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Problems in Classical Potential Theory with Applications to Mathematical Physics

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Problems in Classical Potential Theory
with Applications to
Mathematical Physics

by

Erik Lundberg

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
College of Arts and Sciences
University of South Florida

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References
Abstract

In this thesis we are interested in some problems regarding harmonic functions. The topics are divided into three chapters.

Chapter 2 concerns singularities developed by solutions of the Cauchy problem for a holomorphic elliptic equation, especially Laplace’s equation. The principal motivation is to locate the singularities of the Schwarz potential. The results have direct applications to Laplacian growth (or the Hele-Shaw problem).

Chapter 3 concerns the Dirichlet problem when the boundary is an algebraic set and the data is a polynomial or a real-analytic function. We pursue some questions related to the Khavinson-Shapiro conjecture. A main topic of interest is analytic continuability of the solution outside its natural domain.

Chapter 4 concerns certain complex-valued harmonic functions and their zeros. The special cases we consider apply directly in astrophysics to the study of multiple-image gravitational lenses.
This thesis investigates a few problems in potential theory and complex analysis. The questions span Cauchy and Dirichlet problems for Laplace’s equation, real and complex-valued harmonic functions, singularities and zeros, and we consider both the two-dimensional case and arbitrary dimensions. We are not so much interested in pathologies or in seeking generality for its own sake. Rather the goal has been to gain some insight on a few problems that are physically motivated and simple to state. We divide the topics into three chapters.

**Chapter 2:** concerns singularities of Cauchy’s problem for the Laplace equation and is based on the paper [78], which has been submitted for publication.

**Chapter 3:** concerns singularities and algebraicity of Dirichlet problems and consists of the papers [57], [77], and [79].

**Chapter 4:** concerns complex-valued harmonic functions arising in models of gravitational lensing and consists of the papers [71] and [58].

Although the topics between chapters are somewhat separated, a common thread is an interaction between algebraic geometry and harmonic functions; in two-dimensional instances of such situations, the Schwarz function can often play a role. Therefore, before giving a more detailed overview of each chapter, let us have a brief glimpse of the range of topics by defining the Schwarz function and seeing how it can arise in each context.

Suppose \( \Gamma \) is a real-analytic curve in the plane. Then P. Davis [24] has defined the *Schwarz function* of \( \Gamma \) to be the unique function, complex-analytic in a neighborhood of \( \Gamma \), that coincides with \( \bar{z} \) on \( \Gamma \), where \( \bar{z} \) denotes the complex conjugate of
z. If, for instance, Γ is given algebraically as the zero set of a polynomial $P(x, y)$, we can obtain $S(z)$ by making the complex-linear change of variables $z = x + iy$, $\bar{z} = x - iy$, and then solving for $\bar{z}$ in the equation $P\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = 0$. Let us consider a few examples.

**Example 1:** Suppose Γ is the curve given algebraically by solution set of $(x^2 + y^2)^2 = a^2(x^2 + y^2) + 4\varepsilon^2 x^2$ (“C. Neumann’s oval”). Then for appropriate values of $a$ and $\varepsilon$, Γ is a single closed, bounded curve. Changing variables we have $(z\bar{z})^2 = a^2(z\bar{z}) + \varepsilon^2(z + \bar{z})^2$. Solving for $\bar{z}$ gives $S(z) = \frac{z(a^2 + 2\varepsilon^2) + z\sqrt{4a^4 + 4\varepsilon^4x^2 + 4\varepsilon^2(2z^2)}}{2(z^2 - \varepsilon^2)}$. $S(z)$ has two simple poles in the interior of Γ and a square root branch cut in the exterior.

**Example 2:** Suppose Γ is an ellipse given by the solution set of the equation $x^2/a^2 + y^2/b^2 = 1$. Then changing variables we have $(z\bar{z})^2 - (z - \bar{z})^2 = 4$. Solving for $\bar{z}$ gives $S(z) = \frac{a^2 + b^2}{a^2 - b^2}z + \frac{2ab}{b^2 - a^2}\sqrt{z^2 + b^2 - a^2}$. Thus $S(z)$ has a square root branch cut along the segment joining the foci $\pm\sqrt{a^2 - b^2}$.

**Example 3:** Suppose Γ is the image of the unit circle under the conformal map $f(\zeta) = a\zeta + b\zeta^2$, with $a$ and $b$ both real, $a > 2b > 0$. Then Γ is a single closed, bounded curve. In $\zeta$-plane coordinates, the Schwarz function is simply $f(\frac{1}{\zeta}) = \frac{a}{\zeta} + \frac{b}{\zeta^2}$. Indeed the condition $S(z)|_{\Gamma} = \bar{z}$ can be pulled back: $S(f(\zeta))|_{|\zeta|=1} = \frac{1}{f(\zeta)}$, and on the unit circle $f(\zeta)$ coincides with $f(\frac{1}{\zeta})$ since $a$ and $b$ are real.

In Chapter 2 we discuss a principle that reduces the Laplacian growth or Hele-Shaw moving boundary problem to a simple dynamical description of the singularities of the Schwarz function. This allows the generation of explicit exact solutions of Laplacian growth. For instance, based on the calculations above, one can select one-parameter families of each of the Examples 1 - 3 to obtain exact solutions. In Example 1, fixing $\varepsilon$ and letting $a$ decrease, the Schwarz function has two simple poles inside Γ with fixed positions and decreasing residues. As we will see, this allows us to interpret this one parameter family as the moving boundary of a shrinking domain of oil surrounded by water with suction occurring at each of the “sinks” positioned at $z = \pm \varepsilon$.

The main goal of Chapter 2 is to study higher-dimensional Laplacian growth.
in terms of a generalization of the Schwarz function.

In Chapter 3 we are interested, in particular, in the classical Dirichlet problem posed on an algebraic curve with polynomial data, and the question is if and where the solution develops singularities. If we multiply the Schwarz function in Example 1 by \((z^2 - \varepsilon^2)\), then we obtain a function \(f(z) = (z^2 - \varepsilon^2)S(z)\) analytic in the interior of \(\Gamma\) and coinciding on \(\Gamma\) with \(p(z, \bar{z}) = (z^2 - \varepsilon^2)\bar{z}\), a polynomial. Thus the solution to the Dirichlet problem with polynomial data \(p\) develops singularities at the branch cuts of \(S(z)\). (If we want real-valued data and solution, then we can take the real part of \(f\) and \(p\).)

This gives an illustrative example, but in Chapter 3 we will be considering Dirichlet’s problem posed on classes of curves and surfaces for which this trick is unavailable.

In Chapter 4, in order to calculate the gravitational lensing effect of a massive object, a basic problem that arises is to calculate the Cauchy transform of a two-dimensional domain \(\Omega\).

\[
C_\Omega = \frac{1}{\pi} \int_\Omega \frac{dA(z)}{\zeta - z}
\]

Suppose \(\zeta \in \Omega^c\) and apply Green’s Theorem:

\[
\frac{1}{\pi} \int_\Omega \frac{dA(z)}{\zeta - z} = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{\bar{z}dz}{\zeta - z}
\]

If the boundary of \(\Omega\) has a Schwarz function, then we can replace \(\bar{z}\) with \(S(z)\). Then the Cauchy transform can be determined from the singularities of the Schwarz function in \(\Omega\). For Example 1 above, we have \(C_\Omega = \text{Const}(\frac{1}{\zeta - \xi} + \frac{1}{\zeta + \xi})\). In the context of Chapter 4 this means that the gravitational lens consisting of an object with uniform mass supported over \(\Omega\) produces the same lensing effect (outside its support) as do two point masses. In Chapter 4, we will be more interested, though, in the Cauchy transform of a certain non-uniform mass density supported on an ellipse. The Schwarz function also plays a role in this case.
1.1 Singularities of the Schwarz potential and Laplacian growth

The Schwarz function can be partially generalized to higher dimensions using the following Cauchy problem posed in the vicinity of $\Gamma$, where we assume $\Gamma$ is non-singular and analytic. The solution exists and is unique by the Cauchy-Kovalevskaya Theorem.

\[
\begin{align*}
\Delta w &= 0 \text{ near } \Gamma \\
\left. w \right|_{\Gamma} &= \frac{1}{2}||x||^2 \\
\left. \nabla w \right|_{\Gamma} &= x
\end{align*}
\]  

(1.1.1)

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian.

**Definition 1.1.1** The solution $w(x)$ of the Cauchy problem 1.1.1 is called the Schwarz Potential of $\Gamma$.

In $\mathbb{R}^2$, the Schwarz function can be directly recovered from the Schwarz potential. Consider $S(z) = 2\partial_{\bar{z}}w = w_x - iw_y$. The Cauchy-Riemann equations for $S$ follow from harmonicity of $w$, and $\nabla w = x$ on $\Gamma$ implies $S(z) = \bar{z}$ on $\Gamma$.

**Example:** Let $\Gamma := \{x \in \mathbb{R}^n : ||x||^2 = r^2\}$ be a sphere of radius $r$. When $n = 2$, it is easy to verify that $w(z) = r^2 (\log |z| + 1/2 - \log(r))$ solves the Cauchy Problem (1.1.1), and in higher dimensions the Schwarz potential is $w(x) = -r^n (n-2)||x||^{n-2} + \frac{n}{2(n-2)}r^2$.

This gives a partial generalization of the Schwarz function. The reflection principle associated with the Schwarz function does not generalize to higher dimensions by this or any other means, but the Schwarz potential retains other properties we are interested in. For instance, the Schwarz function plays an important role in the Laplacian growth problem in the plane. To be brief (see Chapter 2 for details), in the Laplacian Growth (LG) problem, the velocity of a moving boundary is determined at each instant by the gradient of its Green’s function (“pressure”) with fixed singularity. The time-derivative of the Schwarz function of the moving boundary coincides with
the $z$-derivative of the pressure, leading to a reformulation of Laplacian growth in terms of a simple description of the time-dependence of singularities of the Schwarz function. This provides a unified view of the many classical exact solutions and has lead to further systematic developments.

In higher dimensions, LG continues to be governed by the singularities of the Schwarz potential. In [78], we gave a simple proof of the following.

**Theorem 1.1.2** If $w(x, t)$ is the Schwarz potential of $\partial \Omega_t$ then $\Omega_t$ solves the Laplacian growth problem if and only if

\[ \frac{\partial}{\partial t} w(x, t) = -nP(x, t) \tag{1.1.2} \]

where $n$ is the spatial dimension. In particular, singularities of the Schwarz potential in the $\Omega_t$ do not depend on time, except for one stationed at the source/sink which does not move but simply changes strength.

In [78] we study the singularities of the Schwarz potential with the goal of obtaining applications to LG using Theorem 1.1.2. In dimensions higher than two, the singularity set of the Schwarz potential is much more mysterious and there is a deficiency of exact solutions to LG, whereas in two dimensions there is an abundance of explicit examples.

Following L. Karp, we consider the special case of axially-symmetric examples in $\mathbb{R}^4$, which can be reduced to the singularities of the Schwarz function of the curve that generates the hypersurface of revolution. We describe some examples of LG in $\mathbb{R}^4$.

For three-dimensional examples, only quadratic surfaces have been understood. G. Johnsson [52] gave the complete description for quadratic surfaces by globalizing the proof of Leray’s principle in this case. In order to study some surfaces of higher degree, in [78] we lift the problem to $\mathbb{C}^n$ and use the globalizing technique of Bony and Schapira [19] combined with the local extension Theorem of Zerner [107] and the
more recent Theorem of Ebenfelt, Khavinson, and Shapiro [31]. We are able to prove the following regarding surfaces of revolution generated by “C. Neumann’s oval”.

**Theorem 1.1.3** Let $W(x)$ be the Schwarz potential of the boundary $\Gamma$ of the domain $\Omega := \{ x \in \mathbb{R}^n : (\sum_{i=1}^n x_i^2)^2 - a^2 \sum_{i=1}^n x_i^2 - 4x_1^2 < 0 \}$. Then $W$ can be analytically continued throughout $\Omega \setminus B$ where $B$ is the segment $\{ x_1 \in [-1, 1], x_j = 0 \text{ for } j = 2, \ldots, n \}$.

In [78] we also consider a generalization of LG to the case when the physical properties of the medium are non-homogeneous. This is the so-called elliptic growth problem. We are able to generalize Theorem 1.1.2 to this case and obtain some novel explicit exact solutions.

Two conjectures in potential theory arise in our study. The first conjecture suggests a response to H. S. Shapiro’s remark that it is not known whether quadrature domains (domains whose Schwarz potential has finitely many finite-order point-singularities in the domain) are always algebraic in dimensions higher than two.

**Conjecture 1.1.4** In dimensions greater than two, there exist quadrature domains that are not algebraic.

The following conjecture generalizes the Schwarz Potential Conjecture formulated by Khavinson and Shapiro. If true, it would naturally separate elliptic growth problems into classes (see Chapter 2 for details). The Schwarz potential Conjecture is easy to prove in the plane, but we do not know if the following is true even in two dimensions.

**Conjecture 1.1.5** Suppose $\alpha > 0$ is entire and that $u$ solves the Cauchy problem on a nonsingular analytic surface for $\text{div}(\alpha \nabla u) = 0$ with entire data. Then the singularity set of $u$ is contained in the singularity set of $v$, where $v$ solves the Cauchy problem with data function $q$, which is a global solution of $\text{div}(\alpha \nabla q) = 1$. 

6
1.2 Algebraic Dirichlet Problems

Let \( \Omega \) be a smoothly bounded domain in \( \mathbb{R}^n \). Consider the Dirichlet Problem (DP) in \( \Omega \) of finding the function \( u \), say, \( \in C^2(\Omega) \cap C(\overline{\Omega}) \) and satisfying

\[
\left\{ \begin{array}{l}
\Delta u = 0 \\
u|_{\Gamma} = v
\end{array} \right. ,
\]

where \( \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \) and \( \Gamma := \partial \Omega, v \in C(\Gamma) \). The solution \( u \) exists and is unique, and if \( \Gamma := \partial \Omega \) is (a component of) a real-analytic hypersurface and the data \( v \) is real-analytic in a neighborhood of \( \Omega \), then \( u \) extends as a real-analytic function across \( \partial \Omega \) into an open neighborhood \( \Omega' \) of \( \overline{\Omega} \).

It is classically known that the Dirichlet problem for Laplace’s equation with polynomial data posed on an ellipsoid has a polynomial solution. D. Khavinson and H. S. Shapiro showed that real-entire data give a real-entire solution for ellipsoids [64]. They formulated a conjecture that, in terms of the Dirichlet problem, ellipsoids are characterized within the class of bounded domains by each of the properties (i) that any polynomial data has a polynomial solution and (ii) any entire data has an entire solution.

Chapter 3 consists of the papers [57], [77], and [79]. In [57] (written jointly with Dmitry Khavinson), we give a detailed survey of work related to conjectures (i) and (ii). The paper [79] (joint work with Hermann Render) relates to conjecture (i), and the paper [77] relates to conjecture (ii). Along the lines of the conjecture (ii), it is natural to ask the following question on a case by case basis for different algebraically-bounded domains:

**Question:** Given a domain with algebraic boundary and polynomial or entire data, does the solution to the Dirichlet problem develop singularities, and if so, how far outside the initial domain are they?

This question sets the tone for the paper [77], where we consider specific examples in the plane and answer the question by specifying data and estimating the
position of the singularities that develop. The estimates use $\mathbb{C}^2$-techniques involving annihilating measures supported on special finite sets of points called “lightning bolts”. This is a complex version of the lightning bolts introduced by Kolmogorov and Arnold to solve Hilbert’s 13th problem. The complex lightning bolts were first used by Hansen and Shapiro [46] to show failure of analytic continuability of harmonic functions but without estimating the location of singularities. In order to use complex lightning bolts to estimate the location of singularities, in [77] we also study the geometry of the complexification of the curve in $\mathbb{C}^2$ in relationship to the real section of the curve and the Vekua hull of certain domains in $\mathbb{R}^2$. This is summarized in the prescription provided by the following Theorem, where $\tilde{\Gamma}$ is the complexification of $\Gamma$, i.e. the zero set in $\mathbb{C}^2$ of the same polynomial, and $\hat{\Omega}$ is the Vekua Hull of $\Omega$.

**Theorem 1.2.1** Let $\Gamma_1$ be a connected component of the algebraic curve $\Gamma$, and let $\Omega$ be a simply connected domain. Suppose $\tilde{\Gamma}$ contains a closed lightning bolt (with respect to the complex-characteristic coordinates $z$ and $w$) of length $2n$. Suppose further that along $\tilde{\Gamma}$ there are paths, also contained in $\hat{\Omega}$, that connect each vertex to $\Gamma_1$. Then, for the Dirichlet problem on $\Gamma_1$, there exist polynomial data of degree $n$ whose solution cannot be analytically continued to all of $\Omega$.

Using this Theorem, we estimate the location of singularities developed by the analytic continuation of solutions with certain data posed on a few different families of curves.

**Example 1:** $\Gamma$ is the solution set of $p(x^4) + q(y^4) = 1$ where $p(x)$ and $q(x)$ are positive for $x \in \mathbb{R}_+$ and satisfy $p(1) + q(0) = p(0) + q(1) = 1$. For the Dirichlet problem with non-harmonic, quadratic data on the curve $p(x^4) + q(y^4) = 1$, the solution develops singularities on the $x$ and $y$ axes no further from the origin than $\max\{|x| + |y| : p(x^4) + q(y^4) = 1\}$. A special case is the “TV-screen” $x^4 + y^4 = 1$ considered by P. Ebenfelt in [30], where the singularities were described exhaustively.

**Example 2:** $\Gamma$ is the zero set of $P(x, y) = x^2 + y^2 - 1 - 2\epsilon(x^3 + xy^2 - x^2 + y^2)$. For the Dirichlet problem with any non-harmonic, quadratic data posed on the bounded
component of \( \Gamma \) (\( \varepsilon \) small), the solution develops singularities no further from the origin than \( \frac{1+\sqrt{1+2\sqrt{2}}}{2\varepsilon(1-2\varepsilon)} \), which is, asymptotically, twice the distance from the bounded component to the unbounded component.

**Example 3:** \( \Gamma \) is the solution set \( 8x(x^2-y^2)+57x^2+77y^2 = 49 \) (a perturbed ellipse). For the Dirichlet problem posed inside the bounded component of \( \Gamma \), there exist cubic data for which the solution develops a singularity on the x-axis no further from the origin than 7.622 (compare to the x-intercept, \((-7,0)\), of the nearest unbounded component). Here, the polynomial defining \( \Gamma \) satisfies the necessary condition posed by Chamberland and Siegel [22], and the cubic data \( y(57x^2 + 77y^2 - 49) \) has the polynomial solution \( 8xy(x^2 - y^2) \).

A recent result of D. Khavinson and N. Stylianopoulos proves the conjecture (i) in two dimensions under an additional assumption on the degree of the solution in terms of the degree of the data. Their result gives an interesting consequence for *Fischer decompositions*. If a polynomial \( f \) has a polynomial decomposition \( f = q\phi + r \), with \( r \) harmonic, then \( u = f - q\phi \) solves the Dirichlet problem with data \( f \) posed on the zero set of \( \phi \). Decomposing \( f \) as \( q\phi + r \) resembles the inductive step in the Euclidean algorithm, except instead of the degree condition \( \deg r < \deg f \), the requirement is that \( r \) is harmonic. In this context, the result of [65] implies in particular that:

**(FD)** Given a polynomial \( \phi \in \mathbb{R}[x,y] \), if every polynomial \( f \) has a Fischer decomposition \( f = q\phi + r \), with the added assumption that \( \deg r < \deg f + C \), for some \( C > 0 \) independent of \( f \), then \( \deg \phi \leq 2 \).

The proof in [65] used a reformulation in terms of so-called “finite-term” recurrence relations for Bergman orthogonal polynomials and applied ratio-asymptotics involving the conformal map to the exterior of the disk. Thus, the proof relies heavily on ideas from complex analysis. In [79] (joint work with Hermann Render), we proved the following version of (FD) in arbitrary dimensions and including a *polyharmonic* case (involving the \( k \)th iterate of the Laplace operator):

**Theorem 1.2.2** Let \( \psi \in \mathbb{R}[x_1,...,x_n] \) be a polynomial. Suppose that there exists a constant \( C > 0 \) such that for any polynomial \( f \in \mathbb{R}[x_1,...,x_n] \) there exists a decompo-
\[ f = \psi q_f + h_f \text{ with } \Delta^k h_f = 0 \text{ and } \]

\[ \deg q_f \leq \deg f + C. \quad (1.2.4) \]

Then \( \deg \psi \leq 2k \).

The proof uses the associated Fischer operator and applies linear algebra and dimension arguments involving harmonic divisors. We also showed for certain classes of examples that the degree condition in Theorem 1.2.2 is satisfied.

**Theorem 1.2.3** Suppose that \( \psi \) is a polynomial of degree \( t > 2 \) and \( \psi = \psi_t + \psi_s + \psi_{s-1} + ... + \psi_0 \) is the decomposition into a sum of homogeneous polynomials. Assume the polynomial \( \psi_s \) is non-zero and contains a non-negative, non-constant factor. Let \( f \) be a polynomial and assume that there exists a decomposition

\[ f = \psi q + h \]

where \( h \) is harmonic and \( q \) is a polynomial. Then \( \deg q \leq 2 - s + \deg f \) and \( \deg h \leq t + 2 - s + \deg f \).

H. Render settled both conjectures (i) and (ii) in arbitrary dimensions for the large class of so-called elliptic surfaces (surfaces whose defining polynomial has a nonnegative leading homogeneous term) [86]. H. S. Shapiro raised a simple non-elliptic example not settled by Render’s results: a circle (or sphere) perturbed by any higher-degree homogeneous harmonic polynomial. Combining Theorems 1.2.2 and 1.2.3 shows that, for this example, there exists polynomial data for which the solution is not a polynomial, confirming conjecture (i) in this case.

### 1.3 Valence of Harmonic Maps and Gravitational Lensing

The strongest test passed by Einstein’s theory of gravitation was the correct prediction of the deflection of starlight as it passes by a massive object. Besides bending or
distorting background sources, a massive object acting as a gravitational lens can create multiple images of a single source. In modeling a gravitational lens, the effect of moving some of the mass parallel to the line of sight is an order of magnitude smaller than the effect of moving it orthogonal to the line of sight. Thus, even extremely non-planar solid mass densities such as a spherical galaxy can be projected onto a *Lensing Plane* orthogonal to the line of sight. This is referred to as the “thin-lens approximation”. Integrating Einstein’s deflection angle against the projected mass density leads to a *lensing map* sending the Lensing Plane to the so-called *Source Plane*. Lensed images can then be identified as pre-images of the source position under the lensing map.

**Main Problem:** *Given a family of gravitational lenses, determine the maximum number of images that can be lensed.*

For light rays passing outside the support of the mass density, the lensing map is harmonic. As a step toward extending the fundamental theorem of algebra, D. Khavinson and G. Neumann (2006) [60] used the theory of planar harmonic maps combined with complex dynamics to prove a bound of $5n - 5$ for the number of zeros of a function of the form $r(z) - \bar{z}$, where $r(z)$ is rational of degree $n$. This turned out to solve a conjecture in astronomy regarding an instance of the Main Problem. In [71] (joint work with Ludwig Kuznia), we investigated the case when $r(z)$ is a Blaschke product. The resulting (sharp) bound is $n + 3$ and the proof is simple. This applies to gravitational lenses consisting of collinear point masses.

In [38], C. D. Fassnacht and C. R. Keeton (astrophysicists) and D. Khavinson posed another instance of the Main Problem and reduced it to the following simple problem in complex analysis:

**Problem (i):** *Give an upper bound for the number of solutions to the following transcendental equation, where $k$ is a real parameter and $w$ is a complex parameter, with the principal branch of arcsin.*

$$\arcsin\left(\frac{k}{\bar{z} + \bar{w}}\right) = z, \quad (1.3.5)$$
Solutions to this equation represent positions of images lensed by an elliptical
galaxy with an “isothermal” density. Astronomers have observed up to four such
images of a single source, but it is not even obvious, at a glance, that the number of
solutions to (1.3.5) is finite. The function appearing is a harmonic map, for which the
increment of the argument counts orientation-reversing zeros with opposite sign, by
itself only giving a lower bound on the total number of zeros. Inverting the equation,
it can then be formulated as a fixed point problem confined to a strip. Self-composing
gives a complex-analytic fixed point problem, but the function has infinitely many
essential singularities. In [58] (joint work with D. Khavinson), we applied a recent
result from complex dynamics [11] to bound the number of attracting fixed points.
This gives an upper bound for orientation-preserving zeros of the original harmonic
map, the important step in obtaining the total estimate in our theorem.

