), (27), (35), (35)(46)$
Q_{227}	$(27)(35)(46), (35)(46), (27), (27), (27), (27), (35)(46)$
Q_{228}	$(27)(35)(46), (35)(46), (27)(46), (27), (27)(46), (27), (35)(46)$
Q_{229}	$(27)(35)(46), (35)(46), (27)(46), (27)(35), (27)(46), (27)(35), (35)(46)$
Q_{230}	$(27)(35)(46), (34)(56), (27), (27), (27), (27), (34)(56)$
Q_{231}	$(27)(35)(46), (34)(56), (26)(47), (25)(37), (26)(47), (25)(37), (34)(56)$
Q_{232}	$(27)(35)(46), (16)(34)(57), (12)(45)(67), (15)(26)(37),$ $(17)(24)(36), (13)(25)(47), (14)(23)(56)$
Q_{233}	$(27)(35)(46), (17)(35)(46), (46), (35), (46), (35), (12)(35)(46)$
Q_{234}	$(267)(345), (13)(45), (154)(267), (135)(267), (143)(267), (14)(35), (15)(34)$
Q_{235}	$(267)(345), (345), (276), (276), (276), (345), (345)$
Q_{236}	$(267)(345), (345), (267), (267), (267), (345), (345)$
Q_{237}	$(267)(345), (164)(357), (147)(256), (152)(376), (136)(274), (175)(243), (123)(465)$
Q_{238}	$(267)(345), (137)(465), (125)(476), (163)(275), (174)(236), (142)(357), (156)(243)$
Q_{239}	$(245367), (173564), (152746), (126375), (147623), (134257), (165432)$
Q_{240}	$(245367), (175634), (157426), (123765), (146273), (132547), (164352)$
Q_{241}	$(25367), (14), (14), (25367), (14), (14), (14)$
Q_{242}	$(25367), (14)(36)(57), (14)(27)(56), (27635), (14)(23)(67), (14)(25)(37), (14)(26)(35)$
Q_{243}	$(267)(35), (14), (14), (267)(35), (14), (14), (14)$
Q_{244}	$(267)(35), (14), (14)(276), (267)(35), (14)(276), (14), (14)$
Q_{245}	$(267)(35), (14), (276), (267)(35), (276), (14), (14)$
Q_{246}	$(267)(35), (14), (267), (267)(35), (267), (14), (14)$
Q_{247}	$(267)(35), (14), (14)(267), (267)(35), (14)(267), (14), (14)$
Q_{248}	$(267)(35), (35), (14), (267)(35), (14), (35), (35)$
Q_{249}	$(267)(35), (35), (276), (1), (276), (35), (35)$
Q_{250}	$(267)(35), (35), (276), (35), (276), (35), (35)$
Q_{251}	$(267)(35), (35), (276), (276), (276), (35), (35)$

Table 17: Quandles of order 7 as disjoint cycles of columns: 252–279 of 298

Quandle X	Disjoint Cycle Notation
Q_{252}	$(267)(35), (35), (276), (276)(35), (276), (35), (35)$
Q_{253}	$(267)(35), (35), (276), (267), (276), (35), (35)$
Q_{254}	$(267)(35), (35), (276), (267)(35), (276), (35), (35)$
Q_{255}	$(267)(35), (35), (267), (1), (267), (35), (35)$
Q_{256}	$(267)(35), (35), (267), (35), (267), (35), (35)$
Q_{257}	$(267)(35), (35), (267), (276), (267), (35), (35)$
Q_{258}	$(267)(35), (35), (267), (267), (267), (35), (35)$
Q_{259}	$(267)(35), (35), (267), (267)(35), (267), (35), (35)$
Q_{260}	$(267)(35), (14)(35), (14), (267)(35), (14), (14)(35), (14)(35)$
Q_{261}	$(267)(35), (14)(35), (14)(276), (267)(35), (14)(276), (14)(35), (14)(35)$
Q_{262}	$(267)(35), (14)(35), (276), (267)(35), (276), (14)(35), (14)(35)$
Q_{263}	$(267)(35), (14)(35), (267), (267)(35), (267), (14)(35), (14)(35)$
Q_{264}	$(267)(35), (14)(35), (14)(267), (267)(35), (14)(267), (14)(35), (14)(35)$
Q_{265}	$(2567), (13), (2567), (1), (13), (13), (13)$
Q_{266}	$(2567), (13), (2567), (13), (13), (13), (13)$
Q_{267}	$(2567), (13), (2567), (13)(2765), (13), (13), (13)$
Q_{268}	$(2567), (13), (2567), (2765), (13), (13), (13)$
Q_{269}	$(2567), (13), (2567), (2567), (13), (13), (13)$
Q_{270}	$(2567), (13), (2567), (13)(2567), (13), (13), (13)$
Q_{271}	$(2567), (13), (2567), (13)(26)(57), (13), (13), (13)$
Q_{272}	$(2567), (13), (2567), (26)(57), (13), (13), (13)$
Q_{273}	$(2567), (143), (2567), (2567), (143), (143), (143)$
Q_{274}	$(2567), (34), (26)(57), (26)(57), (34), (34), (34)$
Q_{275}	$(2567), (1735), (2765), (1), (1236), (1537), (1632)$
Q_{276}	$(2567), (13)(57), (2765), (1), (13)(26), (13)(57), (13)(26)$
Q_{277}	$(2567), (13)(57), (2765), (26)(57), (13)(26), (13)(57), (13)(26)$
Q_{278}	$(2567)(34), (143), (14)(2567), (13)(2567), (134), (143), (134)$
Q_{279}	$(2567)(34), (34), (26)(57), (26)(57), (34), (34), (34)$