**Theorem 1.3.1** *The number of solutions to the equation*

\[
\arcsin\left(\frac{k}{\bar{z} + \bar{w}}\right) = z
\]

*is bounded by 8.*

W. Bergweiler and A. Eremenko [18] improved this result to a bound of 6 and
found an example (a highly eccentric elliptical galaxy) that attains 6. This was a
surprise to the astronomers who only found up to 4 using a model with unbounded
density, of which Eq. (1.3.5) represents a truncation. This urges the question, “Are
the six images in the truncated model the valid consequence of a compactly supported
density or are they an artifact of the sharp edge?” To make the question more precise,
take the simplest approach for removing the sharp edge (jump discontinuity). Namely,
subtract a constant to make the density continuous. This introduces an algebraic term
into Eq. (1.3.5). Then we have the following problem:

**Problem (ii):** *Are there choices of parameters for which the following equation has*
An affirmative answer would be certain to inspire further discussion among mathematicians and astronomers and would perhaps lead to reevaluation of the mainstream models. If there is an example with 6 images then finding it should be feasible, but proving a bound for the number of solutions to Eq. (1.3.6) appears to be a much more difficult case than Eq. (1.3.5). Another natural change to make to Eq. (1.3.5) is to include a tidal force (a linear perturbation) resulting in the following version of the problem.

**Problem (iii):** *Give an upper bound for the number of solutions to the following equation, where $k$ is a real parameter and $\gamma$ and $w$ are complex parameters, with the principal branch of arcsin:

\[
\arcsin \left( \frac{k}{\bar{z} + \bar{w}} \right) + \gamma \bar{z} = z. \quad (1.3.7)
\]

An empirical investigation suggests a sharp upper bound of 8, but including this simple linear perturbation resists a rigorous proof of even a crude upper bound. The different approaches used in [18] and [58] each seem to break down, unless the tidal force is aligned with one of the axes of the elliptical galaxy.
This chapter consists of the paper [78], which has been accepted for publication in Journal of Physics A: Mathematical and Theoretical.

The Schwarz function has played an elegant role in understanding and in generating new examples of exact solutions to the Laplacian growth (or “Hele-Shaw“) problem in the plane. The guiding principle in this connection is the fact that “non-physical” singularities in the “oil domain” of the Schwarz function are stationary, and the “physical” singularities obey simple dynamics. We give an elementary proof that the same holds in any number of dimensions for the Schwarz potential, introduced by D. Khavinson and H. S. Shapiro [62] (1989). A generalization is also given for the so-called “elliptic growth” problem by defining a generalized Schwarz potential.

New exact solutions are constructed, and we solve inverse problems of describing the driving singularities of a given flow. We demonstrate, by example, how $C^n$-techniques can be used to locate the singularity set of the Schwarz potential. One of our methods is to prolong available local extension theorems by constructing “globalizing families”.

\section{Laplacian Growth}

A one-parameter family of decreasing domains, $\{\Omega_t\}$, in $\mathbb{R}^n$ solves the Laplacian growth problem with sink at $x_0 \in \Omega_t$ if the normal velocity, $v_n$ of the boundary $\Gamma_t := \partial \Omega_t$ is determined by a harmonic Green’s function, $P(x,t)$, of $\Omega_t$ as follows.
\[
\begin{align*}
  v_n|_{\Gamma_t} &= -\nabla P \\
  \Delta P &= 0, \text{ in } \Omega_t \\
  P|_{\Gamma_t} &= 0 \\
  P(x \to x_0, t) &\sim -Q \cdot K(x - x_0),
\end{align*}
\] (2.1.1)

where \( K \) is the fundamental solution of the Laplace equation, and \( Q > 0 \) (typically constant) determines the suction rate at \( x_0 \). We can also consider \( Q < 0 \) for the case of a source \( x_0 \) where injection occurs, but this problem is stable (approaching a sphere in the limit) and is sometimes called the “backward-time Laplacian growth”.

This is a \emph{nonlinear} moving boundary problem, ubiquitous as an ideal model (or at least, first approximation) of many growth processes in nature and industry. We stress that we are considering here the ill-posed zero surface-tension case, where the interface can encounter a cusp. The zero surface-tension case has attracted wide and growing attention mainly for two reasons (to be brief): (i) it has direct connections to many other areas such as classical potential theory, integrable systems, soliton theory, and random matrices; (ii) it admits a miraculous complete set of explicit exact solutions in the two-dimensional case.

If the domains \( \Omega_t \) are bounded, with \( Q > 0 \) problem (2.1.1) actually produces a \emph{shrinking} boundary. We get a growth process if \( \Omega_t \) contains infinity, so \( P \) then solves an \emph{exterior} Dirichlet problem. In such a situation, it is common to place the sink at infinity by prescribing asymptotics for \( \nabla P \) so that the flux across neighborhoods of infinity is proportional to \( Q \). In the two-dimensional case this can be realized in the laboratory using a \emph{Hele-Shaw cell}. Two sheets of glass are placed close together with a viscous fluid (“oil”) filling the void between them. A small hole is drilled in the center of the top sheet and an inviscid fluid (“water”) is pumped in at a constant rate. Then problem (2.1.1) serves as an ideal model for the boundary of the growing bubble of water. The harmonic function \( P(x, t) \), in this case, corresponds to the \emph{pressure} in the oil domain. In other physical settings modeled by (2.1.1), \( P(x, t) \) can be a probability, a concentration, an electrostatic field, or a temperature. Because of the huge amount
of literature, we are limited to citing an incomplete list of papers. For a list of over 500 references, see [45].

We are particularly attracted to this problem by the lack of explicit examples in dimensions higher than two. The existence, uniqueness, and regularity theory are well-developed in arbitrary dimensions, and in the plane there is an abundance of explicit, exact solutions. In dimensions higher than two, the only examples are a shrinking sphere (in the case when the “oil domain” $\Omega_t$ is bounded) or the exterior of a homothetically growing ellipsoid (in the case $\Omega_t$ is unbounded). The obvious explanation for this deficiency of explicit examples is a lack of conformal maps in higher dimensions (Liouville’s Theorem) since exact solutions are usually described in terms of a time-dependent conformal map of the domain to the disk. However, exact solutions can be understood using a different tool from complex analysis, the Schwarz function (see [24] and Section 2 below). The following theorem relates to the work of S. Richardson [89] and was first stated in terms of the Schwarz function by R. F. Millar [80]. Also, the discussion given by S. Howison [48] seems to have played an important role in popularizing the use of the Schwarz function in studies of Laplacian growth.

**Theorem 2.1.1 (Dynamics of Singularities: $\mathbb{R}^2$)** Suppose a one-parameter family of domains $\Omega_t$ has smoothly-changing analytic boundary with Schwarz function $S(z, t)$. Then it is a Laplacian growth if and only if

$$\frac{\partial}{\partial t} S(z, t) = -4 \frac{\partial}{\partial z} P(z, t)$$

(2.1.2)

The Schwarz function is only guaranteed to exist in a vicinity of a given analytic curve, and a priori the domain of analyticity for its time-derivative is not any larger. Thus, it is surprising that for a Laplacian growth, $\frac{\partial}{\partial t} S(z, t)$ coincides with a function analytic throughout $\Omega_t$ except at the singularity prescribed at the “sink”. In other words, we can extract from equation (2.1.2) the following elegant description of solutions to problem (2.1.1): *Singularities in $\Omega_t$ of the Schwarz function of $\partial \Omega_t$ do not move except for one simple pole stationed at the sink $x_0$ which decreases in*
strength at the rate $-Q$. Since equation 2.1.2 is given in physical coordinates rather than introducing a uniformized “mathematical plane”, S. Howison [48] has called it an intrinsic description. In recent papers, it is typical to see a combination of the Schwarz function and the conformal map used to derive solutions (e.g. [1]). We will review some familiar examples in Section 4 and understand them completely in terms of Theorem 2.1.1.

The Schwarz function has been partially generalized by D. Khavinson and H. S. Shapiro to higher dimensions by defining a “Schwarz potential”, a solution of a certain Cauchy problem for the Laplace equation [96]. In Section 2, we will review the definition of the Schwarz function and the Schwarz potential before proving the $n$-dimensional version of Theorem 2.1.1. We also give a further generalization to the elliptic growth problem. The rest of the paper is guided by Theorem 2.2.2, which identifies, as the main obstacle, the problem of describing (globally) the singularities of the Schwarz potential. In Section 4 we follow the observation made by L. Karp that the Schwarz potential of four-dimensional, axially-symmetric surfaces can be calculated exactly [53]. We give some explicit examples and also describe some examples of elliptic growth. In Section 5, we use $\mathbb{C}^n$ techniques to understand the Schwarz potential’s singularity set for a nontrivial example in $\mathbb{R}^n$ including the important case $n = 3$. In Section 6, we discuss the connection to quadrature domains and Richardson’s Theorem.

2.2 Dynamics of Singularities

2.2.1 The Schwarz Potential

Suppose $\Gamma$ is a non-singular, real-analytic curve in the plane. Then the Schwarz function $S(z)$ is the function that is complex-analytic in a neighborhood of $\Gamma$ and coincides with $\bar{z}$ on $\Gamma$ (see [24] for a full exposition). If $\Gamma$ is given algebraically as the zero set of a polynomial $P(x, y)$, we can obtain $S(z)$ by making the complex-linear change of variables $z = x + iy$, $\bar{z} = x - iy$, and then solving for $\bar{z}$ in the equation
\[ P\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = 0. \] For instance, suppose \( \Gamma \) is the curve given algebraically by the solution set of the equation \((x^2 + y^2)^2 = a^2(x^2 + y^2) + 4\varepsilon^2x^2\) ("C. Neumann’s oval"). Then changing variables we have \((z\bar{z})^2 = a^2(z\bar{z}) + \varepsilon^2(z + \bar{z})^2\). Solving for \(\bar{z}\) gives

\[ S(z) = \frac{z(a^2 + 2\varepsilon^2) + \sqrt{4a^4 + 4a^2\varepsilon^2 + 4\varepsilon^2(x^2 - \varepsilon^2)}}{2(z^2 - \varepsilon^2)}. \]

Suppose \( \Gamma \) is more generally a nonsingular, analytic hypersurface in \( \mathbb{R}^n \), and consider the following Cauchy problem posed in the vicinity of \( \Gamma \). The solution exists and is unique by the Cauchy-Kovalevskaya Theorem.

\[
\begin{cases}
\Delta w = 0 \text{ near } \Gamma \\
w|_{\Gamma} = \frac{1}{2}||x||^2 \\
\nabla w|_{\Gamma} = x
\end{cases}
\tag{2.2.3}
\]

**Definition 2.2.1** The solution \( w(x) \) of the Cauchy problem 2.2.3 is called the Schwarz Potential of \( \Gamma \).

**Example:** Let \( \Gamma := \{x \in \mathbb{R}^n : ||x||^2 = r^2\} \) be a sphere of radius \( r \). When \( n = 2 \), it is easy to verify that \( w(z) = r^2 (\log |z| + 1/2 - \log(r)) \) solves the Cauchy Problem (2.2.3), and in higher dimensions the Schwarz potential is \( w(x) = -\frac{r^n}{(n-2)||x||^{n-2}} + \frac{n}{2(n-2)}r^2 \).

In \( \mathbb{R}^2 \), the Schwarz function can be directly recovered from the Schwarz potential. Consider \( S(z) = 2\partial_z w = w_x - iw_y \). The Cauchy-Riemann equations for \( S \) follow from harmonicity of \( w \), and \( \nabla w = x \) on \( \Gamma \) implies \( S(z) = \bar{z} \) on \( \Gamma \).

This gives a partial generalization of the Schwarz function. The reflection principle associated with the Schwarz function does not generalize to higher dimensions by this or any other means, but the Schwarz potential retains other desirable properties. In particular, it allows us to generalize Theorem 2.1.1 to higher dimensions.

### 2.2.2 Laplacian growth and the Schwarz potential

The following theorem generalizes Theorem 2.1.1. We consider a family of domains \( \Omega_t \subset \mathbb{R}^n \) so that \( \Omega_t \) has an analytic boundary with analytic time-dependence. Such
a regularity assumption is natural for us since we are in pursuit of explicit, exact solutions. However, we should mention that analyticity of the boundary is a necessary condition for existence of a classical solution, and moreover for an analytic initial boundary, there exists a unique solution remaining analytic with analytic time-dependence for at least some interval of time (see [37] and [104]). Let \( w(x,t) \) denote the Schwarz potential of the boundary \( \Gamma_t \) of \( \Omega_t \).

**Theorem 2.2.2 (Dynamics of Singularities: \( \mathbb{R}^n \))** If \( \Omega_t \) and \( w(x,t) \) are as above then \( \Omega_t \) solves the Laplacian growth problem (2.1.1) if and only if

\[
\frac{\partial}{\partial t} w(x,t) = -nP(x,t)
\]  

(2.2.4)

where \( n \) is the spatial dimension. In particular, singularities of the Schwarz potential in the “oil domain” do not depend on time, except for one stationed at the source (sink) which does not move but simply changes strength.

**Remark 1:** This relates the solution of a “mathematically-posed” Cauchy problem to that of a “physically-posed” Dirichlet problem.

**Remark 2:** Considering the relationship between the Schwarz potential and Schwarz function, in the case of \( n = 2 \), the Theorem says that \( S_t = \frac{\partial}{\partial t}(2\partial_2 w) = -4\partial_2 P \) which is the content of Theorem 2.1.1.

**Remark 3:** This is closely related to the celebrated Richardson’s Theorem [89]. Actually, the connection can be established through the role that the Schwarz potential plays in the theory of quadrature domains (see Section 2.5). Here we are able to give a more elementary proof consisting of two applications of the chain rule.

**Proof.** Assume \( \{\Omega_t\} \) solves the Laplacian growth problem. We will show that for each \( t \), \( w_t(x,t) \) and \( -nP(x,t) \) solve the same Cauchy problem. Then by the uniqueness part of the Cauchy-Kovalevskaya Theorem, they are identical.
First, we will show that \( w_t(x, t)|_{\Gamma_t} = 0 \). Consider a point \( x(t) \) which is on \( \Gamma_t \) at time \( t \). The chain rule gives

\[
\frac{d}{dt} w(x(t), t) = \nabla w(x(t), t) \cdot \dot{x}(t) + w_t(x(t), t)
\]

(2.2.5)

On the other hand, by the first piece of Cauchy data in (2.2.3),

\[
\frac{d}{dt} w(x(t), t) = \frac{1}{2} \frac{d}{dt} (1/2 ||x(t)||^2) = x(t) \cdot \dot{x}(t)
\]

(2.2.6)

By the second piece of Cauchy data,

\[
x(t) \cdot \dot{x}(t) = \nabla w(x(t), t) \cdot \dot{x}(t)
\]

(2.2.7)

Combining (2.2.6) and (2.2.7) with equation (2.2.5) gives

\[
w_t(x, t)|_{\Gamma_t} = 0
\]

(2.2.8)

We are done if we can show that \( \nabla w_t|_{\Gamma_t} = -n \nabla p \). Given some position, \( x \), let \( T(x) \) assign the value of time precisely when the boundary, \( \Gamma_t \), of the growing domain passes \( x \). Then by the Cauchy data defining the Schwarz potential (2.2.3), \( w_{x_k}(x_1, x_2, ..., x_n, T(x_1, x_2, ..., x_n)) = x_k \). Taking the partial with respect to \( x_k \) of the \( k \)th equation gives \( w_{tx_k} T_{x_k} + w_{x_k x_k} = 1 \). Summing these \( k \) equations together gives

\[
\nabla w_t \cdot \nabla T + \Delta w = n.
\]

(2.2.9)

Since \( \Gamma_t \) is the level curve \( T(x) = t \), \( \nabla T \) is orthogonal to \( \Gamma_t \), and \( \nabla T = \frac{v_n}{||v_n||^2} \), where \( v_n \) is the normal velocity of \( \Gamma_t \). Recall, \( v_n = -\nabla P \). Thus, \( \nabla T = \frac{-\nabla P}{||\nabla P||^2} \). Substitution into equation (2.2.9) gives \( \nabla w_t \cdot \nabla P = -n ||\nabla P||^2 \).

By equation (2.2.8), \( w_t|_{\Gamma_t} = 0 \), which implies that \( \nabla w_t \) and \( \nabla P \) are parallel. So, \( \nabla w_t|_{\Gamma_t} = -n \nabla P \).
2.2.3 A Cauchy problem connected to Elliptic Growth

A natural generalization of the Laplacian growth problem is to allow a non-constant “filtration coefficient” $\lambda$ and “porosity” $\rho$. Then instead of Laplace’s equation the pressure satisfies $\text{div}(\lambda \rho \nabla P) = 0$ and should have a singularity at the sink of the same type as the fundamental solution to this elliptic equation. Moreover, the Darcy’s law determining the boundary velocity becomes $v_n = -\lambda \nabla P$. For details, see [59], [75], [76]. Physically, this models the problem in a non-homogeneous medium and also relates to the case of Hele-Shaw cells on curved surfaces (in the absence of gravity) studied in [105, Ch. 7].

We can formulate an equation similar to equation (2.2.4) that relates the pressure function of an elliptic growth to the time-dependence of the solution to a certain Cauchy problem. Let $q(x)$ be a solution of the Poisson equation,

$$\text{div}(\lambda \rho \nabla q) = n \rho, \quad (2.2.10)$$

where $n$ is the spatial dimension. Recall that a solution $q$ can be obtained by taking the convolution of $\rho$ with the fundamental solution of the homogeneous elliptic equation (if one exists). We associate with an elliptic growth having filtration $\lambda$ and porosity $\rho$, the solution $u$ of the following Cauchy problem.

$$\begin{cases}
\text{div} (\lambda \rho \nabla u) = 0 \text{ near } \Gamma \\
u|_{\Gamma} = q \\
\nabla u|_{\Gamma} = \nabla q 
\end{cases} \quad (2.2.11)$$

We can think of $u$ as a “generalized Schwarz potential”. We have the following direct generalization of Theorem 2.2.2. As in Section 2.2, assume $\Omega_t$ has analytic boundary with analytic time-dependence.

**Theorem 2.2.3** If $\Gamma_t = \partial \Omega_t$ and $u(x,t)$ is the solution to 2.2.11 posed on $\Gamma_t$ then $\Omega_t$
is an elliptic growth with pressure function $P(x, t)$ if and only if

$$\frac{\partial}{\partial t} u(x, t) = -nP(x, t) \quad (2.2.12)$$

Proof. As in the proof of Theorem 2.2.2, we show that both sides of 2.2.12 solve the same Cauchy problem.

The first part of the argument is similar in showing that.

$$u_t(x, t)|_{\Gamma_t} = 0 \quad (2.2.13)$$

Consider a point $x(t)$ which is on $\Gamma_t$ at time $t$. The chain rule gives

$$\frac{d}{dt} u(x(t), t) = \nabla u(x(t), t) \cdot \dot{x}(t) + u_t(x(t), t) \quad (2.2.14)$$

On the other hand, by the first piece of Cauchy data in (2.2.11) and the chain rule again,

$$\frac{d}{dt} u(x(t), t) = \frac{d}{dt} q(x(t)) = \nabla q(x(t)) \cdot \dot{x}(t) \quad (2.2.15)$$

By the second piece of Cauchy data,

$$\nabla q(x(t)) \cdot \dot{x}(t) = \nabla u(x(t), t) \cdot \dot{x}(t) \quad (2.2.16)$$

Combining (2.2.15) and (2.2.16) with equation (2.2.14) gives the equation (2.2.13).

We are done if we can show that $\nabla u_t|_{\Gamma_t} = -n \nabla P$.

We again let $T(x)$ assign the value of time when $\Gamma_t$ passes $x$. Then by the Cauchy data defining $u$, $\nabla u(x, T(x)) = \nabla q$. Multiply both sides by $\lambda \rho$ and take the divergence:

$$\lambda \rho \nabla u_t \cdot \nabla T + \text{div}(\rho \lambda \nabla u) = \text{div}(\rho \lambda \nabla q), \quad (2.2.17)$$

which, by definition of $u$ and $q$, simplifies to

$$\lambda \nabla u_t \cdot \nabla T = n. \quad (2.2.18)$$
As in the proof of Theorem 2.2.2, $\nabla T = \frac{v_n}{||v_n||}$, where $v_n$ is the normal velocity of $\Gamma_t$, except now $v_n = -\lambda \nabla P$. Thus, $\nabla T = \frac{-\nabla P}{\lambda ||\nabla P||}$. Substitution into equation (2.2.18) gives $\nabla u_t \cdot \nabla P = -n||\nabla P||^2$.

By equation (2.2.13), $u_t|_{\Gamma_t} = 0$, which implies that $\nabla u_t$ and $\nabla P$ are parallel. So, $\nabla u_t|_{\Gamma_t} = -n\nabla P$.

Let us discuss a special case of the above. Suppose that the Problem (2.2.10) has a solution $q$ that is entire. Let $\alpha$ denote $\lambda \rho$, and suppose $\alpha$ is also entire. For instance, $\lambda = \frac{1}{x^2 + 1}$ and $\rho = x^2 + 1$ gives $\alpha = 1$ and $q = x^4 + x^2$. When $\alpha = 1$ as in this case, the “elliptic growth” is just a Laplacian growth with a variable-coefficient law governing the boundary velocity. The problem (2.2.11) defining $u$ becomes a Cauchy problem for Laplace’s equation with entire data. This is the realm of the Schwarz potential conjecture formulated by Khavinson and Shapiro:

**Conjecture 2.2.4 (Khavinson, Shapiro)** Suppose $u$ solves the Cauchy problem for Laplace’s equation posed on a nonsingular analytic surface $\Gamma$ with real-entire data. Then the singularity set of $u$ is contained in the singularity set of the Schwarz potential $w$.

The conjecture holds in the plane and has been shown to hold “genericall” in higher dimensions [94]. If the conjecture is true, then for the case when $\alpha = 1$, the singularities of $u$ are controlled throughout time by those of $w$. Combining this with Theorems 2.2.2 and 2.2.3 implies that, given a solution of the Laplacian growth problem (2.1.1), the exact same evolution can be generated amid an elliptic growth law with $\alpha = 1$ by a pressure function having singularities at the same locations as those of $w$. The singularities may have different time-dependence and be of a different type.

For instance, consider the plane and the simplest Laplacian growth of suction from the center of a circle so that at time $t$, the Schwarz function is $\frac{1-t}{\pi}$ (a constant rate of suction). Let us determine the pressure required to generate the same process
when $\lambda = \frac{1}{x+1}$ and $\rho = x^2 + 1$. To solve for $u$, we notice that $\partial_z u$ is analytic and coincides with $2x^3 + x$ on the shrinking circle. Since $x = \frac{z + S(z)}{2}$ on the boundary, we have $\partial_z u = (z + \frac{1-t}{z})^3/4 + z/2 + \frac{1-t}{2z}$. This is even true off the boundary since both sides are analytic. The singular terms are $\frac{5-8t+3t^2}{4z}$ and $\frac{1-3t^3+3t^2-t^4}{4z^3}$. Thus, in order to generate the same “movie”, the pressure must have a fundamental solution type singularity along with a weak “multi-pole” at the origin, both diminishing at non-constant rates.

### 2.3 Examples

In this section we will explain some explicit solutions in terms of the Theorems 2.1.1, 2.2.2, and 2.2.3.

#### 2.3.1 Laplacian growth in two dimensions

First we review some familiar examples in the plane, where typically a time-dependent conformal map is introduced. Instead, we work entirely with the Schwarz function and check that Theorem 2.1.1 is satisfied.

**Example 1:** Consider the family of domains $D$ with boundary given by the curves $\{z : z = aw^2 + bw, |w| < 1\}$ with $a, b$ real. The Schwarz function is given by $S(z) = \frac{-2ab}{(a - \sqrt{a^2 + 4bz}) + 4b^3/(a - \sqrt{a^2 + 4bz})^2}$ which has a single-valued branch in the interior of the curve for appropriate parameter values $a$ and $b$. The only singularities of the Schwarz function interior to the curve are a simple pole and a pole of order two at the origin. Given an initial domain from this family we can choose a one-parameter slice of domains so that the simple pole increases (resp. decreases) while the pole of order two does not change. This gives an exact solution to the Laplacian growth problem with injection (resp. suction) taking place at the origin. In the case of injection, the domain approaches a circle. In the case of suction, the domain develops a cusp in finite time.

Instead of just one sink or source $x_0$ with rate $Q$, let us extend problem 2.1.1 by allowing for multiple sinks and/or sources $x_i$ with suction/injection rates $Q_i$. This
is the formulation of the problem which is often made, for instance, see the excellent exposition [105]. The proof of Theorem 2.2.2 carries through without changes so that the time-derivative of the Schwarz potential still coincides with $-nP(x, t)$. The only difference is that now there can be multiple time-dependent point-singularities inside $\Omega_t$.

**Example 2:** We first consider the family of curves mentioned in Section 2.1. The Schwarz function of the boundary is

$$S(z) = \frac{z(a^2 + 2\varepsilon^2) + z\sqrt{4a^4 + 4a^2\varepsilon^2 + 4\varepsilon^2z^2}}{2(z^2 - \varepsilon^2)}$$

which has two simple poles at $z = \pm \varepsilon$ each with residue $(a^2 + 2\varepsilon^2)/2$. In order to satisfy the conditions imposed by Theorem 2.2.2 we choose $\varepsilon = 1$ to be constant. Then we choose $a(t)$ to be decreasing (increasing) to obtain suction (injection) at two sinks (sources). In the case of suction, the oval forms an indentation at the top and bottom and becomes increasingly pinched as the boundary approaches two tangent circles centered at $\pm 1$, the positions of the sinks (see Figure 2.1).

For the next example, we consider the case of Problem (2.1.1) where the “oil domain” $\Omega_t$ is unbounded with a sink at infinity.
Example 3: We recall the Schwarz function for an ellipse given by the solution set of the equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Changing variables we have \( \frac{(z+i\tilde{z})^2}{a^2} - \frac{(z-i\tilde{z})^2}{b^2} = 4 \). Solving for \( \tilde{z} \) gives

\[
S(z) = \frac{a^2 + b^2}{a^2 - b^2} z + \frac{2ab}{b^2 - a^2} \sqrt{z^2 + b^2 - a^2}.
\]

\( S(z) \) has a square root branch cut along the segment joining the foci \( \pm \sqrt{a^2 - b^2} \), but we are only interested in the exterior of the ellipse, where \( S(z) \) is free of singularities. This already guarantees that any evolution of ellipses that has analytic time dependence can be generated by preparing the correct asymptotic pressure conditions to match \( S_t(z, t) \) which is only singular at infinity. In other words, we can use equation 2.1.2 to work backwards in specifying the pressure condition to generate the given flow. Since there are no finite singularities, we only have to specify the conditions at infinity. A realistic case is if the asymptotic condition is steady and isotropic: \( S_t(z \to \infty, t) \approx k/z \) for a constant \( k \) independent of \( t \). Take a homothetic growth with \( a(t) = a\sqrt{t} \) and \( b(t) = b\sqrt{t} \) from some initial ellipse with semi-axes \( a \) and \( b \). Then \( S(z, t) = \frac{a^2 + b^2}{a^2 - b^2} z + \frac{2ab}{b^2 - a^2} \sqrt{z^2 + t(b^2 - a^2)} \), and we have \( S_t(z, t) = k \frac{1}{\sqrt{z^2 + t(b^2 - a^2)}} \), where \( k = 2ab \).

2.3.2 Examples and non-examples in \( \mathbb{R}^4 \)

Next, we consider axially-symmetric, four-dimensional domains. This turns out to be simpler than the more physically relevant \( \mathbb{R}^3 \), and we will see in the next subsection that it is is equivalent to certain cases of elliptic growth in two and three dimensions. Lavi Karp [53] has given a procedure, including several explicit examples, for obtaining the singularities of the Schwarz potential for a domain \( \Omega \) that is the rotation into \( \mathbb{R}^4 \) of a domain in \( \mathbb{R}^2 \) with Schwarz function \( S(z) \). We outline here this procedure for finding the Schwarz potential \( w(x_1, x_2, x_3, x_4) \). Since \( w \) solves a Cauchy problem for axially symmetric data posed on an axially symmetric hypersurface, with, say \( x_1 \) as the axis of symmetry, it can be regarded as a function of two variables. Write \( x = x_1, y = \sqrt{x_2^2 + x_3^2 + x_4^2} \), and \( w(x_1, x_2, x_3, x_4) = U(x, y) \). What makes \( \mathbb{R}^4 \) convenient to work with is the fact that \( V(x, y) = y \cdot U(x, y) \) is a harmonic function of the variables \( x \) and \( y \). Thus, finding \( U(x, y) \) is reduced to solving an algebraic Cauchy problem in the plane, which can be done in terms of the Schwarz function \( S(x + iy) = S(z) \). The
steps for writing this solution are outlined below.

**Step 1:** Write \( f(z) = \frac{1}{2} S(z) \cdot (S(z) - 2z) \).

**Step 2:** Find a primitive function \( F(z) \) for \( f(z) \).

**Step 3:** Write \( V(x, y) = \text{Re}\{F(z)\} \). Then the Schwarz potential for \( \Omega \) is \( U(x, y) = V(x, y) + \text{const.} \).

One of the examples carried through this procedure in [53] is the family of “limacons” from Example 1. The result is that the Schwarz potential can be expanded about the origin as \( w(x_1, x_2, x_3, x_4) = A_2(a, b) \left( \frac{\partial}{\partial x_1} \right)^2 |x|^{-2} + A_1(a, b) \frac{\partial}{\partial x_1} |x|^{-2} + A_0(a, b) |x|^{-2} + H(x) \), where \( H(x) \) is harmonic and \( A_2(a, b) = -b^2a^4/12 \), \( A_1(a, b) = ba^2(a^2 + 2b^2)/2 \), and \( A_0(a, b) = -(a^4 + 6a^2b^2 + 2b^4)/2 \). We can interpret one-parameter slices of this family as a Laplacian growth if we further extend problem 2.1.1 to allow for “multi-poles” (see [36] for a discussion of multi-pole solutions in the plane). If we want a Laplacian growth with just a simple sink then according to the dynamics-of-singularities imposed by Theorem 2.2.2, we need to choose the time-dependence of \( a \) and \( b \) so that the only singularity whose coefficient changes is the fundamental-solution type singularity \( A_0(a, b) |x|^{-2} \). Thus, where \( C_1, C_2 \) are constants, we need to have:

\[
\begin{align*}
A_2(a(t), b(t)) &= C_2 \\
A_1(a(t), b(t)) &= C_1
\end{align*}
\]  

Unfortunately, solutions \( a \) and \( b \) of this system are locally constant so that \( A_0 \) must then be constant and the whole surface does not move at all. The other examples of axially symmetric domains considered in [53] also require introducing multi-poles or even a continuum of singularities, otherwise the conditions imposed by simple sources/sinks leads to a similarly overdetermined system. Roughly speaking, the difficulty is that \( f(z) \) from Step 1 above generally has more singularities than \( S(z) \). Thus, if a class of domains in the plane has enough parameters to control the singularities and obtain a Laplacian growth, then rotation into \( \mathbb{R}^4 \) introduces more singularities which must be controlled with the same number of parameters.

There are exceptions: we can describe some exact solutions involving a simple
source and sink (no multipoles). Consider the hypersurfaces of revolution obtained
by rotation from a family of curves whose Schwarz functions have two simple poles
at \( z = \pm 1 \) (with not necessarily equal residues). This is a two-parameter family
of surfaces; as parameters, we can take the residues of the Schwarz function of the
profile curves. Let \( \Omega \) denote the domain in the plane bounded by the profile curve.
The Schwarz function has the form

\[
S(z) = \frac{A}{z-1} + \frac{B}{z+1} + c(A, B) + d(A, B)z + z^2H(z, A, B),
\]

where \( H(z, A, B) \) is analytic in \( \Omega \). In what follows, we will suppress the dependence
on \( A \) and \( B \) of higher-degree coefficients. Following Steps 1 through 3 above, we have

\[
f(z) = \frac{i}{2} S(z) \cdot (S(z) - 2z)
\]

\[
= \frac{i}{2} \left( \frac{A^2}{(z-1)^2} + \frac{B^2}{(z+1)^2} + \frac{C_1}{z-1} - \frac{C_2}{z+1} + H_1(z) \right),
\]

where \( H_1(z) \) is analytic in \( \Omega \). Then for Step 2 we need a primitive function for \( f(z) \)
which is

\[
F(z) = \frac{i}{2} \left( \frac{A^2}{(z-1)^2} + \frac{B^2}{(z+1)^2} + C_1 \text{Log}(z-1) - C_2 \text{Log}(z+1) + H_2(z) \right),
\]

where \( H_2(z) \) is analytic in \( \Omega \).

Then for Step 3 we have \( V(x, y) = \text{Re}\{F(z)\} = \frac{A^2y}{(x-1)^2+y^2} + \frac{B^2y}{(x+1)^2+y^2} + C_1 \text{arg}(z-1) + C_2 \text{arg}(z+1) + H_3(z). \)

If we can vary \( A \) and \( B \) in a way that keeps \( C_1 \) and \( C_2 \) constant, then the time-
derivative of the Schwarz function will satisfy the dynamics-of-singularities condition.
This seems at first to be another overdetermined problem, but actually \( C_1 \) and \( C_2 \) must
be equal! Otherwise, the two branch cuts of \( C_1 \text{arg}(z-1) \) and \( C_2 \text{arg}(z+1) \) will not
cancel each other outside the interval \([-1, 1]\), and the Schwarz potential will become
singular on the surface itself. This cannot happen since the surface has no points
(in \( \mathbb{R}^n \)) that are characteristic for the Cauchy problem. Therefore, since \( C_1 \) and \( C_2 \)
Figure 2.2: A profile of an axially-symmetric solution in $\mathbb{R}^4$ with injection at one point and suction at another. The initial curve is plotted in bold.

are equal, we spend only one dimension of our parameter space controlling the “non-physical” segment of singularities. This leaves freedom for the “physical” singularities to move, at least locally, along a one-dimensional submanifold of parameters. Figure 2.2 shows the evolution of the profile curve for a typical example that can be obtained in this way.

We omit the cumbersome formulae for the time-dependence of coefficients in the algebraic description of such exact solutions. The two-parameter family of hypersurfaces from which they are selected can be described by the solution set of:

$$\frac{(x_1 + \frac{h}{2(a^2-h^2)}) - h \cdot ((x_1 + \frac{h}{2(a^2-h^2)})^2 + x_2^2 + x_3^2 + x_4^2)^2}{a^2}$$

$$\frac{(4(a^2-h^2)^2 - a^2)(x_2^2 + x_3^2 + x_4^2)}{a^2(a^2-h^2)} = \left(\frac{x_1 + \frac{h}{2(a^2-h^2)}}{x_2^2 + x_3^2 + x_4^2}\right)^2.$$

Similarly, one can obtain examples where the Schwarz function has three or
more simple poles. Again the suction/injection rates will have to occur in a prescribed way or else the time-derivative of the Schwarz potential will have singular segments which are difficult to interpret physically.

**Remark:** The rigidity of the inter-dependence of injection/suction rates in the above example is made less severe by the fact that the initial and final domains only depend on the total quantities injected and removed at the source and sink respectively, and they are independent of the rates and order of work of the source and sink (see [105]: the proof extends word for word to higher dimensions). Thus, injection and suction can happen in any manner, say one at a time, and we will lose the “movie” but retain the final domain.

In the next section we will be interested in examples that correspond to axially symmetric surfaces that do not intersect the axis of symmetry. For instance, to generate a torus, we can choose the profile curve to be a circle of radius $R$ and center $a$, $a > R > 0$. The Schwarz function is $S(z) = \frac{R^2}{z-a} - ai$. Step 1 gives $f(z) = i/2(\frac{R^2}{z-a} - ai)(\frac{R^2}{z-a} - ai - 2z)$. Step 2 gives $F(z) = \frac{-iR^4}{2(z-a)} + 2R^2 a \log(z - ai) + H(z)$, where $H(z)$ is analytic. Step 3 gives $V(z) = \frac{-R^4(y-a)}{2(z-a)^2} + 2R^2 a \log |z - ai| + R(z)$, where $R(z)$ is free of singularities. Finally, the singular part of $U(x, y)$ is $\frac{R^4}{2ay} \frac{\partial}{\partial y} \frac{1}{x^2+y^2} + \frac{2R^2}{y} \log |z - ai|$.

This calculation for the Schwarz potential of the four-dimensional torus was carried out in [2] and discussed in connection with a classical mean-value-property for polyharmonic functions.

### 2.3.3 Examples of elliptic growth

Examples of axially-symmetric, four-dimensional Laplacian growth also solve certain elliptic growth problems in two and three dimensions. The two-dimensional profile solves the planar elliptic growth problem where the filtration coefficient $\lambda = 1$ is constant, and the porosity $\rho(x, y) = y^2$. Indeed, we can check that Theorem 2.2.3 is satisfied. The Schwarz potential $U(x, y)$, reduced to two variables, satisfies the equation $\Delta U + \frac{2U_y}{y} = 0$. Since $\text{div}(y^2 \nabla U) = y^2 \Delta U + 2y U_y$, then $U$ solves the Cauchy problem
\[
\begin{aligned}
div (y^2 \nabla U) &= 0 \text{ near } \Gamma \\
U|_\Gamma &= q \\
\nabla U|_\Gamma &= \nabla q 
\end{aligned}
\tag{2.3.20}
\]

with \(q(x, y) = (x^2 + y^2)/8\) solving the Poisson equation \(\text{div} (y^2 \nabla q) = y^2\).

The three dimensional surfaces of revolution generated by the same profile curves solve a three-dimensional elliptic growth if we choose \(\lambda = 1\) again constant and porosity \(\rho(x, y, z) = \sqrt{y^2 + z^2}\).

It is most interesting when the domain, at least initially, avoids the line \(\{y = 0\}\) where \(\rho(x, y)\) vanishes. Consider, for instance, a circle of radius \(R\) centered at \(ai\). This corresponds to the calculation at the end of Section 2.3 for the four-dimensional torus. Accordingly, a shrinking circle can be generated by a simple source combined with a “dipole flow” positioned at the center of the circle.

A similar calculation applies more generally when \(\lambda(x, y) = y^{2-m}\), \(\rho(x, y) = y^m\), with \(m\) a positive integer, and we can consider more general domains than circles. For instance, a well-known classical solution of the Laplacian growth in the plane involves domains \(\Omega_t\) conformally mapped from the unit disc by polynomials. Physically the solution has a single sink positioned at the image of the origin under the conformal map. The Schwarz function of such an \(\Omega_t\) is meromorphic except at the sink where its highest order pole coincides with the degree of the polynomial. So, \(S(z) = \sum_{i=1}^{k} \frac{a_i}{z_i} + H(z)\), where \(H(z)\) is analytic in \(\Omega_t\). The solution \(q\) of \(\text{div} (y^2 \nabla q) = y^m\) is \(q(x, y) = \frac{y^{m+2}}{(m+2)(m+3)}\). To solve for \(U(x, y)\) we first notice that \(V(x, y) = yU(x, y)\) is harmonic and solves a Cauchy problem with data \(yq(x, y) = \frac{y^{m+3}}{(m+2)(m+3)}\). Thus, \(\partial_z V = -\frac{i}{2} \frac{y^{m+2}}{(m+2)} = -\frac{i}{2} \frac{(z - S(z))^{m+2}}{(m+2)(2i)^m}\) can be analytically continued away from the boundary. As a result, the flow can be generated by a combination of “multipoles” positioned at the same point of order not exceeding \(k(m + 2)\). This resembles the result of I. Loutsenko [75] stating that the same evolution can be generated by multipoles of a certain order under an elliptic growth where \(\rho = 1\) constant and \(\lambda = \frac{1}{y^2}\), with \(p\) a positive integer.

This fails, in an interesting way, for negative values of \(m\). For instance, if
Figure 2.3: An elliptic growth with multi-poles of order up to 3 positioned at \( z = i \). The Schwarz function has a moving singularity.

\[ \lambda(x, y) = y^4 \text{ and } \rho(x, y) = 1/y^2, \]

then a circle of shrinking radius \( R \) centered at \( ai \) is not generated by multipoles positioned at \( ai \). Instead, the generalized Schwarz potential \( U(x, y) \) has singularities at the moving point \( i\sqrt{a^2 - r^2} \). If we instead allow the center of the shrinking circle to move in a way that keeps \( \sqrt{a^2 - r^2} \) constant, then the evolution can be generated by multi-poles at this point of order up to 3 (see figure 2.3). To reiterate, for this evolution of shrinking circles with moving center, the generalized (elliptic) Schwarz potential is singular at a stationary point while the analytic Schwarz function has a moving singularity. Such an example has been anticipated in [59], where a system of nonlinear ODEs was given governing both the strength and the moving position of the Schwarz function’s singularities under an elliptic growth.

### 2.4 The Schwarz potential in \( \mathbb{C}^n \)

The previous sections call for a deeper look into the singularities that can arise from Cauchy’s problem for the Laplace equation. Certain techniques can only be applied if the problem is “complexified”. According to the algebraic form of the initial surface
and data, we can allow each variable to assume complex values. We then consider
the Cauchy problem in $\mathbb{C}^n$ where the original, physical problem becomes a relatively
small slice. We can loosely describe the advantage of a $\mathbb{C}^n$-viewpoint as follows:

Consider first the wave equation in $\mathbb{R}^n$. If the initial surface is non-singular and
algebraic and the data is real-entire, then where can the solution have singularities? A
singularity can propagate to some point if the backwards light-cone from this point is
tangent to the initial surface. The same is true, at least heuristically, for the Laplace
equation, except the “light cone” emanating from a point $x_0$ is the isotropic cone
$\{\sum_{i=1}^n (z_i - x_i^0)^2 = 0\}$, residing in $\mathbb{C}^n$ and only touching $\mathbb{R}^n$ at $x_0$. Thus, the
initial source of the singularity is located on the part of the complexified surface only
visible if the problem is lifted to $\mathbb{C}^n$.

“Leray’s principle” gives the general, precise statement of the above description
of propagation of singularities. It is only known to be rigorously true in a neighbor-
hood of the initial surface. In two dimensions, where the Schwarz potential can be
calculated easily, one can check examples to see if Leray’s principle gives correct global
results (it seems to). At the same time, this gives an appealing geometric “explana-
tion” of the source of singularities and reveals that they are the “foci” of the curve in
the sense of Plücker (see [52, Section 1] and the references therein).

In arbitrary dimensions, G. Johnsson has given a global proof [52] of Leray’s
principle for quadratic surfaces. As Johnsson points out, a major step in the proof
relies on the fact that the gradient of a quadratic polynomial is linear, so that a
certain system of equations can be inverted easily. This becomes much more difficult
for surfaces of higher degree, indeed, perhaps prohibitively difficult even for specific
examples.

In this section we consider a family of surfaces of degree four, the surfaces of
revolution generated by the Neumann ovals from Example 2 in Section 2.3. Leray’s
principle gives an appealing geometric “explanation” for the singularities of the Schwarz
potential in this example, but for the rigorous proof, we apply an ad hoc combination
of other $\mathbb{C}^n$ techniques (actually $\mathbb{C}^2$, after taking into account axial symmetry).
We require the following two local extension Theorems.

**Theorem 2.4.1 (Zerner)** Let $v$ be a holomorphic solution of the equation $Lv = 0$ in a domain $\Omega \subset \mathbb{C}^n$ with $C^1$ boundary, and assume that the coefficients of $L$ are holomorphic in $\overline{\Omega}$. Let $z_0 \in \partial \Omega$. If $\partial \Omega$ is non-characteristic at $z_0$ with respect to $L$ then $v$ extends holomorphically into a neighborhood of $z_0$.

In order to define non-characteristic for a real hypersurface given by the zero set of $\phi$, suppose the polynomial $P(x, \nabla \phi)$ expresses the leading order term of $L$. Then $\Gamma$ is characteristic at $p$ if $P(x, \nabla \phi)$ vanishes at $x = p$. For instance, if $L$ is the Laplacian then the condition for $\{\phi = 0\}$ to be characteristic is $\sum_{i=1}^n \phi_{x_i}^2 = 0$.

In order to state the next theorem (see [31] for the proof), $M$ is a hypersurface (of real codimension one) dividing a domain $\Omega$ into $\Omega_+$ and $\Omega_-$. Also, we suppose the leading order part $P(Z, D)$ of the differential operator $L$ factors as $P(Z, D) = A(Z)Q(Z, D)$, where $A(Z)$ is holomorphic in $\Omega$. Let $X$ denote the everywhere-characteristic zero set of $A(Z)$ (having complex codimension one).

**Theorem 2.4.2 (Ebenfelt, Khavinson, Shapiro)** Assume $M$ non-characteristic for $Q(Z, D)$ at $p_0 \in M$, and that the holomorphic hypersurface $X$ is non-singular at $p_0$ and meets $M$ transversally at that point. Then any holomorphic solution $v$ in $\Omega_-$ of $Lv = 0$ extends holomorphically across $p_0$.

**Theorem 2.4.3** Let $W(x)$ be the Schwarz potential of the boundary $\Gamma$ of the domain $\Omega := \{x \in \mathbb{R}^n : (\sum_{i=1}^n x_i^2)^2 - a^2 \sum_{i=1}^n x_i^2 - 4x_1^2 < 0\}$. Then $W$ can be analytically continued throughout $\Omega \setminus B$ where $B$ is the segment $\{x \in [-1, 1] \times \mathbb{R}^{n-1} \}$.

**Remark (i):** In the plane, it is easily seen that $W$ is only singular at the endpoints of the segment (see Example 2 in Section 2.3). In $\mathbb{R}^4$, it is an example done by L. Karp [53], who showed that the Schwarz potential has two fundamental solution type singularities at the endpoints along with a uniform jump in the gradient across the segment.
Remark (ii): The three-dimensional consequence of this theorem is that if we take the surfaces of revolution generated by the Neumann ovals in Example 2 from Section 2.3 then the resulting evolution is a “Laplacian growth” generated by a pressure function having some distribution of singularities confined to the segment \( \{ x \in [-1, 1], y = 0, z = 0 \} \). This driving mechanism is still rather obscure though, so in the next section we describe an approximation by finitely many simple sinks.

Proof. We first recall that \( W(x) \) is real-analytic in a neighborhood of each nonsingular point of the initial surface (in \( \mathbb{R}^n \)). Indeed, if the surface is nonsingular, \( \nabla \Phi|_{\Gamma} \neq 0 \) so that \( ||\nabla \Phi||_{\Gamma} \neq 0 \) so that \( \Gamma \) is everywhere non-characteristic (in \( \mathbb{R}^n \)) for Laplace’s equation and the Cauchy-Kovalevskaya Theorem applies.

Next we write \( W(x_1, x_2, ..., x_n) = u(x, y) \) where \( y = \sqrt{x_2^2 + x_3^2 + ... + x_n^2} \) and \( x = x_1 \), and we recall the axially-symmetric reduction of Laplace’s equation: \( \Delta u + \frac{(n-2)u_y}{y} = 0 \). Since \( u \) solves a Cauchy problem for which the data and boundary are analytic, the problem can be lifted to \( \mathbb{C}^2 \). So \( u(x, y) \) can be viewed as the restriction to \( \mathbb{R}^2 \) of the solution \( u(X, Y) \), valid for \( X \) and \( Y \) each taking complex values.

We make the linear change of variables \( X = \frac{z+w}{2}, \ Y = \frac{z-w}{2i}; \ u_{zw} + \frac{(n-2)(u_x-u_y)}{z-w} = 0 \). Next we make another change of variables \( z = f(\xi), w = f(\eta), \) using the conformal map
\[
f(\xi) = \frac{(R^4 - 1)\xi}{R(R^2 - \xi^2)}
\]
from the unit disk to the profile of \( \Omega \) (for appropriate value of \( R \)) which is Neumann’s oval (see Figure 2.4).

Write \( v(\xi, \eta) = u(f(\xi), f(\eta)) \). Then \( v_\xi = u_x(f(\xi), f(\eta)) \cdot f'(\xi), \) and the equation satisfied by \( v \) is \( \frac{v_\xi}{f'(\xi)f'(\eta)} + \frac{n-2}{f'(\xi)-f'(\eta)} \left( \frac{v_\xi}{f'(\xi)} - \frac{v_\eta}{f'(\eta)} \right) = 0, \) or \( (f(\xi) - f(\eta))v_\xi + (n-2)f'(\eta)v_\xi - f'(\xi)v_\eta = 0. \) Upon clearing denominators, the leading-order part of the operator is
\[
\frac{(R^4 - 1)}{R}(R^2 - \xi^2)(R^2 - \eta^2) \left( \xi(R^2 - \eta^2) - \eta(R^2 - \xi^2) \right) \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta}.
\]
(2.4.21)

After these transformations, we arrive at a Cauchy problem posed on \( \{ \xi\eta = 1 \} \),
Figure 2.4: The conformal map from the disc to the Neumann oval. This simplifies the $\mathbb{C}^2$ geometry but makes the PDE more complicated.

with data $v = 1/2f(\xi)f(\eta)$, $v_\xi = f(\eta)f'(\xi)$, and $v_\eta = f(\xi)f'(\eta)$. According to the form of the leading-order term 2.4.21, the characteristic points of $\{\xi\eta = 1\}$ are $(\pm 1, \pm 1)$, $(\pm R, \pm 1/R)$, $(\pm 1/R, \pm R)$.