Table 18: Quandles of order 7 as disjoint cycles of columns: 280–298

Quandle X	Disjoint Cycle Notation
Q_{280}	$(267), (13), (267), (1), (13), (13), (13)$
Q_{281}	$(267), (13), (267), (1), (276), (13), (13)$
Q_{282}	$(267), (13), (267), (1), (267), (13), (13)$
Q_{283}	$(267), (13), (267), (13), (13), (13), (13)$
Q_{284}	$(267), (13), (267), (13), (276), (13), (13)$
Q_{285}	$(267), (13), (267), (13), (267), (13), (13)$
Q_{286}	$(267), (13), (267), (276), (276), (13), (13)$
Q_{287}	$(267), (13), (267), (276), (267), (13), (13)$
Q_{288}	$(267), (13), (267), (267), (267), (13), (13)$
Q_{289}	$(267), (143), (267), (267), (1), (143), (143)$
Q_{290}	$(267), (143), (267), (267), (143), (143), (143)$
Q_{291}	$(267), (143), (267), (267), (134), (143), (143)$
Q_{292}	$(267), (143), (267), (267), (134)(276), (143), (143)$
Q_{293}	$(267), (34), (15), (15), (267), (34), (34)$
Q_{294}	$(267), (13)(45), (267), (13), (13), (13)(45), (13)(45)$
Q_{295}	$(267), (13)(45), (267), (276), (276), (13)(45), (13)(45)$
Q_{296}	$(267), (13)(45), (267), (267), (267), (13)(45), (13)(45)$
Q_{297}	$(267), (176), (45), (35), (34), (127), (162)$
Q_{298}	$(267), (13)(45)(67), (276), (276), (267), (13)(27)(45), (13)(26)(45)$

Table 19: Quandles of order 7 with their automorphism groups 1–56 of 298

Quandle X	Inn(X)	Aut(X)	Quandle X	Inn(X)	Aut(X)
Q_1	$\{1\}$	Σ_7	Q_{29}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_2	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \Sigma_4$	Q_{30}	$\mathbb{Z}_6$	D_6
Q_3	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times \Sigma_3$	Q_{31}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$
Q_4	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{32}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_5	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{33}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_6	$\mathbb{Z}_5$	$\mathbb{Z}_5$	Q_{34}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_7	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{35}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_8	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{36}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_9	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times \Sigma_3$	Q_{37}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{10}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{38}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{11}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \Sigma_4$	Q_{39}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4
Q_{12}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	Q_{40}	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times \Sigma_3$
Q_{13}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{41}	$\mathbb{Z}_3$	$\mathbb{Z}_6$
Q_{14}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4	Q_{42}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{15}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{43}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{16}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{44}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{17}	$\mathbb{Z}_3$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{45}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \Sigma_3$
Q_{18}	$\mathbb{Z}_3$	D_6	Q_{46}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{19}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{47}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{20}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{48}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{21}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{49}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{22}	$\mathbb{Z}_4$	D_4	Q_{50}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{23}	$\mathbb{Z}_4$	$\mathbb{Z}_4$	Q_{51}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{24}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{52}	$\mathbb{Z}_4$	D_4
Q_{25}	$\mathbb{Z}_5$	$\mathbb{Z}_{10}$	Q_{53}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{26}	D_3	$D_3 \times \mathbb{Z}_2$	Q_{54}	$\mathbb{Z}_2$	$D_4 \times \Sigma_3$
Q_{27}	$\mathbb{Z}_5$	$\mathbb{Z}_5$	Q_{55}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{28}	$\mathbb{Z}_5$	D_5	Q_{56}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Table 20: Quandles of order 7 with their automorphism groups 57–112 of 298