The restriction of $v$ to the non-holomorphic set $\eta = \bar{\xi}$ corresponds to the original problem. Since $W(x)$ was observed to be analytic near the initial surface, $v(\xi, \eta)$ is analytic in a $\mathbb{C}^2$ neighborhood of the circle $\{\xi\eta = 1, \eta = \bar{\xi}\}$, even at the characteristic points $(\pm 1, \pm 1)$. We analytically continue $v$ from each point on this circle along a radial path toward the origin. Let $P(\theta) = (e^{i\theta}, e^{-i\theta})$. We consider two cases. For the first case, $\theta \neq 0$ and $\neq \pi$, and $v$ can be continued up to the origin. For the second case, when $\theta = 0$ or $= \pi$, the analytic continuation stops at $(1/R, 1/R)$ and $(-1/R, -1/R)$ respectively. Thus, $v$ can be analytically continued to the disk minus the segment joining these two points. This transforms (by inverting the conformal map) to the statement we are trying to prove about $W$. For each case we construct a globalizing family in a similar manner to the proof of the Bony-Schapira Theorem [19].

CASE 1: Suppose $\theta \neq 0$ and $\neq \pi$ so that $(e^{i\theta}, e^{-i\theta})$ is not on the pre-image of the axis of symmetry of $\Omega$. Let $0 < s < 1$ be arbitrary. We establish the continuability of $v$ to a neighborhood of the segment $\{tP(\theta), s \leq t \leq 1\}$. Consider the path $\gamma_\theta := \{(re^{i\theta}, \frac{1}{re^{i\theta}}), s \leq r \leq \frac{1}{s}\}$ which is on the initial surface $\{\xi\eta = 1\}$, and passes through none of the points that are characteristic for the operator 2.4.21. Thus, by
the Cauchy-Kovalevskaya Theorem, \( v \) is analytic in a neighborhood of each point on \( \gamma_\theta \). Choose \( \varepsilon_1 > 0 \) small enough so that \( v \) is analytic in a \( \varepsilon_1 \)-neighborhood of \( \gamma_\theta \). Let \( \Omega_0 \) denote this tubular (\( \mathbb{C}^2 \)) domain of analyticity.

For \( 1 \geq T \geq s \), let \( N_{\varepsilon_2}(T) \) denote the \( \varepsilon_2 \)-neighborhood of the segment \( \{tP(\theta), T \leq t \leq 1\} \). Since for each \( 1 \geq t \geq s \), the characteristic (for the operator 2.4.21) lines through \( tP(\theta) \) also intersect \( \gamma_\theta \), then for a small enough \( \varepsilon_2 \), any characteristic line that intersects \( N_{\varepsilon_2}(T) \) also intersects \( \Omega_0 \). Let \( \Omega_T \) be the set \( \left( \text{co}(\Omega_0 \cup N_{\varepsilon_2}(T)) \setminus \text{co}(\Omega_0) \right) \cup \Omega_0 \), where \( \text{co}(S) \) denotes the convex hull of a set \( S \).

**Claim 2.4.4** For points on \( \partial \Omega_T \setminus \partial \Omega_0 \), the tangent plane is a supporting hyperplane for \( \Omega_T \).

*Proof.* [proof of Claim] By definition, \( \Omega_T \subset \text{co}(\Omega_0 \cup N_{\varepsilon_2}(T)) \), and these two sets share a boundary near points \( p \in \partial \Omega_T \setminus \partial \Omega_0 \). The tangent plane at \( p \in \partial \Omega_T \setminus \partial \Omega_0 \) is also a tangent plane for \( \partial \text{co}(\Omega_0 \cup N_{\varepsilon_2}(T)) \). By convexity, it must be a supporting hyperplane for \( \text{co}(\Omega_0 \cup N_{\varepsilon_2}(T)) \). It is then also a supporting hyperplane for the subset \( \Omega_T \).

Let \( E := \{T : v \text{ can be analytically continued to } \Omega_T\} \). Since \( 1 \in E \), \( E \) is non-empty. We will show that \( E \) is both open and closed relative to \( [s, 1] \) and is therefore equal to \([s, 1]\). The fact that \( E \) is closed follows from the fact that the domains \( \Omega_T \) are continuous and nested. To see that \( E \) is open, we apply Zerner’s Theorem. Suppose \( T \in E \), i.e., \( v \) extends to \( \Omega_T \). By the Claim, the tangent plane \( P \) to \( \Omega_T \) at \( p \in \partial \Omega_T \setminus \partial \Omega_0 \) is a supporting hyperplane. We must have that \( P \) passes through \( N_{\varepsilon_2}(s) \). Otherwise, \( P \) is a supporting hyperplane for both \( \Omega_0 \) and \( N_{\varepsilon_2}(s) \) and, therefore, for any segment joining points in each of these sets (a contradiction). Since \( P \) passes through \( N_{\varepsilon_2}(s) \) and not \( \Omega_0 \), it is non-characteristic. By Theorem 2.4.1, \( v \) extends to a neighborhood of \( p \).

**CASE 2:** Suppose \( \theta = 0 \) or \( \theta = \pi \). For specificity, say \( \theta = 0 \). Then \( \gamma_0 := \{(r, \frac{1}{r}), s \leq r \leq \frac{1}{s}\} \) passes through the characteristic point \((1, 1)\). We have already observed, though, that \( v \) is analytic in a neighborhood of the point \((1, 1)\). If \( s \leq 1/R \),

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then $\gamma_0$ also passes through the characteristic points $(R, 1/R)$, and $(1/R, R)$. So, we let $s > 1/R$. Then we can still choose an $\varepsilon_1 > 0$ small enough that $v$ is analytic in a $\varepsilon_1$-neighborhood of $\gamma_0$. We use $\Omega_0$ again to denote this domain of analyticity. We can proceed in the same way as in the previous case, defining $N_{\varepsilon_2}(T)$ and $\Omega_T$, except now the set of characteristic points $\xi = \eta$ intersects the advancing boundary of $\Omega_T$ for every value of $T$. Zerner’s Theorem fails at this point of intersection, but Theorem 2.4.2 applies since the complex line $\xi = \eta$ is transversal to each of the boundaries $\partial \Omega_T$. Thus, we can again prove that the set $E$ is open and closed relative to $[s, 1]$, but recall that we assumed $s > 1/R$.

The method of proof can clearly be applied to other examples having axial symmetry. In a future study, we hope to apply $\mathbb{C}^n$ techniques to some surfaces of degree four that do not have axial-symmetry, such as the family of examples in $\mathbb{R}^3$, 

$$(x^2 + y^2 + z^2)^2 - (a^2x^2 + b^2y^2 + c^2z^2) = 0,$$ 

with $a > b > c > 0$. These are three-dimensional versions of the Neumann oval without axial-symmetry.

### 2.5 Quadrature domains

In order to limit the number of definitions in the exposition of our main results, we have so far avoided explicit mention of “quadrature domains”, but it would be remiss not to discuss this important connection. Also, this will allow us to give a detailed approximate description of the second remark made after the statement of Theorem 2.4.3.

First we consider the plane. A domain $\Omega$ is a quadrature domain if it admits a formula expressing the area integral of any analytic function $f$ belonging to, say $L^1(\Omega)$, as a finite sum of weighted point evaluations of the function and its derivatives. i.e.

$$\int_\Omega f dA = \sum_{m=1}^N \sum_{k=0}^{n_k} a_{mk} f^{(k)}(z_m)$$

where $z_i$ are distinct points in $\Omega$ and $a_{mk}$ are constants independent of $f$. 

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Suppose \( \Omega \) is a bounded, simply-connected domain with non-singular, analytic boundary. Then the following are equivalent. Moreover, there are simple formulas relating the details of each.

(i) \( \Omega \) is a quadrature domain.

(ii) The exterior logarithmic potential of \( \Omega \) is equivalent to that which is generated by finitely many interior points (allowing multipoles).

(iii) The Schwarz function of \( \partial \Omega \) is meromorphic in \( \Omega \).

(iv) The conformal map from the disk to \( \Omega \) is rational.

For the equivalence of (i) and (iii), see [24, Ch. 14]. For the equivalence of (i), (ii), and (iv), see [105, Ch. 3].

In higher dimensions, one simply replaces “analytic” with “harmonic” in the definition of quadrature domain. In condition (ii), “logarithmic” becomes “Newtonian”. In higher dimensions, “multipole” refers to a finite-order partial derivative of the fundamental solution to Laplace’s equation. In condition (iii), “Schwarz function” becomes “Schwarz potential”, and instead of “meromorphic” the Schwarz potential must be real-analytic except for finitely many “multipoles” (as described above). Then the equivalence of (i), (ii), and (iii) persists in higher dimensions (see [62, Ch. 4]). Condition (iv) of course does not extend.

If the initial domain of a Laplacian growth is a quadrature domain, then it will stay a quadrature domain by virtue of the equivalence of (i) and (iii) combined with Theorem 2.2.2. Moreover, according to the formulas (omitted here) relating the details of (i) and (iii), the consequent time-dependence of the quadrature is the content of Richardson’s Theorem. In the plane, the quadrature domain can be reconstructed from its quadrature formula, and quadrature domains are dense within natural classes of Jordan curves; the smoother the class, the stronger the topology in which they are dense (see [16] and the references therein).

**Theorem 2.5.1 (Richardson)** If \( \Omega_t \) is a Laplacian growth with \( m \) sinks located at
$x_i$ with rates $Q_i$, then for any harmonic function $u$

$$\frac{d}{dt} \int_{\Omega_t} u dV = - \sum_{i=1}^{m} Q_i u(x_i)$$

If the initial domain is not a quadrature domain, then the connection of Theorem 2.2.2 to Richardson’s Theorem requires defining quadrature domains in the wide sense, allowing the quadrature formula to consist of a distribution with compact support contained in $\Omega$ (see [62] and [96]). For such generalized quadrature domains, a distribution with minimal support is called a “mother body” for the domain. The singularity set of the Schwarz potential gives a supporting set for the “mother body”.

Work of Gustafsson and Sakai guarantees existence of a quadrature domain in $\mathbb{R}^n$ satisfying a prescribed quadrature formula, but besides the special examples in $\mathbb{R}^4$ the only explicit example for $n > 2$ is a sphere. Moreover, little qualitative information is known about quadrature domains in higher dimensions besides that the boundary is analytic. For instance, it is not even known whether quadrature domains are generally algebraic (in the plane, it follows from condition (iv). We make the following conjecture, where we mean “quadrature domain” in the classical, restricted sense (otherwise the statement is trivial, since any analytic, non-singular surface is a quadrature domain in the wide sense):

**Conjecture 2.5.2** In dimensions greater than two, there exist quadrature domains that are not algebraic.

For the three-dimensional example from Theorem 2.4.3, we were able to isolate the singularities for the Schwarz potential to a segment inside. Thus, $\Omega$ is a quadrature domain in the wide sense and has a mother body supported on this segment. We approximate the distribution using a finite number of points on this segment. Choosing the points $x_k = -1 + k/2$, $k = 0, 1, .., 4$, we numerically integrate 20 harmonic basis functions (writing them in terms of Legendre polynomials) over $\Omega$. If we assume a quadrature formula involving point evaluations at the points $(x_k, 0, 0)$, then we have an overdetermined linear system for the coefficients (20 equations and 5 unknowns).
We take two surfaces, and solve the least squares problem for the coefficients (using the same 5 points). Then the two surfaces can be approximately described as the boundaries of initial and final domains driven by sinks at these points, where the total amount removed is given by the decrease in quadrature weight.

![Image](image.png)

Figure 2.5: The profile of a supposed initial \((a = 1)\) and final \((a = 2)\) domain. The driving mechanism to generate the smaller domain starting from the larger can be approximated by certain amounts of suction at the indicated points.

Suppose \(\Omega_{\text{initial}}\) is given by \(a = 2\) (see statement of Theorem 2.4.3) and \(\Omega_{\text{final}}\) is given by \(a = 1\). Then of the total volume extracted, according to the approximate description 81% is removed at the points \((\pm 1, 0, 0)\), 15% at the points \((\pm 1/2, 0, 0)\), and 4% at the origin (See Figure 2.5). The accuracy of this description is reflected in the fact that the norm of the error vector for both least squares problems is on the order of \(10^{-4}\).

2.6 Concluding remarks

1. The equivalent definitions of quadrature domains listed in Section 2.5 indicate the possible reformulations of the Laplacian growth problem either in terms of potential theory or in terms of holomorphic PDEs. The potential theory approach has attracted more attention and has certain advantages such as weak formulations of Laplacian
growth. We have focused on the holomorphic PDE approach, and in Section 2.4 we
gave a glimpse of its main advantage: $C^n$ techniques.

2. The remarks at the end of Section 2.2 mention a consequence of the Schwarz
potential conjecture regarding Laplacian growth. It would be interesting if one could
obtain a partial result in the other direction along the lines of “Surfaces satisfying the
SP conjecture are preserved by Laplacian growth”. This would only be interesting in
higher dimensions, since the conjecture is already known to be true in the plane.

3. In Section 2.2, the discussion centered around the case when $\alpha = \lambda \rho = 1$ is
constant. It is natural to consider when $\alpha$ is a (fixed) non-constant entire function,
and ask if the solution $q$ to $\text{div}(\alpha \nabla q) = 1$ generalizes the data $\frac{1}{2}||x||^2$ in the Schwarz
potential conjecture. We make the following “elliptic Schwarz potential conjecture”.

**Conjecture 2.6.1** Suppose $\alpha > 0$ is entire and that $u$ solves the Cauchy problem on a
nonsingular analytic surface for $\text{div}(\alpha \nabla u) = 0$ with entire data. Then the singularity
set of $u$ is contained in the singularity set of $v$, the solution of the Cauchy problem
with data $q$, where $q$ is a solution of $\text{div}(\alpha \nabla q) = 1$.

One might object to generalizing unresolved conjectures. We should point out that
the Schwarz potential conjecture is true in the plane and simple to prove, whereas we
do not know if Conjecture 2.6.1 is true in the plane. One piece of evidence for the SP
conjecture is that the Schwarz potential develops singularities at every characteristic
point of the initial surface [56, Proposition 11.3]. A similar proof shows that this is
also true for $v$, where $\{\phi = 0\}$ being characteristic for the elliptic operator means
$\nabla \alpha \cdot \nabla \phi + \alpha \nabla \phi \cdot \nabla \phi = 0$.

4. At the end of Section 2.3 we have mentioned the fact that “injection is inde-
pendent of the order of work of sources and sinks”. In other words, the Laplacian
growths driven by different sources and sinks “commute” with each other. We can even
consider, say hypothetically, injection at each of infinitely many interior points of a
domain. Then we have infinitely many processes that commute with each other. This,
and especially its infinitesimal version which follows from the Hadamard variational
formula, has the form of an “integrable hierarchy”. To use the preferred language in this setting, we have a “commuting set of flows with respect to infinitely many generalized times” (the “times” are the amounts that have been injected into each of infinitely many sources). This holds in arbitrary dimensions but has recently attracted attention in two dimensions where it is directly connected to certain integrable hierarchies in soliton theory (see [81], [70] and [103]). Aspects of the higher-dimensional case and possible connections to other integrable systems seem completely unexplored.

5. Quadrature domains have also appeared, often only implicitly, in solutions of Euler’s equations. Physically, this area of fluid dynamics is much different, involving inviscid flow with vorticity. D. Crowdy has given a survey [23] of his own work and others’ (mainly in the two-dimensional case) where quadrature domains have been applied to vortex dynamics.

The ellipsoid is an example of a quadrature domain in the wide sense for which the mother body has been calculated (see [62, Ch. 5]). The exterior gravitational potential of a uniform ellipsoid coincides with that of a non-uniform density supported on the two-dimensional “focal ellipse” of the ellipsoid. This fact was used by Dritschel et al [25] as a main step in developing a model for interaction of “quasi-geostrophic” meteorological vortices. Actually, they didn’t use the exact density of the mother body, but only the location of its support in order to choose a small number of point vortices that generate a velocity field approximating that of an ellipsoid of uniform vorticity. Determining the strength of the approximating point vortices is nothing more than interpolating the quadrature formula. Our calculation at the end of Section 2.5, and similar calculations, could have promise for extending the model in [25] to examples of non-ellipsoidal vortices. An important missing ingredient here is a stability analysis, which has been carried out for ellipsoids.

6. Our intuition for Conjecture 2.5.2 is based on two suspicions regarding the axially-symmetric case. (1) According to the singularities of the four dimensional rotation of a limacon considered in Section 2.3, the quadrature formula involves point evaluations up to a second-order partial derivative. On the basis of L. Karp’s procedure described
in Section 2.3, it seems that an axially-symmetric example involving only a point evaluation of the function and a first-order partial with respect to $x$ will have to be generated by a curve whose Schwarz function has an essential singularity at the origin. Then, the conformal map would be transcendental. (2) In $\mathbb{R}^3$ we expect the situation to be at least as bad. Following [44, Ch.s 4 and 5], one can write an integral formula involving a Gauss hypergeometric function for the solution of a Cauchy problem for an $n$-dimensional axially-symmetric potential. The three dimensional case of the formula has the same form as the four-dimensional case, except the involved hypergeometric function is transcendental instead of rational.
3 Algebraic Dirichlet Problems

3.1 The Search for Singularities of Solutions to the Dirichlet Problem: Recent Developments

This section is taken from the survey article [57] written jointly with Dmitry Khavinson and based on an invited talk delivered by Dmitry Khavinson at the CRM workshop on Hilbert Spaces of Analytic Functions held at CRM, Université de Montreal, December 8-12, 2008.

3.1.1 The main question

Let Ω be a smoothly bounded domain in \( \mathbb{R}^n \). Consider the Dirichlet Problem (DP) in Ω of finding the function \( u \), say, \( \in C^2(\Omega) \cap C(\overline{\Omega}) \) and satisfying

\[
\begin{align*}
\Delta u &= 0 \\
\left. u \right|_{\Gamma} &= v,
\end{align*}
\]  

(3.1.1)

where \( \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \) and \( \Gamma := \partial \Omega, v \in C(\Gamma) \). It is well known since the early 20th century from works of Poincare, C. Neumann, Hilbert, and Fredholm that the solution \( u \) exists and is unique. Also, since \( u \) is harmonic in \( \Omega \), hence real-analytic there, no singularities can appear in \( \Omega \). Moreover, assuming \( \Gamma := \partial \Omega \) to consist of real-analytic hypersurfaces, the more recent and difficult results on “elliptic regularity” assure us that if the data \( v \) is real-analytic in a neighborhood of \( \Omega \) then \( u \) extends as a real-analytic function across \( \partial \Omega \) into an open neighborhood \( \Omega' \) of \( \overline{\Omega} \). In two dimensions, this can be done using the reflection principle. In higher dimensions, the boundary
can be biholomorphically “flattened”, but this leads to a general elliptic operator for which the reflection principle does not apply. Instead, analyticity must be shown by directly verifying convergence of the power series representing the solution through difficult estimates on the derivatives (see [41]).

**Question** Suppose the data \( v \) is a restriction to \( \Gamma \) of a “very good” function, say an entire function of variables \( x_1, x_2, ..., x_n \). In other words, the data presents no reasons whatsoever for the solution \( u \) of (3.1.1) to develop singularities.

(i) Can we then assert that all solutions \( u \) of (3.1.1) with entire data \( v(x) \) are also entire?

(ii) If singularities do occur, they must be caused by geometry of \( \Gamma \) interacting with the differential operator \( \Delta \). Can we then find data \( v_0 \) that would force the worst possible scenario to occur? More precisely, for any entire data \( v \), the set of possible singularities of the solution \( u \) of (3.1.1) is a subset of the singularity set of \( u_0 \), the solution of (3.1.1) with data \( v_0 \).

### 3.1.2 The Cauchy Problem

An inspiration to this program launched by H. S. Shapiro and D. Khavinson in [64] comes from reasonable success with a similar program in the mid 1980’s regarding the analytic Cauchy Problem (CP) for elliptic operators, in particular, the Laplace operator. For the latter, we are seeking a function \( u \) with \( \Delta u = 0 \) near \( \Gamma \) and satisfying the initial conditions

\[
\begin{align*}
(u - v)|_{\Gamma} &= 0 \\
\nabla (u - v)|_{\Gamma} &= 0,
\end{align*}
\]

(3.1.2)

where \( v \) is assumed to be real-analytic in a neighborhood of \( \Gamma \). Suppose as before that the data \( v \) is a “good” function (e.g. a polynomial or an entire function). In that context, the techniques developed by J. Leray [73] in the 1950’s (and jointly with L. Garding and T. Kotake [42]) together with the works of P. Ebenfelt [33], G. Johnsson [52], and, independently, by B. Sternin and V. Shatalov [94] in Russia and
their school produced a more or less satisfactory understanding of the situation. To
mention briefly, the answer (for the CP) to question (i) in two dimensions is essentially
“never” unless \( \Gamma \) is a line while for (ii) the data mining all possible singularities of
solutions to the CP with entire data is \( v_0 = |x|^2 = \sum x_j^2 \) (see [62], [55], [93], and [56]
and references therein).

3.1.3 The Dirichlet problem: When does entire data imply entire solution?

Let us raise Question (i) again for the Dirichlet Problem: Does (real) entire data \( v \)
imply entire solution \( u \) of (3.1.1)?

In this section and the next, \( P \) will denote the space of polynomials and \( P_N \)
the space of polynomials of degree \( \leq N \). The following pretty fact goes back to the
19th century and can be associated with the names of E. Heine, G. Lamé, M. Ferrers,
and probably many others (cf. [56]). The proof is from [64] (cf. [10], [12]).

**Proposition 3.1.1** If \( \Omega := \{ x : \sum x_j^2 - 1 < 0, a_1 > ... > a_n > 0 \} \) is an ellipsoid, then
any DP with a polynomial data of degree \( N \) has a polynomial solution of degree \( \leq N \).

**Proof.** Let \( q(x) = \sum \frac{x_j^2}{a_j^2} - 1 \) be the defining function for \( \Gamma := \partial \Omega \). The (linear) map
\( T : P \rightarrow \Delta(qP) \) sends the finite-dimensional space \( P_N \) into itself. \( T \) is injective (by
the maximum principle) and, therefore, surjective. Hence, for any \( P, \deg P \geq 2 \) we
can find \( P_0, \deg P_0 \leq \deg P - 2 \). \( TP_0 = \Delta(qP_0) = \Delta P \). \( u = P - qP_0 \) is then the
desired solution.

The following result was proved in [64].

**Theorem 3.1.2** Any solution to DP (3.1.1) in an ellipsoid \( \Omega \) with entire data is also
desire.

Later on, D. Armitage sharpened the result by showing that the order and
the type of the data are carried over, more or less, to the solution [7]. The following
conjecture has also been formulated in [64].
**Conjecture 3.1.3** Ellipsoids are the only bounded domains in $\mathbb{R}^n$ for which Theorem 3.1.2 holds, i.e., ellipsoids are the only domains in which entire data implies entire solution for the DP (3.1.1).

In 2005, H. Render [86] proved this conjecture for all algebraically bounded domains $\Omega$ defined as bounded components of $\{ \phi(x) < 0, \phi \in \mathbb{P}_N \}$ such that $\{ \phi(x) = 0 \}$ is a bounded set in $\mathbb{R}^n$ or, equivalently, the senior homogeneous part $\phi_N(x)$ of $\phi$ is elliptic, i.e., $|\phi_N(x)| \geq C|x|^N$ for some constant $C$. For $n = 2$, an easier version of this result was settled in 2001 by M. Chamberland and D. Siegel [22]. Below we outline their argument, which establishes similar results as Render’s for the following modified conjecture.

**Conjecture 3.1.4** Ellipsoids are the only surfaces for which polynomial data implies polynomial solution.

**Remark:** We will return to Render’s Theorem below. For now let us note that, unfortunately, it already tells us nothing even in 2 dimensions for many perturbations of a unit disk, e.g., $\Omega := \{ x \in \mathbb{R}^2 : x^2 + y^2 - 1 + \varepsilon h(x, y) < 0 \}$ where, say, $h$ is a harmonic polynomial of degree $> 2$.

### 3.1.4 When does polynomial data imply polynomial solution?

Let $\gamma = \{ \phi(x) = 0 \}$ be a bounded, irreducible algebraic curve in $\mathbb{R}^2$. If the DP posed on $\gamma$ has polynomial solution whenever the data is a polynomial, then as Chamberland and Siegel observed, (a) $\gamma$ is an ellipse or (b) there exists data $f \in \mathbb{P}$ such that the solution $u \in \mathbb{P}$ of DP has $\deg u > \deg f$.

In case (b) $u - f|_\gamma = 0$ implies that $\phi$ divides $u - f$ by Hilbert’s Nullstellensatz, and, since $\deg u = M > \deg f$, $u_M = \phi_k g_l$ where $\phi_k$ and $u_M$ are the senior homogeneous terms of $\phi$ and $u$ respectively. The senior term of $u$ must have the form $u_M = a z^M + b \bar{z}^M$ since $u_M$ is harmonic. Hence, $u_M$ factors into linear factors and so must $\phi_k$. Hence $\gamma$ is unbounded. This gives the following result [22].
Theorem 3.1.5 Suppose $\deg \phi > 2$ and $\phi$ is square-free. If the Dirichlet problem posed on $\{\phi = 0\}$ has a polynomial solution for each polynomial data, then the senior part of $\phi$, which we denote by $\phi_N$, of order $N$, factors into real linear terms, namely,

$$\phi_N = \prod_{j=0}^{n} (a_j x - b_j y),$$

where $a_j, b_j$ are some real constants and the angles between the lines $a_j x - b_j y = 0$, for all $j$, are rational multiples of $\pi$.

This theorem settles Conjecture 3.1.4 for bounded domains $\Omega \subseteq \{\phi(x) < 0\}$ such that the set $\{\phi(x) = 0\}$ is bounded in $\mathbb{R}^2$. However, the theorem leaves open simple cases such as $x^2 + y^2 - 1 + \varepsilon(x^3 - 3xy^2)$.

Example: The curve $y(y - x)(y + x) - x = 0$ (see figure 3.1) satisfies the necessary condition imposed by the theorem. Moreover, any quadratic data can be matched on it by a harmonic polynomial. For instance, $u = xy(y^2 - x^2)$ solves the interpolation problem (it is misleading to say “Dirichlet” problem, since there is no bounded component) with data $v(x, y) = x^2$. On the other hand, one can show (non-trivially) that the data $x^3$ does not have polynomial solution.

3.1.5 Dirichlet’s Problem and Orthogonal Polynomials

Most recently, D. Khavinson and N. Stylianopoulos showed that if for a polynomial data there always exists a polynomial solution of the DP (3.1.1), with an additional constraint on the degree of the solution in terms of the degree of the data (see below), then $\Omega$ is an ellipse [65]. This result draws on the 2007 paper of M. Putinar and N. Stylianopoulos [85] that found a simple but surprising connection between Conjecture 3.1.4 in $\mathbb{R}^2$ and (Bergman) orthogonal polynomials, i.e. polynomials orthogonal with respect to the inner product $\langle p, q \rangle_{\Omega} := \int_{\Omega} p \overline{q} \, dA$, where $dA$ is the area measure. To understand this connection let us consider the following properties:

1. There exists $k$ such that for a polynomial data of degree $n$ there always exists a polynomial solution of the DP (3.1.1) posed on $\Omega$ of degree $\leq n + k$. 

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2. There exists $N$ such that for all $m, n$, the solution of (3.1.1) with data $\bar{z}^m z^n$ is a harmonic polynomial of degree $\leq (N-1)m+n$ in $z$ and of degree $\leq (N-1)n+m$ in $\bar{z}$.

3. There exists $N$ such that orthogonal polynomials $\{p_n\}$ of degree $n$ on $\Omega$ satisfy a (finite) $(N+1)$-recurrence relation, i.e.

$$zp_n = a_{n+1,n}p_{n+1} + a_{n,n}p_n + \ldots + a_{n-N+1}p_{n-N+1},$$

where $a_{n-j,n}$ are constants depending on $n$.

4. The Bergman orthogonal polynomials of $\Omega$ satisfy a finite-term recurrence relation, i.e., for every fixed $k > 0$, there exists an $N(k) > 0$, such that $a_{k,n} = \langle zp_n, p_k \rangle = 0$, $n \geq N(k)$.

5. Conjecture 3.1.4 holds for $\Omega$.

Putinar and Stylianopoulos noticed that with the additional minor assumption that polynomials are dense in $L^2_\alpha(\Omega)$, properties (4) and (5) are equivalent. Thus,
they obtained as a corollary (by way of Theorem 3.1.5 from the previous section) that the only bounded algebraic sets satisfying property (4) are ellipses. We also have (1) \( \Rightarrow \) (2), (2) \( \Leftrightarrow \) (3), and (3) \( \Rightarrow \) (4). Khavinson and Stylianopoulos used the equivalence of properties (2) and (3) to prove the following theorem which has an immediate corollary.

**Theorem 3.1.6** Suppose \( \partial \Omega \) is \( C^2 \)-smooth, and orthogonal polynomials on \( \Omega \) satisfy a (finite) \( (N + 1) \)-recurrence relation, in other words property (3) is satisfied. Then, \( N = 2 \) and \( \Omega \) is an ellipse.

**Corollary 3.1.7** Suppose \( \partial \Omega \) is a \( C^2 \)-smooth domain for which there exists \( N \) such that for all \( m, n \), the solution of (3.1.1) with data \( \bar{z}^m z^n \) is a harmonic polynomial of degree \( \leq (N - 1)m + n \) in \( z \) and of degree \( \leq (N - 1)n + m \) in \( \bar{z} \). Then \( N = 2 \) and \( \Omega \) is an ellipse.

**Proof.** [Sketch of proof] First, one notes that all the coefficients in the recurrence relation are bounded. Divide both sides of the recurrence relation above by \( p_n \) and take the limit of an appropriate subsequence as \( n \to \infty \). Known results on asymptotics of orthogonal polynomials (see [101]) give \( \lim_{n \to \infty} \frac{p_{n+1}}{p_n} = \Phi(z) \) on compact subsets of \( \mathbb{C} \setminus \bar{\Omega} \), where \( \Phi(z) \) is the conformal map of the exterior of \( \Omega \) to the exterior of the unit disc. This leads to a finite Laurent expansion at \( \infty \) for \( \Psi(w) = \Phi^{-1}(w) \). Thus, \( \Psi(w) \) is a rational function, so \( \bar{\Omega} := \mathbb{C} \setminus \bar{\Omega} \) is an unbounded quadrature domain, and the Schwarz function (cf. [24], [97]) of \( \partial \Omega \), \( S(z) = \bar{z} \) on \( \partial \Omega \) has a meromorphic extension to \( \bar{\Omega} \). Suppose, for the sake of brevity and to fix the ideas, for example, that \( S(z) = cz^d + \sum_{j=1}^{M} \frac{c_j}{z-z_j} + f(z) \), where \( f \in H^\infty(\bar{\Omega}) \), and \( z_j \in \bar{\Omega} \). Since our hypothesis is equivalent to \( \Omega \) satisfying property (2) discussed above, the data \( \bar{z}P(z) = \bar{z} \prod_{j=1}^{n} (z - z_j) \) has polynomial solution, \( g(z) + h(\bar{z}) \) to the DP. On \( \Gamma \) we can replace \( \bar{z} \) with \( S(z) \). Write \( \bar{h}(\bar{z}) = h^\#(\bar{z}) \), where \( h^\# \) is a polynomial whose coefficients are complex conjugates of their counterparts in \( h \). We have on \( \Gamma \)

\[
S(z)P(z) = g(z) + h^\#(S(z)), \quad (3.1.3)
\]
which is actually true off $\Gamma$ since both sides of the equation are analytic. Near $z_j$, the
left-hand-side of this equation tends to a finite limit (since $S(z)P(z)$ is analytic in
$\hat{\Omega} \setminus \infty$!) while the right-hand-side tends to $\infty$ unless the coefficient $c_j$ is zero. Thus,

$$S(z) = cz^d + f(z). \quad (3.1.4)$$

Using property (2) again with data $|z|^2 = z\bar{z}$ we can infer that $d = 1$. Hence, $\hat{\Omega}$ is a
null quadrature domain. Sakai’s theorem [91] implies now that $\Omega$ is an ellipse.

**Remark:** It is well-known that families of orthogonal polynomials on the line satisfy
a 3-term recurrence relation. P. Duren in 1965 [26] already noted that in $\mathbb{C}$ the only
domains with real-analytic boundaries in which polynomials orthogonal with respect
to arc-length on the boundary satisfy 3-term recurrence relations are ellipses. L.
Lempert [72] constructed peculiar examples of $C^\infty$ non-algebraic Jordan domains in
which no finite recurrence relation for Bergman polynomials holds. Theorem 3.1.6
shows that actually this is true for all $C^2$-smooth domains except ellipses.

**3.1.6 Looking for singularities of the solutions to the Dirichlet Problem**

Once again, inspired by known results in the similar quest for solutions to the Cauchy
problem, one could expect, e.g., that the solutions to the DP (3.1.1) exhibit behavior
similar to those of the CP (3.1.2). In particular, it seemed natural to suggest that the
singularities of the solutions to the DP outside $\Omega$ are somehow associated with the
singularities of the Schwarz potential (function) of $\partial \Omega$ which does indeed completely
determine $\partial \Omega$ (cf. [62], [97]). It turned out that singularities of solutions of the DP are
much more complicated than those of the CP. Already in 1992 in his thesis, P. Ebenfelt
showed [30] that the solution of the following “innocent” DP in $\Omega := \{x^4 + y^4 - 1 < 0\}$
(the “TV-screen”)

$$\begin{cases} 
\Delta u = 0 \\
u|_{\partial \Omega} = x^2 + y^2 
\end{cases} \quad (3.1.5)$$
Figure 3.2: A plot of the “TV screen” \( \{ x^4 + y^4 = 1 \} \) along with the first eight singularities (plotted as circles) encountered by analytic continuation of the solution to DP (3.1.5).

has an infinite discrete set of singularities (of course, symmetric with respect to 90° rotation) sitting on the coordinate axes and running to \( \infty \) (see figure 3.2).

To see the difference between analytic continuation of solutions to CP and DP, note that for the former

\[
\frac{\partial u}{\partial z} \mid_{\Gamma} = v_z(z, \bar{z}) = v_z(z, S(z)),
\]

and since \( \frac{\partial u}{\partial z} \) is analytic, (3.1.6) allows \( u_z \) to be continued everywhere together with \( v \) and \( S(z) \), the Schwarz function of \( \partial \Omega \). For the DP we have on \( \Gamma \)

\[
\frac{\partial u}{\partial \bar{z}} = v(\bar{z})
\]
for \( u = f + \bar{g} \) where \( f \) and \( g \) are analytic in \( \Omega \). Hence, (3.1.7) becomes

\[
f(z) + g(S(z)) = v(z, S(z)). \tag{3.1.8}
\]

Now, \( v(z, S(z)) \) does indeed (for entire \( v \)) extend to any domain free of singularities of \( S(z) \), but (3.1.8), even when \( v \) is real-valued so that \( g = f \), presents a non-trivial functional equation supported by a rather mysterious piece of information that \( f \) is analytic in \( \Omega \). (3.1.8) however gives an insight as to how to capture the DP-solution’s singularities by considering the DP as part of a Goursat problem in \( \mathbb{C}^2 \) (or \( \mathbb{C}^n \) in general). The latter Goursat problem can be posed as follows (cf. [95]).

Given a complex-analytic variety \( \hat{\Gamma} \) in \( \mathbb{C}^n \), \( \hat{\Gamma} \cap \mathbb{R}^n = \Gamma := \partial \Omega \), find \( u : \sum_{j=1}^{n} \partial_{x_j}^2 u = 0 \) near \( \hat{\Gamma} \) (and also in \( \Omega \subset \mathbb{R}^n \)) so that \( u|_{\hat{\Gamma}} = v \), where \( v \) is, say, an entire function of \( n \) complex variables. Thus, if \( \hat{\Gamma} := \{ \phi(z) = 0 \} \), where \( \phi \) is, say, an irreducible polynomial, we can, e.g., ponder the following extension of Conjecture 3.1.3:

**Question:** For which polynomials \( \phi \) can every entire function \( v \) be split (Fischer decomposition) as \( v = u + \phi h \), where \( \Delta u = 0 \) and \( u, h \) are entire functions (cf. [39], [95])?

### 3.1.7 Render’s breakthrough

Trying to establish Conjecture 3.1.3, H. Render [86] made the following ingenious step. He introduced the real version of the Fischer space norm

\[
\langle f, g \rangle = \int_{\mathbb{R}^n} f \bar{g} e^{-|x|^2} dx, \tag{3.1.9}
\]

where \( f \) and \( g \) are polynomials. Originally, the Fischer norm (introduced by E. Fischer [39]) requires the integration to be carried over all of \( \mathbb{C}^n \) and has the property that multiplication by monomials is adjoint to differentiation with the corresponding multi-index (e.g., multiplication by \( \sum_{j=1}^{n} x_j^2 \) is adjoint to the differential operator \( \Delta \)). This
property is only partially preserved for the real Fischer norm. More precisely [86],

\[ \langle \Delta f, g \rangle = \langle f, \Delta g \rangle + 2(\deg(f) - \deg(g))\langle f, g \rangle \tag{3.1.10} \]

for homogeneous \( f, g \).

Suppose \( u \) solves the DP with data \(|x|^2\) on \( \partial \Omega \subseteq \{ P = 0 : \deg(P) = 2k, k > 1 \} \). Then \( u - |x|^2 = Pq \) for analytic \( q \), and thus \( \Delta^k(Pq) = 0 \). Using (3.1.10), this (non-trivially) implies that the real Fischer product \( \langle (Pq)_{m+2k}, q_m \rangle \) between all homogeneous parts of degree \( m + 2k \) and \( m \) of \( Pq \) and \( q \), respectively, is zero. By a tour de force argument, Render used this along with an added assumption on the senior term of \( P \) (see below) to obtain estimates from below for the decay of the norms of homogeneous parts of \( q \). This, in turn yields an if-and-only-if criterion for convergence in the real ball of radius \( R \) of the series for the solution \( u = \sum_{m=0}^{\infty} u_m \), \( u_m \) homogeneous of degree \( m \). Let us state Render’s main theorem.

**Theorem 3.1.8** Let \( P \) be an irreducible polynomial of degree \( 2k, k > 1 \). Suppose \( P \) is elliptic, i.e. the senior term \( P_{2k} \) of \( P \) satisfies \( P_{2k}(x) \geq c_P|x|^{2k} \), for some constant \( c_P \). Let \( \phi \) be real analytic in \( \{|x| < R\} \), and \( \Delta^k(P\phi) = 0 \) (at least in a neighborhood of the origin). Then, \( R \leq C(P, n) < +\infty \), where \( C \) is a constant depending on the polynomial \( P \) and the dimension of the ambient space.

**Remark:** The assumption in the theorem that \( P \) is elliptic is equivalent to the condition that the set \( \{ P = 0 \} \) is bounded in \( \mathbb{R}^n \).

**Corollary 3.1.9** Assume \( \partial \Omega \) is contained in the set \( \{ P = 0 \} \), a bounded algebraic set in \( \mathbb{R}^n \). Then, if a solution of the DP (3.1.1) with data \(|x|^2\) is entire, \( \Omega \) must be an ellipsoid.

**Proof.** Suppose not, so \( \deg(P) = 2k > 2 \), and the following (Fischer decomposition) holds: \(|x|^2 = P\phi + u, \Delta u = 0 \). Hence, \( \Delta^k(P\phi) = 0 \) and \( \phi \) cannot be analytically continued beyond a finite ball of radius \( R = C(P) < \infty \), a contradiction. \( \blacksquare \)
Caution: We want to stress again that, unfortunately, the theorem still tells us nothing for say small perturbations of the circle by a non-elliptic term of degree \( \geq 3 \), e.g., \( x^2 + y^2 - 1 + \varepsilon (x^3 - 3xy^2) \).

### 3.1.8 Back to \( \mathbb{R}^2 \): lightning bolts

Return to the \( \mathbb{R}^2 \) setting and consider as before our boundary \( \partial \Omega \) of a domain \( \Omega \) as (part of) an intersection of an analytic Riemann surface \( \hat{\Gamma} \) in \( \mathbb{C}^2 \) with \( \mathbb{R}^2 \). Roughly speaking, if say \( \partial \Omega \) is a subset of the algebraic curve \( \Gamma := \{(x, y) : \phi(x, y) = 0\} \), where \( \phi \) is an irreducible polynomial, then \( \hat{\Gamma} = \{(X, Y) \in \mathbb{C}^2 : \phi(X, Y) = 0\} \). Now look at the Dirichlet problem again in the context of the Goursat problem: Given, say, a polynomial data \( P \), find \( f, g \) holomorphic functions of one variable near \( \hat{\Gamma} \) (a piece of \( \hat{\Gamma} \) containing \( \partial \Omega \subseteq \hat{\Gamma} \cap \mathbb{R}^2 \)) such that

\[
\Delta u = 4 \frac{\partial^2}{\partial z \partial w} = 0
\]

where we have made the linear change of variables \( z = X + iY, \ w = X - iY \) (so \( \bar{w} = z \) on \( \mathbb{R}^2 = \{(X, Y) : X, Y \text{ are both real}\} \)). Obviously, \( \Delta u = 4 \frac{\partial^2}{\partial z \partial w} = 0 \) and \( u \) matches \( P \) on \( \partial \Omega \). Thus, the DP in \( \mathbb{R}^2 \) has become an interpolation problem in \( \mathbb{C}^2 \) of matching a polynomial on an algebraic variety by a sum of holomorphic functions in each variable separately. Suppose that for all polynomials \( P \) the solutions \( u \) of (3.1.11) extend as analytic functions to a ball \( B_{\Omega} = \{|z|^2 + |w|^2 < R_{\Omega}\} \) in \( \mathbb{C}^2 \). Then, if \( \hat{\Gamma} \cap B_{\Omega} \) is path connected, we can interpolate every polynomial \( P(z, w) \) on \( \hat{\Gamma} \cap B_{\Omega} \) by a holomorphic function of the form \( f(z) + g(w) \). Now suppose we can produce a compactly supported measure \( \mu \) on \( \hat{\Gamma} \cap B_{\Omega} \) which annihilates all functions of the form \( f(z) + g(w) \), \( f, g \) holomorphic in \( B_{\Omega} \) and at the same time does not annihilate all polynomials \( P(z, w) \). This would force the solution \( u \) of (3.1.11) to have a singularity in the ball \( B_{\Omega} \) in \( \mathbb{C}^2 \). Then, invoking a theorem of Hayman [47] (see also [56]), we would be able to assert that \( u \) cannot be extended as a real-analytic function to the real disk \( B_R \) in \( \mathbb{R}^2 \) containing \( \Omega \) and of radius \( \geq \sqrt{2}R \). An example of such annihilating measure supported by the vertices of a “quadrilateral”
was independently observed by E. Study [100], H. Lewy [74], and L. Hansen and H. S. Shapiro [46]. Indeed, assign alternating values ±1 for the measure supported at the four points $p_0 := (z_1, w_1)$, $q_0 := (z_1, w_2)$, $p_1 := (z_2, w_2)$, and $q_1 := (z_2, w_1)$. Then

$$
\int (f + g) d\mu = f(z_1) + g(w_1) - f(z_1) - g(w_2) + f(z_2) + g(w_2) - f(z_2) - g(w_1) = 0
$$

for all holomorphic functions $f$ and $g$ of one variable. This is an example of a closed lightning bolt (LB) with four vertices. Clearly, the idea can be extended to any even number of vertices.

**Definition 3.1.10** A complex closed lightning bolt (LB) of length $2(n + 1)$ is a finite set of points (vertices) $p_0, q_0, p_1, q_1, ..., p_n, q_n, p_{n+1}, q_{n+1}$ such that $p_0 = p_{n+1}$, and each complex line connecting $p_j$ to $q_j$ or $q_j$ to $p_{j+1}$ has either $z$ or $w$ coordinate fixed and they alternate, i.e., if we arrived at $p_j$ with $w$ coordinate fixed then we follow to $q_j$ with $z$ fixed etc.

For “real” domains lightning bolts were introduced by Arnold and Kolmogorov in the 1950s to study Hilbert’s 13th problem (see [67] and the references therein).

The following theorem has been proved in [14] (see also [15]).

**Theorem 3.1.11** Let $\Omega$ be a bounded simply connected domain in $\mathbb{C} \cong \mathbb{R}^2$ such that the Riemann map $\phi : \Omega \to \mathbb{D} = \{|z| < 1\}$ is algebraic. Then all solutions of the DP with polynomial data have only algebraic singularities which occur only at branch points of $\phi$ with the branching order of the former dividing the branching order of the latter iff $\phi^{-1}$ is a rational function. This in turn is known to be equivalent to $\Omega$ being a quadrature domain.

**Proof.** [Idea of proof:] The hypotheses imply that the solution $u = f + \bar{g}$ extends as a single-valued meromorphic function into a $\mathbb{C}^2$-neighborhood of $\hat{\Gamma}$. By another theorem of [14], one can find (unless $\phi^{-1}$ is rational) a continuous family of closed LBs on $\hat{\Gamma}$ of bounded length avoiding the poles of $u$. Hence, the measure with alternating values ±1 on the vertices of any of these LBs annihilates all solutions $u = f(z) +$
Figure 3.3: A Maple plot of the cubic $8x(x^2 - y^2) + 57x^2 + 77y^2 - 49 = 0$, showing the bounded component and one unbounded component (there are two other unbounded components further away).

$g(w)$ holomorphic on $\hat{\Gamma}$, but does not, of course, annihilate all polynomials of $z, w$. Therefore, $\phi^{-1}$ must be rational, i.e. $\Omega$ is a quadrature domain [95].

The author of this thesis ([77] or see Section 3.2 below) has recently constructed some other examples of LBs on complexified boundaries of planar domains which do not satisfy the hypothesis of Render’s theorem. The LBs validate Conjecture 3.1.3 and produce an estimate regarding how far into the complement $\mathbb{C}\setminus\overline{\Omega}$ the singularities may develop. For instance, the complexification of the cubic, $8x(x^2 - y^2) + 57x^2 + 77y^2 - 49 = 0$ has a lightning bolt with six vertices in the (non-physical) plane where $z$ and $w$ are real, i.e., $x$ is real and $y$ is imaginary (see figure 3.3 for a plot of the cubic in the plane where $x$ and $y$ are real and see figure 3.4 for the “non-physical” slice including the lightning bolt). If the solution with appropriate cubic data is analytically continued in the direction of the closest unbounded component of the curve defining $\partial\Omega$, it will have to develop a singularity before it can be forced to match the data on that component.
Figure 3.4: A lightning bolt with six vertices on the cubic \(2(z + w)(z^2 + w^2) + 67zw - 5(z^2 + w^2) = 49\) in the non-physical plane with \(z\) and \(w\) real, i.e. \(x\) real and \(y\) imaginary.
3.1.9 Further questions

In two dimensions one of the main results in [14] yields that disks are the only domains for which all solutions of the DP with rational (in \(x, y\)) data \(v\) are rational. The fact that in a disk every DP with rational data has a rational solution was observed in a senior thesis of T. Fergusson at U. of Richmond [90]. On the other hand, algebraic data may lead to a transcendental solution even in disks (see [32], also cf. [34]). In dimensions 3 and higher, rational data on the sphere (e.g., \(v = \frac{1}{x_1-a}, |a| > 1\)) yields transcendental solutions of (3.1.1), although we have not been able to estimate the location of singularities precisely (cf. [32]).

It is still not clear on an intuitive level why ellipsoids play such a distinguished role in providing "excellent" solutions to DP with "excellent" data. A very similar question, important for applications, (which actually inspired the program launched in [64] on singularities of the solutions to the DP) goes back to Raleigh and concerns singularities of solutions of the Helmholtz equation ([\(\Delta - \lambda^2\]u = 0, \(\lambda \in \mathbb{R}\)) instead. (The minus sign will guarantee that the maximum principle holds and, consequently, ensures uniqueness of solutions of the DP.) To the best of our knowledge, this topic remains virtually unexplored.

3.2 Dirichlet’s Problem and Complex Lightning Bolts

This section is taken from the paper [77] published in the journal “Computational Methods and Function Theory”. We investigate some of the topics surveyed in the previous section.

We consider the Dirichlet problem in the plane with entire data on algebraic curves. More specifically we will be interested in where singularities develop when a solution is continued analytically. Our approach involves finitely supported annihilating measures supported on finite sets called lightning bolts. Lightning bolts were first used in the real setting by Kolmogorov and Arnold to solve Hilbert’s 13th problem. After [77] was published, H. Render pointed out that the expository example
involving the wave equation is treated more generally in the paper of F. John [51].

### 3.2.1 Algebraically posed Dirichlet problems

Consider the Dirichlet problem posed inside (a connected component of) an algebraic curve with real-entire data. We can extend the solution to at least a neighborhood by “reflection” using the curve’s Schwarz function (see [97]). In this section, we treat the question of how far from the curve of initial data we can analytically continue the solution.