Quandle X	Inn(X)	Aut(X)	Quandle X	Inn(X)	Aut(X)
Q_{57}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{85}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3$
Q_{58}	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{86}	$\mathbb{Z}_3$	$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
Q_{59}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{87}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{60}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{88}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{61}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Σ_4	Q_{89}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{62}	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_4 \times \Sigma_3$	Q_{90}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{63}	$\mathbb{Z}_3 \times \Sigma_3$	$\mathbb{Z}_3 \times \Sigma_3$	Q_{91}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{64}	Σ_3	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	Q_{92}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{65}	Σ_3	Σ_3	Q_{93}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{66}	Σ_3	$D_4 \times \Sigma_3$	Q_{94}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{67}	Σ_3	$\Sigma_3 \times \Sigma_4$	Q_{95}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{68}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \Sigma_4$	Q_{96}	$\mathbb{Z}_3 \times \Sigma_3$	$\mathbb{Z}_3 \times \Sigma_3$
Q_{69}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{97}	$\mathbb{Z}_3 \times \Sigma_3$	$\mathbb{Z}_3 \times \Sigma_3$
Q_{70}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{98}	$\mathbb{Z}_3 \times \Sigma_3$	$\mathbb{Z}_3 \times \Sigma_3$
Q_{71}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{99}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{72}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{100}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{73}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{101}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{74}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{102}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \Sigma_3$
Q_{75}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{103}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
Q_{76}	D_6	D_6	Q_{104}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_3 \times D_4$
Q_{77}	D_6	D_6	Q_{105}	A_4	$\mathbb{Z}_3 \times A_4$
Q_{78}	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times \Sigma_4$	Q_{106}	$A_4 \times \mathbb{Z}_2$	$A_4 \times \mathbb{Z}_2$
Q_{79}	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times \Sigma_3$	Q_{107}	A_4	$A_4 \times \Sigma_3$
Q_{80}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{108}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \Sigma_5$
Q_{81}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{109}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{82}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{110}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{83}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	Q_{111}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{84}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	Q_{112}	$\mathbb{Z}_6$	$\mathbb{Z}_6$

Table 21: Quandles of order 7 with their automorphism groups 113–168 of 298

Quandle X	Inn(X)	Aut(X)	Quandle X	Inn(X)	Aut(X)
Q_{113}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{141}	$\mathbb{Z}_2 \times \Sigma_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$
Q_{114}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Q_{142}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{115}	$\mathbb{Z}_2 \times Z_2$	$\mathbb{Z}_2 \times D_4$	Q_{143}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{116}	$\mathbb{Z}_2 \times Z_2$	$\mathbb{Z}_2 \times D_4$	Q_{144}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{117}	$\mathbb{Z}_2 \times Z_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{145}	$A_4 \times \mathbb{Z}_2$	$A_4 \times \mathbb{Z}_2$
Q_{118}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{146}	$A_4 \times \mathbb{Z}_2$	$A_4 \times \mathbb{Z}_2$
Q_{119}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$Z_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{147}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{120}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{148}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{121}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{149}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{122}	$\mathbb{Z}_6$	D_6	Q_{150}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{123}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{151}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{124}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{152}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{125}	$\mathbb{Z}_6$	D_6	Q_{153}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{126}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	Q_{154}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{127}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{155}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{128}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{156}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{129}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{157}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{130}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{158}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{131}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{159}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{132}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{160}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{133}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	Q_{161}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{134}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{162}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{135}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{163}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{136}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{164}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{137}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{165}	D_3	D_6
Q_{138}	$\mathbb{Z}_2 \times \Sigma_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	Q_{166}	D_3	D_6
Q_{139}	$\mathbb{Z}_2 \times \Sigma_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	Q_{167}	D_6	D_6
Q_{140}	$\mathbb{Z}_2 \times \Sigma_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	Q_{168}	D_6	D_6