This question was investigated in [30] where Ebenfelt described the set of singularities developed by solutions with quadratic data on the curve $x^4 + y^4 = 1$. In a broader setting, he made the first step in confirming the conjecture of Khavinson and Shapiro ([64]) that the ellipse is the only curve for which all solutions with entire data are entire. Recently Render confirmed this conjecture in all dimensions for a large class of even-degree varieties ([86]). His method relies on difficult estimates from below on the elliptic operator acting on homogeneous parts to bound the radius of convergence. An upper bound for the maximum disc of convergence gives an upper bound for the maximum disc of analyticity.

We pursue here the technique used in [15] to show failure of analyticity of solutions. We combine this approach with the concept of the Vekua hull to develop a method to locate (at least in principle) singularities. Then we give a simple proof of the proximity of singularities to the initial curve for Ebenfelt’s example, and in finally we discuss examples not covered by Render’s approach. First, we introduce the ideas in the real setting where we will stray slightly from our main interest in order to illustrate the method.

### 3.2.2 Real Lightning Bolts and Ill-posed Problems for the Wave Equation

Recall the definition, given in the previous section, of a lightning bolt. Briefly, the points (hereafter, vertices) in a lightning bolt are given by the vertices of a polygonal
arc whose segments are parallel to the coordinate axes. For example, (2,1),(2,4),(-1,4),(-1,1) is a lightning bolt. In this particular case, the sequence obtained by augmenting this lightning bolt with its first vertex, (2,1), is again a lightning bolt. We say that a lightning bolt is closed if it remains a lightning bolt when augmented with its first vertex. Clearly, a closed lightning bolt always has an even number of vertices.

Measures constructed on lightning bolts were used by Kolmogorov and Arnold to solve Hilbert’s 13th problem regarding the solution of 7th degree equations using functions of two parameters. Lightning bolts played a central role in determining when a function of several real variables can be represented as a superposition of functions of fewer variables. The superposition problem was further developed in approximation theory ([67]). In the case when a function of two variables is to be matched on a closed curve by a sum of functions in each variable separately, this problem is related to solving a boundary value problem for the wave equation in one spatial dimension.

In order to illustrate the use of real lightning bolts, we pursue the wave equation with Dirichlet-type data. If the propagation speed has been normalized so that the wave operator has the form \( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \), then changing variables to the characteristic coordinates \( \xi = x + t, \eta = x - t \) converts the operator to \( \frac{\partial^2}{\partial \xi \partial \eta} \), and all solutions, \( u \), to the homogeneous wave equation \( \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \) then take the form \( u(\xi, \eta) = f(\xi) + g(\eta) \) (see [50]).

This reformulates the problem into one of matching a two-variable function (the data) on the boundary curve by a sum of functions in each variable separately. This places us nearly in the same setting as the superposition problem. Many results carry over immediately, perhaps in a weaker form. For instance, on the triangle with vertices \( (0, 0), (1/2, 0), (1, 1) \) (which clearly has no closed lightning bolts), any continuous function can be uniformly approximated (but in some cases not represented) by sums of continuous functions in each variable separately (for a proof see [67]). Thus, we can “solve” the wave equation with solutions that approximately match any continuous boundary data.
Necessary conditions and negative answers are provided by the following (simple) theorem and the ideas in its proof.

**Theorem 3.2.1** Let $Q$ be a subset of $\mathbb{R}^2$. If $Q$ contains a closed lightning bolt of length $2n$, then there is a polynomial $P(\xi, \eta)$ of degree $n$ which cannot be approximated on $Q$ by functions of the form $f(\xi) + g(\eta)$.

**Proof.** Without loss of generality, suppose $\xi_1 = \xi_2$. Let $P(\xi, \eta) = (\eta - \eta_1) \prod_{j=2}^{n} (\xi - \xi_{2j})$. Then $P(\xi, \eta)$ is of degree $n$ and is zero at each vertex except the second. Consider the measure which assigns the value $(-1)^k$ to the $k$th vertex, $(\xi_k, \eta_k)$, in the lightning bolt. Integrating $P(\xi, \eta)$ against this measure produces the value $P(\xi_2, \eta_2)$. Integrating any $f(\xi) + g(\eta)$ against this measure gives $\sum_k (-1)^k [f(\xi_k) + g(\eta_k)] = \sum_k (-1)^k f(\xi_k) + \sum_k (-1)^k g(\eta_k)$. Each of these sums telescopes to zero. By the Hahn-Banach Theorem, $P(\xi, \eta)$ cannot be represented or even uniformly approximated by sums $f(\xi) + g(\eta)$.

**Example:** In [49] a boundary value problem for the non-homogeneous wave equation is posed on a triangle with, using $(\xi, \eta)$-coordinates, vertices $(0, 0)$, $(2, 0)$, $(2, 2)$. Data is prescribed to be zero along the hypotenuse and to satisfy $u(t, 0) = -u(2, t)$ along the characteristic edges. It is shown that this restriction is sufficient to guarantee existence and uniqueness. For the homogeneous problem on this triangle, the proof of Theorem 3.2.1 indicates that a necessary condition for existence of solutions is that the boundary data must be annihilated by every alternating measure constructed on a closed lightning bolt residing on the boundary. For the family of lightning bolts $(t, 0), (2, 0), (t, t), (2, t)$ this condition is $u(t, 0) - u(2, 0) + u(2, t) - u(t, t) = 0$. Combining this with the restriction $u(t, 0) = -u(2, t)$ implies that $u(\xi, \eta)$ is zero. This indicates how uniqueness of solutions for the non-homogeneous equation follows from the restriction on the boundary values. Indeed, the difference of two solutions for the non-homogeneous equations is a solution for the homogeneous equation that still satisfies the restriction on the boundary, so that it must be identically zero.
In this example, the closed lightning bolts are just characteristic parallelograms (see [50]). Next, consider an example of a closed lightning bolt with more than four vertices.

Place data on the rectangle with \((x,t)\)-coordinates \((0,0), (0,b), (a,b), (0,a)\). The \((\xi,\eta)\) coordinates for the vertices are \((0,0), (b,-b), (a+b,a-b), (a,-a)\), which is a rectangle tilted 45°. Studying lightning bolts on a rectangle with this particular tilt becomes a simple problem in dynamical billiards, since successive vertices are the same as the wall-collisions of a particle with the same initial conditions.

Figure 3.5: For the rectangle with side ratio 1:2, there are infinitely many lightning bolts with six vertices. The horizontal line first returns after three intersections giving the first half of the lightning bolt.

Finding a closed lightning bolt corresponds to finding a periodic orbit (except in the case when the orbit hits a corner). Tiling the plane with successive reflections of this rectangle over its sides reduces the bookkeeping of a particle’s orbit to that of noting the crossing points of a horizontal line (see figure 3.5). If the side lengths of the rectangle have rational ratio, any horizontal line will eventually cross two “tiles” at the same point (giving a periodic orbit), in which case we have a closed lightning bolt. To see why, suppose the rectangle has sides of length 1 and \(q\) where \(q\) is rational. Then the crossing positions on the sides of length 1 are obtained by adding, modulo 1, \(q\) to the previous crossing position. We obtain infinitely many closed lightning bolts of the same length. By Theorem 3.2.1, in order to have existence of a solution to the homogeneous wave equation the data must be annihilated by each of these closed lightning bolts. It seems that one could use this restriction as a guide for finding
a condition (corresponding to the previous example) that guarantees existence and uniqueness for the non-homogeneous equation posed on the rectangle, but we leave the discussion here and return to the Dirichlet problem.

### 3.2.3 Complex Lightning Bolts and the Vekua hull

From now on, in $\mathbb{C}^2$, we understand lightning bolt to mean complex lightning bolt, which we define by complete analogy to the real case, but allowing the vertices to have complex coordinates. The method of annihilating measures supported on lightning bolts was first used in the complex setting in [46] (actually the authors used the name “2-sets”). Notice that, in the complex setting, Theorem 3.2.1 and its proof hold without any modifications.

We reformulate the Dirichlet problem in a standard way similar to what was done with the wave equation in the previous section. With $P(x, y)$ a polynomial, let $\Gamma := \{(x, y) \in \mathbb{R}^2 : P(x, y) = 0\}$ be the algebraic curve that contains the bounded component, $\Gamma_1$, where the data is posed. We obtain the complexification of $\Gamma$, denoted $\tilde{\Gamma}$, by allowing complex values $X, Y$ in the zero set $\{(X, Y) \in \mathbb{C}^2 : P(X, Y) = 0\}$. The change of variables to $z = X + iY$ and $w = X - iY$ converts the Laplacian to $\Delta = 4 \frac{\partial^2}{\partial z \partial w}$. Solutions of $\Delta u = 0$ then take the form $u(z, w) = f(z) + g(w)$, where $f$ and $g$ are holomorphic. The Dirichlet problem becomes a task of matching the data on $\tilde{\Gamma}$ with a sum $f(z) + g(w)$ holomorphic in a $\mathbb{C}^2$ neighborhood of a simply connected domain containing $\Gamma_1$. For a sequence of points in $\mathbb{C}^2$, the property of being a lightning bolt depends heavily on the coordinate system. We will be interested in lightning bolts with respect to $(z, w)$-coordinates.

For a domain $\Omega$ in the complex plane, the Vekua hull (see [63]) of $\Omega$, denoted $\hat{\Omega}$, is a set in $\mathbb{C}^2$ defined by $\{(z, w) : z \in \Omega, w \in \Omega^*\}$, where $\Omega^* := \{\zeta : \bar{\zeta} \in \Omega\}$. The relevant property of the Vekua hull for us is that if $f(z)$ and $g(\bar{z})$ are analytic and anti-analytic (resp.) in $\Omega$, then the harmonic function $f(z) + g(w)$ is analytic in $\hat{\Omega}$ as a function of two complex variables. Thus, failure to extend a solution of the reformulated Dirichlet problem to the $\mathbb{C}^2$ domain, $\hat{\Omega}$, implies failure to extend the
solution in the real plane (retrieved by setting $w = \bar{z}$) to the domain $\Omega$.

**Theorem 3.2.2** Let $\Gamma_1$ be a connected component of the algebraic curve $\Gamma$, and let $\Omega$ be a simply connected domain. Suppose $\tilde{\Gamma}$ contains a closed lightning bolt (with respect to coordinates $z$ and $w$) of length $2n$. Suppose further that along $\tilde{\Gamma}$ there are paths, also contained in $\hat{\Omega}$, that connect each vertex to $\Gamma_1$. Then, for the Dirichlet problem on $\Gamma_1$, there exist polynomial data of degree $n$ whose solution cannot be analytically continued to all of $\Omega$.

**Proof.** Let $P(z, w)$ be the polynomial of degree $n$ furnished by Theorem 3.2.1. For each vertex of the lightning bolt, consider its “connecting path” – the path that connects the vertex to $\Gamma_1$. Take a tube-like neighborhood of this path thin enough to be contained in $\hat{\Omega}$. Intersect this with $\tilde{\Gamma}$ to get a “strip”, $S$ (of the Riemann surface, $\tilde{\Gamma}$), which contains both an arc of $\Gamma_1$ and the vertex under consideration. Suppose the solution $f(z) + g(\bar{z})$ for data $P(z, \bar{z})$ is analytic in $\Omega$. Then $f(z) + g(w)$ is analytic in $\hat{\Omega}$ and, in particular, in $S$, so that $P(z, w) - f(z) - g(w)$ is also analytic in $S$ and, vanishing on an arc ($\Gamma_1$), must be zero on all of $S$. Thus, $P(z, w) = f(z) + g(w)$ at each vertex of the lightning bolt, implying that $f(z) + g(w)$ is not annihilated by the alternating measure constructed in Theorem 3.2.1, a contradiction.

3.2.4 Ebenfelt’s Example Revisited

Let us apply the method of Theorem 3.2.2, letting $\Gamma = \Gamma_1$ be the “TV screen” curve whose equation is $x^4 + y^4 = 1$. The singularities developed by the function that is harmonic inside and matches the data $x^2 + y^2$ on the curve, $x^4 + y^4 = 1$, make up a discrete infinite set residing on the coordinate axes. Following [30], one can calculate that the closest singularities are situated on the boundary of the disc of radius $2^{\frac{1}{4}}$.

In $(z, w)$-coordinates, $\tilde{\Gamma}$ is given by the zero set of $\psi(z, w) = \frac{1}{2}((z + w)^4 + (z - w)^4) - 8 = z^4 + 6z^2w^2 + w^4 - 8$, which carries the closed lightning bolt

$\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. 66
Using coordinates $X = \frac{z + w}{2}$ and $Y = \frac{z - w}{2i}$, the vertices are $v_1 = (1, 0)$, $v_2 = (0, -i)$, $v_3 = (-1, 0)$, and $v_4 = (0, i)$ (see figure 3.6).

In order to use Theorem 3.2.2, we must find paths along $\bar{\Gamma}$ connecting each vertex to the real plane $\{w = \bar{z}\} = \{X \in \mathbb{R}, Y \in \mathbb{R}\}$. Notice that $(1, 0)$ and $(-1, 0)$ are already in the real plane. In $(X, Y)$ coordinates, the equation for the complexification takes its original form $\{(X, Y) : X^4 + Y^4 = 1\}$. Consider the cross section cut by $\mathbb{R} \times i\mathbb{R}$ (X pure real and Y pure imaginary). Writing $X = x$ and $Y = iy$ we see that it is a copy of the slice from the real plane (see figure 3.6 for a three dimensional slice containing both copies). Connect $v_4 = (0, -i)$ to $v_1 = (1, 0)$ using $\gamma_{4,1}(t) = (t, i\sqrt[4]{1 - t^4})$ which travels along the vertical slice in figure 3.6. We can also connect $v_4 = (0, -i)$ to $v_3 = (-1, 0)$ using the path $\gamma_{4,3}(t) = (-t, i\sqrt[4]{1 - t^4})$. Similarly, we get paths $\gamma_{2,1}(t)$ and $\gamma_{2,3}(t)$ in the slice $\mathbb{R} \times i\mathbb{R}$ that connect $v_2 = (0, -i)$ to $v_1 = (1, 0)$ and $v_3 = (-1, 0)$ (resp.).

Figure 3.6: A 3-d slice of the complexified TV screen that contains all four vertices and both connecting paths.

These paths are completely contained in the Vekua hull of the open disc with any radius larger than $2^{\frac{3}{4}}$. Indeed, the requirement is that both $|X + iY| \leq 2^{\frac{3}{4}}$ and $|X - iY| \leq 2^{\frac{3}{4}}$. Consider the path connecting $v_4 = (0, -i)$ to $(1, 0)$ (the other cases are similar). Along this path, $X$ and $iY$ are each real and positive so that $|X + iY| = X + iY$ always exceeds $|X - iY|$. The maximum of $X + iY$ is taken when
\( X = iY = 2^{-\frac{1}{4}} \) which gives \( X + iY = 2^{\frac{3}{4}} \). Thus, by Theorem 3.2.2, there is quadratic data, \( P(z, w) \), for which the solution to the Dirichlet problem with this data develops a singularity no further than \( 2^{\frac{3}{4}} \) from the origin. Since any quadratic polynomial can be obtained from \( P(z, w) \) by addition of a harmonic quadratic, we can actually say "for any non-harmonic quadratic data, the solution develops a singularity within \( 2^{\frac{3}{4}} \) of the origin".

We can be more specific about the location of the singularities. The paths \( \gamma_{4,1}(t) \) and \( \gamma_{2,1}(t) \) are each contained in the Vekua hull of any thin neighborhood of the segment \([-1, 2^{\frac{3}{4}}]\). The paths \( \gamma_{4,3}(t) \) and \( \gamma_{2,3}(t) \) are contained in the Vekua hull of any thin neighborhood of the segment \([-2^{\frac{3}{4}}, 1]\). There are no singularities inside the curve, so we conclude that there are singularities on the positive and negative \( x \)-axis no further from the origin than \( 2^{\frac{3}{4}} \). We can also locate singularities on the \( y \)-axis no further from the origin than \( 2^{\frac{3}{4}} \) by repeating these steps for the lightning bolt (in \((z, w)\)-coordinates), \{\((i, -i), (-i, -i), (-i, i), (i, i)\}\}.

It should be mentioned that, although Ebenfelt's method is more complicated, it produces an exhaustive description of the whole infinite set of singularities.

We summarize our results for the curve \( \Gamma := x^4 + y^4 = 1 \), in the following theorem.

**Theorem 3.2.3** For the Dirichlet problem with any non-harmonic, quadratic data on the curve \( \Gamma := x^4 + y^4 = 1 \), the solution develops singularities on the positive and negative \( x \) and \( y \) axes no further from the origin than \( 2^{\frac{3}{4}} \).

Many other curves contain a similar closed lightning bolt. For instance, Ebenfelt generalized his proof to curves with complexified form \( (\alpha - \beta)z^{2k} + 2(\alpha + \beta)z^kw^k + (\alpha - \beta)w^{2k} = 4 \), with \( \alpha > \beta > 0 \). These curves carry a lightning bolt with vertices \((\pm(\frac{1}{\alpha})^{\frac{1}{2k}}, \pm(\frac{1}{\alpha})^{\frac{1}{2k}})\). We generalize Theorem 3.2.3 (and its proof) in a different direction.

**Theorem 3.2.4** Suppose polynomials \( p(x) \) and \( q(x) \) are positive for \( x \in \mathbb{R}_+ \) and satisfy \( p(1) + q(0) = p(0) + q(1) = 1 \). Then for the Dirichlet problem with non-harmonic,
quadratic data on the curve \( p(x^4) + q(y^4) = 1 \), the solution develops singularities on the \( x \) and \( y \) axes no further from the origin than \( \max\{|x| + |y| : p(x^4) + q(y^4) = 1\} \).

**Proof.** The same closed lightning bolts \( \{(1, 1), (1, -1), (-1, -1), (-1, 1)\} \) and \( \{(i, -i), (-i, -i), (i, i)\} \) (in \( z,w \)-coordinates) lie on the complexification of \( p(x^4) + q(y^4) = 1 \). In \( (X,Y) \)-coordinates these are \( \{(0, 1), (0, -i), (-1, 1), (0, i)\} \) and \( \{(i, 1), (i, -1), (0, 0), (-1, i)\} \). The curve, \( p(x^4) + q(y^4) = 1 \), also has the same symmetry as the TV screen in the \( X \) pure real and \( Y \) pure real (3d) slices. Thus, we can construct connecting paths similar to \( \gamma_{i,j}(t) \) by traveling along the \( R \times iR \) slice (or the \( iR \times R \) slice for the second lightning bolt. Along the paths in \( R \times iR \), \( X + iY \) and \( X - iY \) are pure real, and bounded by \( \max\{|x| + |y| : p(x^4) + q(y^4) = 1\} \). Along the paths in the \( iR \times R \) slice, the same estimate holds but \( X + iY \) and \( X - iY \) are pure imaginary.

We use thin neighborhoods of intervals on the \( x \) and \( y \) axes to obtain one choice of \( \Omega \) for each lightning bolt. The proof is finished by an application of Theorem 3.2.2.

**Remark:** It is natural to choose discs for \( \Omega \) in Thereom 3.2.2 if we are only interested in bounding the distance of a solution’s singularities to the initial curve. If we want more detailed information, we can use additional lightning bolts, alternative connecting paths, and choices for \( \Omega \) whose Vekua hulls more frugally catch the connecting paths. In proving Theorems 3.2.3 and 3.2.4, “more frugally” meant choosing \( \Omega \) to be a thin neighborhood of a segment containing the \( z \) and \( w \) projections of the connecting path. Generally, though, the \( z \) and \( w \) projections of a connecting path form two different paths with common end points, so that choosing \( \Omega \) to be a thin neighborhood violates the assumption in Theorem 3.2.2 that \( \Omega \) is simply connected.

### 3.2.5 A Family of Curves Not Covered by Render’s Theorem

We consider the zero sets of the following family of cubic perturbations of the unit circle which have an oval component and an unbounded component (see figure 3.7).

\[
P(x, y) = x^2 + y^2 - 1 - 2\epsilon(x^3 + xy^2 - x^2 + y^2)
\]  

(3.2.12)
Figure 3.7: A plot for the perturbed circle with unbounded component: $\varepsilon = .1$

(This family is chosen following the idea in [14] so that the complexification of this family of curves takes the simple form $\psi(z, w) = (\varepsilon z^2 - z)(\varepsilon w^2 - w) - (\varepsilon z^2 - 1)(\varepsilon w^2 - 1).$)

Rewrite $\psi(z, w) = 0$ as $(\varepsilon z^2 - z)(\varepsilon w^2 - w) = (\varepsilon z^2 - 1)(\varepsilon w^2 - 1),$ and abbreviate as $\phi(z)\phi(w) = 1,$ where $\phi(z) = \frac{\varepsilon z^2 - z}{\varepsilon z^2 - 1}.$ Choosing any complex number $a$ and solving $\phi(z) = a$ and $\phi(w) = \frac{1}{a}$ gives coordinates $(z(a), w(a))$ for a point on the complexified curve. Since we generally get two solutions each, there are infinitely many closed lightning bolts.

For instance, with $a = -1 = \frac{1}{a},$ $z = w = \frac{1 + \sqrt{1 + 8\varepsilon}}{4\varepsilon},$ giving the closed lightning bolt (abbreviate $1 + 8\varepsilon$ with $\cdot$),

\[
\left(\frac{1 - \sqrt{\cdot}}{4\varepsilon}, \frac{1 - \sqrt{\cdot}}{4\varepsilon}\right), \left(\frac{1 - \sqrt{\cdot}}{4\varepsilon}, \frac{1 + \sqrt{\cdot}}{4\varepsilon}\right), \left(\frac{1 + \sqrt{\cdot}}{4\varepsilon}, \frac{1 + \sqrt{\cdot}}{4\varepsilon}\right), \left(\frac{1 + \sqrt{\cdot}}{4\varepsilon}, \frac{1 - \sqrt{\cdot}}{4\varepsilon}\right)
\]

The first vertex is in the real plane ($\{w = \bar{z}\}$) on the bounded component of $\psi(z, \bar{z}) = 0.$ The last is, incidentally, in the real plane on the unbounded component. This conforms to our intuition that the presence of a second component of the zero set is an obstacle to analytic continuation of solutions. For instance, if a solution to the Dirichlet problem posed on one component of an irreducible curve is entire...
then it must match the data on any other component(s), “accidentally” solving an overdetermined boundary value problem.

We produce paths connecting the vertices to the bounded component of the curve by first parameterizing a single path $a(t)$ and then considering $\left( z(a(t)), w\left(1/\overline{a(t)}\right) \right)$ and $\left( z(1/\overline{a(t)}), w(a(t)) \right)$. Our path $a(t)$ will be a closed path with $a(0) = a(1) = -1$. As $a(t)$ traverses this path, the expression $1 - 4\varepsilon a(1-a)$ (inside the square root in the formula for $z(a) = w(a)$) will wind exactly once around zero, switching the branch of square root. The resulting paths depend on the initial choice of the branch of the square root. We use the paths $P_{\pm,\pm}(t) = (z_{\pm}(a(t)), w_{\pm}(1/\overline{a(t)}))$ and $Q_{\pm,\pm}(t) = (z_{\pm}(1/\overline{a(t)}), w_{\pm}(a(t)))$, where the subscripts indicate which branch of the square root is used at $t = 0$ to obtain $z(-1)$ and $w(-1)$ (“+” indicates the branch which is positive for positive reals). The sequence of paths $Q_{-,+}(t), P_{-,+}(t), Q_{+,+}(t), P_{+,+}(t)$ connect the vertices consecutively, and ultimately each vertex to the first, which resides on the bounded component of the curve in the real plane, $\{w = \bar{z}\}$.

Notice that $1 - 4\varepsilon a(1-a)$ has two zeros $\frac{1}{2}(1 \pm i \sqrt{\frac{1}{\varepsilon} - 1})$. Have $a(t)$ travel along the unit circle from $-1$ to $-i$. Then from there along the imaginary axis to $\frac{\varepsilon}{2} \sqrt{\frac{1}{\varepsilon} - 1}$ and finally to $\frac{1}{2}(1 - i \sqrt{\frac{1}{\varepsilon} - 1})$ where we will replace this endpoint with a tiny circle around it before retracing our steps back to $a = -1$. Since $a(t)$ winds around a root of multiplicity one, the image, $1 - 4\varepsilon a(t)(1-a(t))$ winds around the origin exactly once (by the argument principle), switching branches of the square root. As $a$ traces this arc, $\frac{1}{a}$ travels along the unit circle from $-1$ to $i$ then along the imaginary axis to $2i \sqrt{\frac{1}{1-\varepsilon}}$ and from there to $(\frac{1}{2}(1 - i \sqrt{\frac{1}{\varepsilon} - 1}))^{-1}$. We need to estimate $|\phi^{-1}(a)|$ and $|\phi^{-1}(\frac{1}{a})|$ along this path.

For the piece along the unit circle, we use the fact that $a$ has modulus 1 and distance from 1 at least $\sqrt{2}$. Thus, $|z(a)| = \left|\frac{1 \pm \sqrt{1 - 4\varepsilon a(1-a)}}{2\varepsilon(1-a)}\right| \leq \frac{1 + \sqrt{1 - 4\varepsilon a(1-a)}}{2\sqrt{2\varepsilon}} \leq \frac{1 + \sqrt{1 + 8\varepsilon}}{2\sqrt{2\varepsilon}}$. We get the same estimate for $|\phi^{-1}(\frac{1}{a})|$ since, along this part of the path, $\frac{1}{a}$ also has modulus 1 and distance from 1 at least $\sqrt{2}$.

Along the path where $a$ and $\frac{1}{a}$ are pure imaginary, they each have distance from 1 at least 1. Since we also have $|\frac{1}{a}| \leq 1$, writing $\frac{1}{a} = xi$ with $x \in (0, 1]$, we estimate
\[|\phi^{-1}(\frac{1}{a})| \leq \frac{1+\sqrt{|1-a^2-4\epsilon x^2|}}{2\epsilon} \leq \frac{1+\sqrt{1+2\epsilon}}{2\epsilon}. \]

Similarly, since \(a \in \left[\frac{-i\sqrt{1}}{2}, -i\right]\) write \(a = -ix\) with \(x \in [1, \frac{1}{2}\sqrt{1-\epsilon}]\). Then

\[|\phi^{-1}(a)| \leq \frac{1+\sqrt{|1+a^2-4\epsilon x^2|}}{2\epsilon} \leq \frac{1+\sqrt{1+4\epsilon}}{2\epsilon}. \]

Along the last bit of segment, \(a = \frac{1}{2}(x - i\sqrt{1-\epsilon})\), with \(x \in [0, 1]\). Since (for small \(\epsilon\)) \(|a(1-a)| = \frac{x^2}{4}(1 - \frac{x^2}{2}) + (x-1)^2(\frac{1}{2}\sqrt{1-\epsilon}-1)|\) is greatest when \(x = 0\), we can use the same estimate for \(|\phi^{-1}(a)|\) as we did along the imaginary axis. We can also use the same estimate for the numerator of \(|\phi^{-1}(\frac{1}{a})|\). For the denominator,

\[2\epsilon|1 - \frac{2}{1-i\sqrt{1-\epsilon}}| = 2\epsilon|1 - 2\epsilon(1 + i\sqrt{1-\epsilon})| \geq 2\epsilon(1 - 2\epsilon).\]

Thus, the connecting paths are contained in the Vekua hull of the disc centered at the origin with radius \(\frac{1+\sqrt{1+2\sqrt{\epsilon}}}{2\epsilon(1-2\epsilon)}\). We have now proven the following.

**Theorem 3.2.5** For the Dirichlet problem with any non-harmonic, quadratic data posed on the bounded component of \(x^2 + y^2 - 1 - 2\epsilon(x^3 + xy^2 - x^2 + y^2) = 0\) (\(\epsilon\) small), the solution develops singularities no further from the origin than \(\frac{1+\sqrt{1+2\epsilon}}{2\epsilon(1-2\epsilon)}\), which is, asymptotically, twice the distance from the bounded component to the unbounded component.

Relevant to our study, authors in [22] investigated when polynomial solutions are guaranteed by polynomial data. They give the necessary (but by no means sufficient) condition that the senior term of the curve’s equation must divide a homogeneous harmonic polynomial. The cubic, \(8x(x^2 - y^2) + 57x^2 + 77y^2 = 49\) (see figure 3.8) satisfies this necessary condition, since \(8xy(x^2 - y^2)\) is a homogeneous harmonic. Moreover, \(8xy(x^2 - y^2)\) solves the Dirichlet problem for the cubic data \(g(x, y) = y[57x^2 + 77y^2 - 49]\).

In \((z, w)\)-coordinates, the complexified form is

\[2(z + w)(z^2 + w^2) + 67zw - 5(z^2 + w^2) = 49,\]

which has a nontrivial slice in the plane with \(z\) and \(w\) pure real. The curve in this plane includes an oval component (see figure 3.9) containing all six vertices of the
Figure 3.8: A maple plot of the cubic $8x(x^2 - y^2) + 57x^2 + 77y^2 = 49$, showing the bounded component and one unbounded component (there are two other unbounded components further away).

closed lightning bolt,

$$\{(-1, -1), (-1, -7/2), (-7, -7/2), (-7, -7), (-7/2, -7), (-7/2, -1)\}.$$ 

We immediately obtain paths connecting the vertices to the bounded component of the real ($\{w = \bar{z}\}$) curve by traveling along the oval to the point $(-1, -1)$ which is on the bounded component of the real curve. The $z$ and $w$ projections of the oval are each contained in the interval $[-7.622, -1]$. We can choose a thin neighborhood of the $x$-axis segment $[-7.622, -1]$ for our domain $\Omega$ in Theorem 3.2.2. Thus, traveling along the negative $x$-axis we encounter a singularity, if not before crossing the unbounded component (at $x = -7$), no further than 10% of the distance from the bounded component to the unbounded component. We summarize this final result.

**Theorem 3.2.6** For the Dirichlet problem posed inside the bounded component of the perturbed ellipse, $8x(x^2 - y^2) + 57x^2 + 77y^2 = 49$, there exist cubic data for which the solution develops a singularity on the $x$-axis no further from the origin than 7.622 (compare to the $x$-intercept, $(-7, 0)$, of the nearest unbounded component).

**Remarks:** One hopes for a theorem giving the existence of closed lightning bolts on the complexification of all irreducible algebraic curves of degree greater than two that
Figure 3.9: A lightning bolt with six vertices on the cubic \(2(z + w)(z^2 + w^2) + 67zw - 5(z^2 + w^2) = 49\) in the plane with \(z\) and \(w\) real.
have a bounded component. This would prove (in two dimensions) the conjecture of Khavinson and Shapiro that the ellipse is the only curve for which entire data gives entire solutions. General theorems on the existence of closed lightning bolts are given in [14], [15], but they rely heavily on the hypothesized form of the Riemann map of the interior domain. Thus, a modified construction is needed for more general algebraic curves.

3.3 The Khavinson-Shapiro Conjecture and Polynomial Decompositions

This section is taken from the paper [79] written jointly with Hermann Render which was published in The Journal of Mathematical Analysis and its Applications.

The main result states the following: Let \( \psi \) be a polynomial in \( n \) variables. Suppose that there exists a constant \( C > 0 \) such that any polynomial \( f \) has a polynomial decomposition \( f = \psi q_f + h_f \) with \( \Delta^k h_f = 0 \) and \( \deg q_f \leq \deg f + C \). Then \( \deg \psi \leq 2k \). Here \( \Delta^k \) is the \( k \)th iterate of the Laplace operator \( \Delta \). As an application, new classes of domains in \( \mathbb{R}^n \) are identified for which the Khavinson-Shapiro conjecture holds.

It is instructive to consider the case \( k = 1 \) in the statement above.

3.3.1 Algebraic Dirichlet problems and Polyharmonic Decompositions

A real-valued function \( h \) defined on an open set \( U \) in \( \mathbb{R}^n \) is called \( k \)-harmonic or polyharmonic of order \( k \) if \( h \) is differentiable up to the order \( 2k \) and satisfies the equation \( \Delta^k h (x) = 0 \) for all \( x \in U \). Here \( \Delta \) denotes the Laplacian \( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \) and \( \Delta^k \) is the \( k \)th iterate of the Laplace operator \( \Delta \). Polyharmonic functions have been studied extensively in [8], and they are useful in many branches in mathematics, see [68]. For example, in elasticity theory and dynamics of slow, viscous fluids polyharmonic functions of order 2, or more briefly, biharmonic functions, are very important.

Before discussing our main results we still need some notation. By \( \mathbb{R}[x_1, \ldots, x_n] \) we denote the space of all polynomials with real coefficients in the variables \( x_1, \ldots, x_n \).
Frequently we use the fact that any polynomial $\psi$ of degree $m$ can be expanded into a sum of homogeneous polynomials $\psi_j$ of degree $j$ for $j = 0, \ldots, m$, and we write shortly $\psi = \psi_0 + \ldots + \psi_m$; here $\psi_m \neq 0$ is called the principal part or leading part of the polynomial $\psi$. The degree of a polynomial $\psi$ is denoted by $\deg \psi$.

In this article we will be concerned with a conjecture (see below) which arises naturally from the following statement proven in [86, Theorem 3] (for $k = 1$ see also [10]):

**Theorem 3.3.1** Let $\psi \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial of degree $2k$ such that the leading part $\psi_{2k}$ is non-negative. Then for any polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ there exist unique polynomials $q_f$ and $h_f$ in $\mathbb{R}[x_1, \ldots, x_n]$ such that

$$f = \psi q_f + h_f \quad \text{and} \quad \Delta^k (h_f) = 0. \quad (3.3.13)$$

Moreover, the decomposition is degree preserving, meaning that $\deg h_f \leq \deg f$ and, consequently, $\deg q_f \leq \deg f - 2k$.

Theorem 3.3.1 is related to the polynomial solvability of Dirichlet-type problems. For example, let us consider the polynomial

$$\psi_0 (x) = \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} - 1, \quad (3.3.14)$$

so $E_0 := \{ x \in \mathbb{R}^d : \psi_0 (x) < 0 \}$ is an ellipsoid. Then the decomposition (3.3.13) (where $k = 1$) shows the well known and old fact that for any polynomial $f$, restricted to the boundary $\partial E_0$, there exists a harmonic polynomial $h$ which coincides with the data function $f$ on $\partial E_0$. In other words, the solutions for polynomial data functions of the Dirichlet problem for the ellipsoid are again polynomials, see [9], [12], [22], or [64].

In [64] D. Khavinson and H.S. Shapiro formulated the following two conjectures (i) and (ii) for bounded domains $\Omega$ for which the Dirichlet problem is solvable:
Ω is an ellipsoid if for every polynomial $f$ the solution of the Dirichlet problem $u_f$ is (i) a polynomial and, respectively, (ii) entire.

Conjectures (i) and (ii) are still open, but important contributions have been made by several authors. Most of the results are proven for the two-dimensional case, see e.g. [22], [30], [77] and [46]. M. Putinar and N. Stylianopoulos have shown recently in [85] that the conjecture (i) for a simply connected bounded domain $\Omega$ in the complex plane is true if and only if the Bergman orthogonal polynomials satisfy a finite recurrence relation. D. Khavinson and N. Stylianopoulos proved among other things that the Bergman orthogonal polynomials satisfy a recurrence relation of order $N + 1$ if and only if conjecture (i) holds and a degree condition for the solution $u_f$ is satisfied, for details and further discussion see [65]. In [86] H. Render has given a solution for (i) and (ii) for arbitrary dimension and for a large but not exhaustive class of domains.

We believe that the validation of the following conjecture for the case $k = 1$ would be an important step for proving the Khavinson-Shapiro conjecture (e.g. confer the proof of Theorem 27 in [86]):

**Conjecture 3.3.2** Suppose $\psi \in \mathbb{R}[x_1, ..., x_n]$ is a polynomial, such that every polynomial $f \in \mathbb{R}[x_1, ..., x_n]$ has a decomposition $f = \psi q_f + h_f$, where $h_f$ is polyharmonic of order $k$. Then $\deg \psi \leq 2k$.

We are able to prove the conjecture if we add a degree condition on the involved polynomials which is in the spirit of the above-mentioned work [65]. More precisely, the main result of the present paper is the following:

**Theorem 3.3.3** Let $\psi \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial. Suppose that there exists a constant $C > 0$ such that for any polynomial $f \in \mathbb{R}[x_1, ..., x_n]$ there exists a decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and

$$\deg q_f \leq \deg f + C. \quad (3.3.15)$$

Then $\deg \psi \leq 2k$. 77
Theorem 3.3.3 will be a consequence of a somewhat stronger result proved after a short discussion of harmonic divisors. In passing we note that the conjecture 3.3.2 does not hold for polynomials $\psi$ with complex coefficients, see [55].

It is a natural question to ask under which conditions at the given polynomial $\psi(x)$ the degree condition in Theorem 3.3.3 is automatically satisfied. In other words, can we conclude from the equation

$$f = \psi q_f + h_f$$

with $\Delta^k h_f = 0$

that a restriction must exist on the degree of $q_f$ or $h_f$ in terms of the degree of $f$? For the case $k = 1$ we shall prove in Section 4 that the degree condition (3.3.15) is satisfied if (i) the leading part $\psi_t$ of $\psi$ contains a non-negative non-constant factor or (ii) $\psi$ has a homogeneous expansion of the form $\psi = \psi_t + \psi_s + ... + \psi_0$ where $\psi_s \neq 0$ contains a non-negative, non-constant factor. An extension of these results for arbitrary $k$ is also given. These results allow us to identify new types of domains in $\mathbb{R}^n$ for which the Khavinson-Shapiro conjecture is true.

### 3.3.2 Fischer operators and harmonic divisors

For $Q \in \mathbb{R}[x_1, ..., x_n]$ let us define $Q(D)$ as the differential operator replacing a monomial $x^\alpha$ appearing in $Q$ by the differential operator $\partial^\alpha / \partial x^\alpha$, where $\alpha$ is a multi-index. For two polynomials $Q$ and $\psi$ we call the operator $F_Q^\psi : \mathbb{R}[x_1, ..., x_n] \to \mathbb{R}[x_1, ..., x_n]$ defined by

$$F_Q^\psi(q) := Q(D)(\psi q)$$

the Fischer operator; for the significance of this notion we refer to the excellent exposition [95], or [10], [86]. We shall need the following result due to E. Fischer [39] which is in a slightly modified form valid for polynomials with complex coefficients, see [95]:

**Theorem 3.3.4** Let $Q \in \mathbb{R}[x_1, ..., x_n]$ be a homogeneous polynomial. Then the operator $q \mapsto Q(D)(Qq)$ is bijective.
At first we observe that the conjecture 3.3.2 is equivalent to the surjectivity of the Fischer operator with \( Q = \left( \sum_{i=1}^{n} x_i^2 \right)^k \); this fact is well known, but for the convenience of the reader, we include the short proof.

**Proposition 3.3.5** Suppose \( k \in \mathbb{N} \) and \( \psi \) is a polynomial. The operator

\[
F_{\psi}^k(q) := \Delta^k(\psi q)
\]

is surjective if and only if every polynomial \( f \) can be decomposed as \( f = \psi q_f + h_f \), where \( h \) is polyharmonic of order \( k \).

**Proof.** Taking \( \Delta^k \) of both sides of \( f = \psi q + h \) gives \( \Delta^k f = F_{\psi}^k(q) \). Given \( g \) we can find \( f \) such that \( g = \Delta^k f \), showing \( F_{\psi}^k \) is surjective. Conversely, if \( F_{\psi}^k \) is surjective, then given \( f \) there is a \( q \) such that \( \Delta^k f = F_{\psi}^k(q) \), showing that \( h = f - \psi q \) is polyharmonic of order \( k \).

A polynomial \( f_m \) is called *homogeneous* of degree \( m \) if \( f_m(rx) = r^m f_m(x) \) for all \( r > 0 \) and for all \( x \in \mathbb{R}^n \). We will use \( \mathbb{P}^N \) to denote the space of polynomials of degree at most \( N \), and \( \mathbb{P}^N_{\text{hom}} \) the space of homogeneous polynomials of degree \( N \). For a homogeneous polynomial \( \psi \) we define the space of all homogeneous \( k \)-harmonic divisors of degree \( m \) of \( \psi \) by

\[
D_{\psi}^k(m) = \{ q \in \mathbb{P}^m_{\text{hom}} : \Delta^k(\psi q) = 0 \}.
\]

For \( k = 1 \) we obtain the definition of a harmonic divisor (of degree \( m \)) which arises in the investigation of stationary sets for the wave and heat equation, see [4], [5], and the injectivity of the spherical Radon transform, see [6], [3].

It is an interesting but difficult problem to compute the dimension of the space \( D_{\psi}^k(m) \) and to describe how it depends on the polynomial \( \psi \). In the proof of our main result Theorem 3.3.3 we shall use the rough upper estimate provided in the next proposition and the remarks following:
Proposition 3.3.6 Let $\psi \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial. Then

$$\dim D^m_k(\psi) \leq \dim \left\{ f \in P^m_{\text{hom}} : \Delta^k f = 0 \right\}.$$

Proof. Let $q \in D^m_k(\psi)$. Then $q \in P^m_{\text{hom}}$ and $q\psi = h$ for some $h \in P^{m+t}_{\text{hom}}$ with $\Delta^k h = 0$, where $t$ is the degree of $\psi$. Clearly we have $\psi(D)(\psi q) = \psi(D) h$ and

$$0 = \psi(D)(\Delta^k h) = \Delta^k(\psi(D) h). \quad (3.3.17)$$

By Theorem 3.3.4, the operator $F$ defined by $F(q) = \psi(D)(\psi q)$ is bijective, and from $\psi q = h$ we infer that $q = F^{-1}(w)$ with $w := \psi(D) h$. Equation (3.3.17) shows that $w \in \left\{ f \in P^m_{\text{hom}} : \Delta^k f = 0 \right\}$. Thus

$$D^m_k(\psi) \subset F^{-1}\left( \left\{ f \in P^m_{\text{hom}} : \Delta^k f = 0 \right\} \right).$$

Since $F^{-1}$ is a bijective operator, the claim is now obvious.

Let us define $H^m_k := \left\{ f \in P^m_{\text{hom}} : \Delta^k f = 0 \right\}$. By Theorem 3.3.4 for $Q(x) = |x|^{2k}$ it follows that any polynomial $f$ has a Fischer decomposition $f = |x|^{2k}q + h$ where $h$ is $k$-harmonic. Moreover, $h$ and $q$ are homogeneous iff $f$ is. So we have

$$P^m_{\text{hom}} = |x|^{2k} P^{m-2k}_{\text{hom}} \oplus H^m_k.$$

Thus we obtain

$$\dim D^m_k(\psi) \leq \dim H^m_k = \dim P^m_{\text{hom}} - \dim P^{m-2k}_{\text{hom}}. \quad (3.3.18)$$

The following question was posed by M. Agranovsky for the case $k = 1$ in [3], where it was also answered in the case that $\psi$ factors completely into linear factors.

Question 3.3.7 What is the asymptotic behavior of $\dim D^m_k(\psi)$, as $m \to \infty$?
We expect that a full answer to this question would allow us to relax the assumption on degree appearing in Theorem 3.3.3.

3.3.3 Proof of the main result

Assume that \(2k \leq t\) and let \(\psi\) be a polynomial of degree \(\leq t\) and let \(F^k_\psi\) be the Fischer operator defined in Proposition 3.3.5. The following technical notion is a crucial tool for proving our main result Theorem 3.3.3: For a natural number \(M\) define \(S_i \subset P^i\) as the subspace whose image under \(F^k_\psi\) is contained in \(P^{M+t-2k}\), i.e.,

\[
S_i := \{q \in P^i : F^k_\psi(q) \in P^{M+t-2k}\}
\]

for \(i \in \mathbb{N}_0\). Since \(\psi\) has degree \(\leq t\) it follows that

\[
P^M = S_M \subset S_{M+1} \subset \ldots \subset S_{M+j}
\]

for all \(j \geq 1\).

**Proposition 3.3.8** Let \(\psi = \psi_t + \ldots + \psi_0\) be a polynomial of degree \(\leq t\) and let \(M\) be a natural number. Then for all \(j \in \mathbb{N}\)

\[
\dim S_{M+j} \leq \dim S_{M+j-1} + \dim D^{M+j}_k(\psi_t).
\]

**Proof.** For given \(j \in \mathbb{N}\) we will construct a space \(Q_j\) such that \(S_{M+j} = S_{M+j-1} \oplus Q_j\), and \(\dim Q_j \leq \dim D^{M+j}_k(\psi_t)\). First define \(Q_{H,j} := \{q_{M+j} : q_{M+j} \text{ is the degree-}(M+j)\) homogeneous term of some \(q \in S_{M+j}\}\). Choose (finitely many) polynomials in \(S_{M+j}\) whose leading terms form a basis for \(Q_{H,j}\), and define \(Q_j\) to be the subspace of \(S_{M+j}\) spanned by these polynomials. Suppose \(\hat{q} \in S_{M+j}\). The degree-(\(M+j\)) homogeneous term \(\hat{q}_{M+j}\) (possibly zero) can be matched by the leading homogeneous term of some \(q \in Q_j\) so that \(\hat{q} - q \in S_{M+j-1}\). This shows that \(S_{M+j} = S_{M+j-1} \oplus Q_j\).

Now, we will establish \(\dim Q_j \leq \dim D^{M+j}_k(\psi_t)\). It suffices to show that \(Q_{H,j} \subset D^{M+j}_k(\psi_t)\), since \(\dim Q_j = \dim Q_{H,j}\) by construction. Suppose \(q_{M+j} \in Q_{H,j}\).
is nonzero, i.e., there is a $q \in S_{M+j}$ and $\deg q = M + j$ such that $q_{M+j}$ is the leading homogeneous term of $q$. Since $F^k_\psi(q) \in \mathbb{P}^{M+t-2k}$, we have $\deg(\Delta^k(\psi q)) \leq M + t - 2k$. This implies that the leading term, $\Delta^k(\psi q_{M+j})$, of $\Delta^k(\psi q)$ is zero (since it has degree $M + j + t - 2k$). i.e., $\psi q_{M+j}$ is $k$-harmonic. Therefore, $Q_{H,j} \subset D^M_{k+j}$.

The main result of this paper, Theorem 3.3.3, follows now from the following more general result by taking $\alpha = 1$:

**Theorem 3.3.9** Let $\psi$ be a polynomial. Suppose that there exist constants $\alpha \geq 1$, $C > 0$ such that for any polynomial $f$ there exists a decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and

$$
\deg q_f \leq \alpha \deg f + C.
$$

Then $\deg \psi \leq 2k \cdot \alpha^{n-1}$.

**Proof.** Let $t$ be the degree of $\psi$, and suppose $t \geq 2k$. (If $t < 2k$, there is nothing to prove.) Let $f \in \mathbb{P}^{M+t-2k}$ and suppose that $M > 2k$. Choose a polynomial $g \in \mathbb{P}^{M+t}$ with $\Delta^k g = f$. By assumption there exists $q_f$ and $h_f$ with $g = \psi q_f + h_f$ and $\Delta^k h_f = 0$ and $\deg q_f \leq \alpha (M + t) + C$. Then $f = \Delta^k g = F^k_\psi(q_f)$ and we infer the inclusion

$$
\mathbb{P}^{M+t-2k} \subset F^k_\psi(\mathbb{P}^{B_M})
$$

(3.3.19)

with $B_M := \alpha M + \alpha t + C \geq M$. Using the above notation $S_{B_M} = \{q \in \mathbb{P}^{B_M} : F^k_\psi(q) \in \mathbb{P}^{M+t-2k}\}$ we see that (3.3.19) implies that $\mathbb{P}^{M+t-2k} \subset F^k_\psi(S_{B_M})$. Since $F^k_\psi$ is a linear operator, we have

$$
\dim \mathbb{P}^{M+t-2k} \leq \dim F^k_\psi(S_{B_M}) \leq \dim S_{B_M}.
$$

(3.3.20)

Applying Proposition 3.3.8 inductively we obtain

$$
\dim S_{B_M} \leq \dim(\mathbb{P}^M) + \sum_{j=M+1}^{B_M} \dim D^j_k(\psi_t)
$$

(3.3.21)
Since $\mathbb{P}^{M+t-2k} = \mathbb{P}^M \oplus \mathbb{P}_{\text{hom}}^{M+1} \oplus \ldots \oplus \mathbb{P}_{\text{hom}}^{M+t-2k}$ and $\dim \mathbb{P}_{\text{hom}}^{M+1} \leq \dim \mathbb{P}_{\text{hom}}^{M+j}$ for $j \geq 1$ we infer from (3.3.20) and (3.3.21) the interesting formula

$$(t - 2k) \dim \mathbb{P}_{\text{hom}}^{M+1} \leq \sum_{j=M+1}^{B_M} \dim D_k^j (\psi_t). \quad (3.3.22)$$

Further we know from (3.3.18) that $\dim D_k^j (\psi_t) \leq \dim \mathbb{P}_j^{\text{hom}} - \dim \mathbb{P}_{j-2k}^{\text{hom}}$. Thus the right hand side in (3.3.22) is a telescoping sum. Using that $\dim \mathbb{P}_j^{\text{hom}} \leq \dim \mathbb{P}_{B_M}^{\text{hom}}$ for $j = B_M - 2k + 1, \ldots, B_M$ and $\dim \mathbb{P}_{\text{hom}}^{M+2k} \leq \dim \mathbb{P}_j^{\text{hom}}$ for the lower indices we can estimate

$$\sum_{j=M+1}^{B_M} \dim D_k^j (\psi_t) \leq 2k \dim \mathbb{P}_{\text{hom}}^{B_M} - 2k \dim \mathbb{P}_{\text{hom}}^{M+2k}.$$

Thus we infer from (3.3.22) and the well known fact

$$\dim \mathbb{P}_{\text{hom}}^{M+1} = \binom{n + M}{n - 1} = \binom{n + M}{M + 1},$$

proven in [9] that

$$(t - 2k) \frac{(M + 2) \ldots (M + n)}{(n - 1)!} \leq 2k \frac{(B_M + 1) \ldots (B_M + n - 1) - (M + 2 - 2k) \ldots (M + n - 2k)}{(n - 1)!}.$$

Clearly the term $(n - 1)!$ can be canceled in the inequality. Divide the inequality by $M^{n-1}$ on both sides and recall that $B_M = \alpha M + \alpha t + C$. Now take the limit $M \to \infty$ and we obtain

$$t - 2k \leq 2k \left( \alpha^{n-1} - 1 \right).$$

This implies $t \leq 2k \alpha^{n-1}$ and the proof is complete.