Table 22: Quandles of order 7 with their automorphism groups 169–224 of 298

Quandle X	Inn(X)	Aut(X)	Quandle X	Inn(X)	Aut(X)
Q_{169}	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$	Q_{197}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{170}	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$	Q_{198}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{171}	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$	Q_{199}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{172}	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$	Q_{200}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{173}	D_5	$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$	Q_{201}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{174}	D_5	$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$	Q_{202}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{175}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$Z_2 \times A_4$	Q_{203}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{176}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{204}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{177}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4	Q_{205}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_4 \times \Sigma_3$
Q_{178}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{206}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{179}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{207}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{180}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{208}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{181}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	Q_{209}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{182}	$\Sigma_3 \times \Sigma_3$	$(\Sigma_3 \times \Sigma_3) \rtimes \mathbb{Z}_2$	Q_{210}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{183}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	$D_4 \times \Sigma_3$	Q_{211}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{184}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{212}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{185}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{213}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{186}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{214}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{187}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{215}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{188}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{216}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
Q_{189}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{217}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$
Q_{190}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \Sigma_4$	Q_{218}	D_4	D_4
Q_{191}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{219}	D_4	D_4
Q_{192}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{220}	Σ_3	D_6
Q_{193}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times Z_2$	Q_{221}	Σ_4	Σ_4
Q_{194}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{222}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$
Q_{195}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	D_4	Q_{223}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times A_4$
Q_{196}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{224}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Table 23: Quandles of order 7 with their automorphism groups 225–280 of 298

Quandle X	Inn(X)	Aut(X)	Quandle X	Inn(X)	Aut(X)
Q_{225}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{253}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{226}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{254}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{227}	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	Q_{255}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{228}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{256}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{229}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \Sigma_4$	Q_{257}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{230}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Q_{258}	$\mathbb{Z}_6$	$\mathbb{Z}_6$
Q_{231}	D_6	D_6	Q_{259}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{232}	D_7	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Q_{260}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{233}	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Sigma_3$	$D_4 \times \Sigma_3$	Q_{261}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
Q_{234}	A_4	A_4	Q_{262}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{235}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	Q_{263}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{236}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \Sigma_3$	Q_{264}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_3 \times D_4$
Q_{237}	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Q_{265}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{238}	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Q_{266}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{239}	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Q_{267}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{240}	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	Q_{268}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{241}	$\mathbb{Z}_{10}$	$\mathbb{Z}_{10}$	Q_{269}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{242}	D_5	D_5	Q_{270}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{243}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{271}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{244}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{272}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{245}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{273}	$\mathbb{Z}_{12}$	Z_{12}
Q_{246}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{274}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{247}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{275}	Σ_4	Σ_4
Q_{248}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{276}	D_4	D_4
Q_{249}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{277}	D_4	D_4
Q_{250}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{278}	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$
Q_{251}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{279}	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
Q_{252}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{280}	$\mathbb{Z}_6$	$\mathbb{Z}_6$

Table 24: Quandles of order 7 with their automorphism groups 281–298

Quandle X	Inn(X)	Aut(X)	Quandle X	Inn(X)	Aut(X)
Q_{281}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{290}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3$
Q_{282}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{291}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3$
Q_{283}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \ Z_2$	Q_{292}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \Sigma_3$
Q_{284}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{293}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{285}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{294}	$\mathbb{Z}_6 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_2$
Q_{286}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \ Z_2$	Q_{295}	$\mathbb{Z}_6$	$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
Q_{287}	$\mathbb{Z}_6$	$\mathbb{Z}_6$	Q_{296}	$\mathbb{Z}_6$	$\mathbb{Z}_3 \times \ D_4$
Q_{288}	$\mathbb{Z}_6$	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Q_{297}	$A_4 \times \Sigma_3$	$A_4 \times \Sigma_3$
Q_{289}	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \Sigma_3$	Q_{298}	Σ_3	D_6

Chapter 3

Conclusion

After reading Ho and Nelsen's *Matrices and finite quandles*, [13], we thought to continue where they left off. In 2006, Ricardo Restrepo wrote a program in *matlab* which generated all quandles up to eight elements. Then in 2010, Restrepo rewrote the program in C which generates all quandles up to nine elements. A partial description of this algorithm is in [8]. With the help of Edwin Clark, we wrote a program using the software *Maple* to find as much information about the automorphism and inner automorphism as possible.

Throughout the process of finding these groups we were able to identify certain patterns. Specifically, we saw that for the dihedral quandle, R_n, of order n the inner automorphism group is isomorphic to the dihedral group $D_{\frac{m}{2}}$ of order m, where $m = lcm(n, 2)$. Again looking at the dihedral quandles, we saw that the cardinality of the automorphism group of R_n is $n\phi(n)$. These observations led to the first two theorems of Chapter 2.

After the automorphism groups and the inner automorphism groups of the dihedral quandles of orders 6 and 7 were computed, a discussion between Elhamdadi and Professor Hou occurred. As a result of that discussion Professor Hou's attention was drawn to the automorphism group of Alexander quandles. Hou computes and gives the structure of the automorphism group of Alexander quandles in [14].

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