3.3.4 Criteria for degree-related decompositions

We are now turning to the question under which conditions the degree condition is automatically satisfied. The first criterion is simple to prove:
Proposition 3.3.10 Suppose that $\psi$ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \ldots + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial $\psi_t$ contains a non-negative, non-constant factor. Let $f$ be a polynomial and assume that there exists a decomposition

$$f = \psi q + h$$

where $h$ is harmonic and $q$ is a polynomial. Then $\deg q \leq \deg f - t$ and $\deg h \leq \deg f$.

Proof. Write $q = q_M + \ldots + q_0$ with homogeneous polynomials $q_j$ of degree $j = 0, \ldots, M$. Expand the product $\psi q$ into a sum of homogeneous polynomials, so $\psi q = \psi_t q_M + R(x)$ where $R(x)$ is a polynomial of degree $< M + t$. Suppose that $M + t > \deg f$. Since $\Delta f = \Delta (\psi q)$ we conclude that $\Delta (\psi_t q_M) = 0$, so $\psi_t q_M$ is harmonic. By the Brelot-Choquet theorem, a harmonic polynomial cannot have non-negative factors, see [20]. Thus $\psi_t q_M = 0$, and we obtain a contradiction.

The next criterion is more difficult to prove and uses again ideas from the proof of the Brelot-Choquet theorem:

Theorem 3.3.11 Suppose that $\psi$ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + \ldots + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial $\psi_s$ is non-zero and contains a non-negative, non-constant factor. Let $f$ be a polynomial and assume that there exists a decomposition

$$f = \psi q + h$$

where $h$ is harmonic and $q$ is a polynomial. Then $\deg q \leq 2 - s + \deg f$ and $\deg h \leq t + 2 - s + \deg f$.

Before proving Theorem 3.3.11 we notice the following conclusion:
Corollary 3.3.12 Suppose that $\psi$ is a polynomial with a non-zero second-highest degree term that contains a non-negative factor. If every polynomial $f$ has a Fischer decomposition $f = \psi q_f + h_f$ with $h_f$ harmonic, then $\deg(\psi) \leq 2$.

Proof. Suppose $\deg(\psi) > 2$. By Theorem 3.3.11, $\deg q_f - \deg f$ is bounded. Now we can apply Theorem 3.3.3, to obtain $\deg \psi \leq 2$.

The following lemma is needed for the proof of Theorem 3.3.11:

Lemma 3.3.13 Suppose that $\psi$ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + \ldots + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume that $g \in \mathbb{P}^m$ and $q$ is a polynomial of degree $M$ such that $F^k_\psi(q) := \Delta(\psi q) = g$ and $M + s > m$. Then for every $p \in \mathbb{P}^{s-1}$,

$$\int_{S^{n-1}} q_M^2 \cdot \psi_s \cdot p \, d\theta = 0,$$

where $q_M \neq 0$ is the leading homogeneous part of $q$.

Proof. (of Lemma): Write $q = q_M + \ldots + q_0$ with homogeneous polynomials $q_j$ of degree $j = 0, \ldots, M$. Expand the product $\psi q$ into a sum of homogeneous polynomials,

$$\psi q = \psi_t q_M + \ldots + \psi_t q_{M-t+s+1} + (\psi_t q_{M-t+s} + \psi_s q_M) + R(x) \quad (3.3.23)$$

where $R(x)$ is a polynomial of degree $< M + s$. Since $\Delta(\psi q) = g$ and $M + s > m$, we conclude that $\Delta(\psi_t q_M) = 0$ and $\Delta(\psi_t q_{M-t+s} + \psi_s q_M) = 0$. Thus, we can write

$$\psi_t q_M = h_{M+t} \quad (3.3.24)$$

$$\psi_t q_{M-t+s} + \psi_s q_M = h_{M+s}, \quad (3.3.25)$$

where $h_{M+t}$ and $h_{M+s}$ are homogeneous harmonic polynomials.
Take \( p \in \mathbb{P}^{s-1} \), and multiply equation (3.3.25) by \( q_{MP} \) and integrate over the unit sphere, \( \mathbb{S}^{n-1} \). Then

\[
\int_{\mathbb{S}^{n-1}} \psi_t q_{M-t+s} \cdot q_{MP} \, d\theta + \int_{\mathbb{S}^{n-1}} \psi_s q_M^2 \cdot p \, d\theta = \int_{\mathbb{S}^{n-1}} h_{M+s} \cdot q_{MP} \, d\theta.
\]

Since \( \deg(q_{MP}) < M + s \) and \( h_{M+s} \) is harmonic, the integral on the right-hand side is zero. Indeed, homogeneous harmonics of different degree are orthogonal in the space \( L^2(\mathbb{S}^{n-1}) \) (see [9]), and, moreover, \( q_{MP} \) can be matched on \( \mathbb{S}^{n-1} \) by a harmonic polynomial of not higher degree. Substituting equation 3.3.24 into the first integral on the left-hand side gives \( \int_{\mathbb{S}^{n-1}} h_{M+t} \cdot p \cdot q_{M-t+s} \, d\theta \), which is also zero, since \( \deg(pq_{M-t+s}) < M + t \).

\[
0 = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot P \, d\theta = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \phi \cdot P^2 \, d\theta.
\]

Since \( P \neq 0 \), \( \phi \neq 0 \), and \( \phi(\theta) \geq 0 \) for all \( \theta \in \mathbb{S}^{n-1} \), we have the contradiction \( q_M = 0 \).

\[
0 = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot P \, d\theta = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \phi \cdot P^2 \, d\theta.
\]

\[
0 = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot P \, d\theta = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \phi \cdot P^2 \, d\theta.
\]

The following instructive example is due to L. Hansen and H.S. Shapiro [46]; it was also suggested in [57] as a simple example for which the Khavinson-Shapiro conjecture is unresolved (whenever \( \varphi \) is a cubic): Let \( \varphi \in \mathbb{R}[x_1, \ldots, x_n] \) be a homogeneous harmonic polynomial of degree > 2, in particular \( \varphi \) does not contain a nonnegative non-constant factor, see [20]. We perturb the equation for the unit ball \( |x|^2 - 1 \) by \( \varepsilon \varphi \), i.e. we consider

\[
\psi_\varepsilon (x) := |x|^2 - 1 + \varepsilon \varphi (x) \text{ for } \varepsilon > 0.
\]
If $\varepsilon > 0$ is small enough, then the component of $E_\varepsilon := \{\psi_\varepsilon < 0\}$ containing 0 is a bounded domain in $\mathbb{R}^d$. Then the Dirichlet problem for the data function $|x|^2 = x_1^2 + \ldots + x_n^2$ restricted to $\partial E_\varepsilon$ has a harmonic polynomial solution $u_f(x) = 1 - \varepsilon \varphi(x)$ since

$$|x|^2 = \psi_\varepsilon(x) \cdot 1 + 1 - \varepsilon \varphi(x).$$

Note that in this example the degree of the solution $u_f$ for the Dirichlet problem is higher than the degree of the data function $f$.

The question arises whether any polynomial data function may have a polynomial solution. If this is the case, and $\psi_\varepsilon$ is irreducible and changes the sign in a neighborhood of some point in $\partial E_\varepsilon$ then the proof of Theorem 27 in [86] implies that for any polynomial $f$ there exists a decomposition $f = \psi_\varepsilon q_f + h_f$ where $h_f$ is harmonic. By Corollary 3.3.12 $\deg \psi_\varepsilon \leq 2$. Thus we have proved that for this class of examples the Khavinson-Shapiro conjecture is true.

In the rest of this section we extend Theorem 3.3.11 to the case $k \geq 1$. We consider the following inner product

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x)g(x)e^{-|x|^2} \, dx \quad (3.3.27)$$

and the following orthogonality condition established in [86].

**Theorem 3.3.14** Suppose that $f$ is a homogeneous polynomial, and let $k \in \mathbb{N}$ with $2(k - 1) \leq \deg f$. Then $\Delta^k f = 0$ if and only if $\langle f, g \rangle = 0$ for all polynomials $g$ with $2(k - 1) + \deg g < \deg f$.

**Theorem 3.3.15** Suppose that $\psi$ is a polynomial of degree $t > s$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + \ldots + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial $\psi_s \neq 0$ is non-negative. If the polynomial $f$ has the decomposition

$$f = \psi q + h$$

where $h$ is $k$-harmonic, then $\deg(q) \leq 2k - s + \deg f$. 

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Proof. Suppose that $M + s > 2k + \deg f$, where $f = \psi q + h$ and $M = \deg q$. We will derive a contradiction. We proceed as in the proof of Lemma 3.3.13 writing $q = q_1 + ... + q_0$ with homogeneous polynomials $q_j$ of degree $j = 0, ..., M$. Expand the product $\psi q$ as in (3.3.23). Then we conclude that $\Delta^k(\psi_t q_M) = 0$ and $\Delta^k(\psi_t q_M - t + s + \psi_s q_M) = 0$. Thus, we can write

$$\psi_t q_M = H_{M+t}$$ (3.3.28)

$$\psi_t q_M - t + s + \psi_s q_M = H_{M+s},$$ (3.3.29)

where $H_{M+t}$ and $H_{M+s}$ are homogeneous $k$-harmonic polynomials. Next take the inner product (3.3.27) of both sides of equation (3.3.29) with $q_M$. Then

$$\langle q_M - t + s, q_M \psi_t \rangle + \langle \psi_s, q_M^2 \rangle = \langle H_{M+s}, q_M \rangle$$

Using equation 3.3.28, we arrive at $\langle q_M - t + s, H_{M+t} \rangle + \langle \psi_s, q_M^2 \rangle = \langle H_{M+s}, q_M \rangle$. Now we use Theorem 3.3.14. Since $\deg H_{M+t} > \deg q_M - t + s + 2(k - 1)$ and $\deg H_{M+s} > \deg q_M + 2(k - 1)$, the first term on the left and the term on the right are both zero. Thus, $\langle \psi_s, q_M^2 \rangle = 0$ implies $q_M = 0$ (since $\psi \neq 0$ is non-negative), a contradiction.

$\blacksquare$
4 Valence of Harmonic Maps and Gravitational Lensing

4.1 Fixed Points of Conjugated Blaschke Products with Applications to Gravitational Lensing

This section is taken from the joint paper [71] with Ludwig Kuznia published in “Computational Methods and Function Theory”.

A conjecture in astronomy was recently resolved as an accidental corollary of a theorem regarding zeros of certain planar harmonic maps. As a step toward extending the fundamental theorem of algebra, the theorem gave a bound of \(5n - 5\) for the number of zeros of a function of the form \(r(z) - \bar{z}\), where \(r(z)\) is rational of degree \(n\). In this section, we will investigate the case when \(r(z)\) is a Blaschke product. The resulting (sharp) bound is \(n + 3\) and the proof is simple. We discuss an application to gravitational lenses consisting of collinear point masses.

The strongest test passed by Einstein’s general relativity was the prediction of the deflection of starlight famously confirmed by Eddington during a solar eclipse [29]. Besides perturbing or magnifying light from a distant star, a gravitational field can create multiple images or even an elliptical ring from a single source (see [35] for early speculations made by Einstein himself). In order to model this phenomenon, we will assume that the so-called gravitational lens consists entirely of point-masses residing in a common plane perpendicular to our line of sight (modest violations of this assumption are not severe—we can project the outliers to the lens plane). The path of a light ray under the influence of gravity follows geodesics of a space-time metric which in turn is found by solving a system of nonlinear PDE’s. As an exception,
the space-time metric arising from a single star can be calculated exactly (see the
discussion on the Schwarzschild solution in [40]) and leads to the Einstein deflection
angle, $\alpha = 4MG/r$, measured between entry and
exit asymptotes of a passing photon in terms of
the mass, $M$, and distance, $r = |\xi|$, at the point
of closest approach. This is the only result we
require from general relativity. Basic geometry
does the rest.

Consider first a single point mass at the
origin of the lens plane. Suppose a star with po-
sition $w$ in the source plane emits a light ray that
enters the lens plane and is deflected toward our
telescope. The small angle approximation gives
so for the length of the vector, $v$, which has the
same direction as $\xi$. Thus, $v = 4MG\frac{\xi}{|\xi|^2}$. Now the similarity of the two right triangles
in Figure 4.1 leads to the relationship $w + 4sMG\frac{\xi}{|\xi|^2} = \frac{l+s}{t}\xi$. Write $z = \frac{l+s}{t}\xi$ and
use units which subsume the resulting constant in front of $M\frac{z}{|z|^2}$. Then the lensing
equation for a point mass is

$$z = w + M\frac{z}{|z|^2} \quad (4.1.1)$$

Now suppose the lens plane contains $n$ deflectors with positions $z_i$ and masses
$m_i$. If the interactions among point masses contriving the lens are weak enough
(indicating that nonlinear terms in the field equations are negligible), we can follow
our temptation to take the superposition of the Einstein deflection angles due to
individual masses. The lensing equation then becomes

$$z = w + \sum_{i=1}^{n} m_i \frac{z - z_i}{|z - z_i|^2} \quad (4.1.2)$$

The replacement $\frac{z - z_i}{|z - z_i|^2} = \frac{1}{z - z_i}$ invites a complex-variable point of view:
\[ z = w + \sum_{i=1}^{n} \frac{m_i}{\bar{z} - z_i} \] (4.1.3)

Although it can be shown mathematically that such a configuration can have at most \( n^2 + 1 \) images, in 1997 Mao, Petters, and Witt [82] suggested the bound was actually linear in \( n \). Rhie refined this in 2001 [87], conjecturing that a gravitational lens consisting of \( n \) point masses cannot create more than \( 5n - 5 \) images of a given source. In 2003, she constructed point mass configurations for which this bound is attained [88]. One year later, Khavinson and Neumann [60] proved a bound of \( 5n - 5 \) zeros for harmonic mappings of the form \( r(z) - \bar{z} \), where \( r(z) \) is rational of deg \( n > 1 \). Notice that conjugating both sides of (4.1.3) puts it in the form \( \bar{z} = r(z) \). (See [61] for the exposition and further details.)

In the next section we will consider a case when \( 5n - 5 \) is not the best possible. Namely, we require that \( r(z) = B(z) \) is a finite Blaschke product. We will prove a sharp bound for this case using introductory-level complex variables. This will not immediately give any insight into gravitational lensing, though, because all residues of \( r(z) \) must be real and positive in order for \( z = r(z) \) to coincide with a lensing equation. This almost never happens in the case of Blaschke products. In the third section, we bring a class of physical examples into the picture using a familiar Möbius transformation.

4.1.1 Case \( r(z) = B(z) \)

Throughout this paper, we assume that any Blaschke product is nontrivial (i.e. \( B(z) \neq z \)). The goal of this section is to show that maps of the form \( B(z) - \bar{z} \) can have at most \( n + 3 \) zeros, where \( B(z) \) is a finite Blaschke product. Recall that a finite Blaschke product is a function of the form \( \prod_{i=1}^{n} \frac{z - a_i}{1 - \bar{a}_i z} \), where \( |a_i| < 1 \) for \( i = 1, \ldots, n \). First, we notice that solutions to \( B(z) = \bar{z} \) are symmetric with respect to the unit circle. This fact is verified via simple algebra.
Lemma 4.1.1 Let $B(z)$ be a Blaschke product and $z_0 \in \mathbb{D}$, then $B(z_0) = \overline{z_0}$ if and only if $B(1/z_0) = 1/z_0$, where $1/0 = \infty$ by the usual convention.

The next lemma, whose proof is a standard exercise in a graduate course of complex analysis (see for example [43], page 265), gives some flavor of the importance of Blaschke products, stating that they are the only analytic, boundary-preserving, self-maps of the disc. (One recognizes this as a trivial case of the factorization theorem from $H_p$ theory.)

Lemma 4.1.2 Suppose $f(z)$ is an analytic map of $\mathbb{D}$ into itself (continuous up to the boundary) which sends the boundary to the boundary, then $f(z)$ has finitely many zeros in $\mathbb{D}$. Furthermore, $f$ is a Blaschke product with $n$ factors, where $n$ is the number of zeros of $f(z)$ in $\mathbb{D}$.

Proof. We first show that $f$ has finitely many zeros in $\mathbb{D}$. Since $f$ is continuous in $\overline{\mathbb{D}}$ and maps the boundary to the boundary, there exists an $r < 1$ such that $f$ has no zeros in the annulus $\{z : r < |z| \leq 1\}$. If $f$ had infinitely many zeros in $\mathbb{D}$, then they would all be in $\{z : |z| \leq r\}$. However, this would imply that $f \equiv 0$. Hence $f$ has finitely many zeros. Now form a Blaschke product, $B(z) = \prod_{i=1}^{n} \frac{z-a_i}{1\overline{a_i} z}$, using the zeros, $a_i$, of $f(z)$ which lie in $\mathbb{D}$. Notice that $g(z) = \frac{f(z)}{B(z)}$ is analytic and nonvanishing in $\mathbb{D}$. Therefore, $u(z) = \text{Re}\{\log(g(z))\} = \log|g(z)|$ is harmonic throughout $\mathbb{D}$. Moreover, on $\partial\mathbb{D}$, $|g(z)| = \frac{|f(z)|}{|B(z)|} = 1$ so that $u(z)|_{\partial\mathbb{D}} = 0$. It then follows from the maximum principle that $u(z)$ is identically zero. Therefore, $\log(g(z))$, as a purely imaginary, analytic function, must be constant (consider the Cauchy-Riemann equations). Hence, $f(z)$ is a unimodular multiple (i.e. a rotation) of $B(z)$.

Lemma 4.1.3 If $B(z)$ is a Blaschke product, then $B(z) = z$ has at most one solution in $\mathbb{D}$.

Proof. Suppose that $z_0 \in \mathbb{D}$ is a fixed point of $B$. Let $\phi(z)$ be the disc automorphism $\frac{z_0 - z}{1 - \overline{z_0} z}$ and set $f = \phi^{-1} \circ B \circ \phi$, then $f : \mathbb{D} \rightarrow \mathbb{D}$ and $f(0) = 0$. Thus, by the Schwarz
lemma, if $f$ has another fixed point, then $f(z) \equiv z$. Since $f$ is not the identity map and fixed points of $B$ correspond to fixed points of $f$, it follows that $B$ can have at most one fixed point in $\mathbb{D}$.

Now we prove our main theorem.

**Theorem 4.1.4** Let $B(z)$ be a Blaschke product with $n$ factors; if $B(z) \neq z$ then $B(z) = \overline{z}$ has at most $n + 3$ solutions in $\overline{\mathbb{C}}$. In particular, there are $n + 1$ solutions on $\partial \mathbb{D}$, and there is a solution in $\mathbb{D}$ if and only if there is a solution in $\mathbb{C} - \overline{\mathbb{D}}$.

**Proof.** We begin by showing that $B(z) = \overline{z}$ has $n + 1$ solutions on $\partial \mathbb{D}$. Notice that $\overline{z} = 1/z$ if $z \in \partial \mathbb{D}$, thus $B(z) = \overline{z}$ becomes $B(z) = 1/z$. Therefore, it is equivalent to solve $zB(z) = 1$ for $z \in \partial \mathbb{D}$. Now $zB(z)$ is a Blaschke product with $n + 1$ factors, hence it is an $n + 1$ fold covering of $\mathbb{D}$ without ramification points over $\partial \mathbb{D}$. Therefore, $zB(z) = 1$ has $n + 1$ distinct solutions on $\partial \mathbb{D}$. Next we investigate the interior. Notice that if $z_0$ is such that $B(z_0) = \overline{z_0}$, then $\overline{B(B(z_0))} = z_0$. This leads us to consider $\overline{B(B(z))} = z$. We note that $\overline{B(B(z))}$ is a rational function, analytic in $\mathbb{D}$, that maps $\mathbb{D}$ to itself and has modulus one on the boundary. Hence, by Lemma 4.1.2, $\overline{B(B(z))}$ is a Blaschke product. By Lemma 4.1.3, $\overline{B(B(z))} = z$ has at most one solution in $\mathbb{D}$, and hence $B(z) = \overline{z}$ has at most one solution in $\mathbb{D}$. Moreover, by Lemma 4.1.1, there is a solution in $\mathbb{C} - \overline{\mathbb{D}}$ if and only if there is a solution in $\mathbb{D}$.

It is evident from the proof of Theorem 4.1.4 that $B(z) = \overline{z}$ will always have $n + 1$ solutions on $\partial \mathbb{D}$. Interestingly, there are examples having $n + 1$ solutions and those having $n + 3$ solutions. For example, $zA(z) = \overline{z}$, where $A(z)$ is a Blaschke product with $n - 1$ factors, will have a solution at $z = 0$ and the corresponding solution in $\mathbb{C} - \overline{\mathbb{D}}$, $z = \infty$. We postpone an example with exactly $n + 1$ solutions until later.

We now move on to a corollary of Theorem 4.1.4.
Corollary 4.1.5 Let $\Omega$ be a simply connected Jordan domain. Suppose $f$ is an anti-analytic map of $\Omega$ to itself which maps $\partial\Omega$ to itself ($f$ is “proper”). Then $f(w) - w$ has at most $n + 2$ zeros in $\Omega$, where $n$ is the degree of $f(w)$ as a self-map of $\partial\Omega$ (i.e. for each $w \in \partial\Omega$, $f^{-1}(w)$ has $n$ elements).

Proof. Let $\phi$ be the Riemann map of $\Omega$ onto $\mathbb{D}$. Then $\overline{\phi(f(\phi^{-1}(z)))}$ is an analytic, boundary-preserving self-map of $\mathbb{D}$ with $n$ zeros (by the argument principle). By Lemma 4.1.2, $\phi(f(\phi^{-1}(z))) = B(z)$, where $B(z)$ is a Blaschke product with $n$ factors. Thus, $\phi(f(\phi^{-1}(z))) = \overline{B(z)}$ which has at most $n + 2$ fixed points in $\bar{\mathbb{D}}$ by Theorem 4.1.4. We have $\phi(f(\phi^{-1}(z_0))) = z_0$ iff $f(w_0) = w_0$, where $w_0 = \phi^{-1}(z_0)$. Thus, $f(w)$ has at most $n + 2$ fixed points in $\Omega$.

### 4.1.2 Gravitational Lensing by Collinear Point Masses

Suppose the positions of the masses $z_i$ in (4.1.3) along with (projection of) the source $w$ are collinear. Then without loss of generality we may assume they are on the real axis. Then the lensing map sends the real line to itself, the upper half plane to itself, and the lower half plane to itself. As in the proof of Corollary 4.1.5, we conjugate $f(z) = w + \sum_{i=1}^{n} \frac{m_i}{z - z_i}$ with the Möbius transformation, $\phi(z) = \frac{z - i}{z + i}$, which sends the upper half plane to $\mathbb{D}$. It is in fact true that all rational functions corresponding to finite Blaschke products have the form $bz + w + \sum_{i=1}^{n} \frac{m_i}{z - z_i}$, where $m_i$, $w$, and $z_i$ are as above and $b$ is nonnegative, and the degree of the Blaschke product is $n$ if $b = 0$ and $n + 1$ if $b \neq 0$. (For a more information on this more general result on the correspondence between Blaschke products and rational functions see [92]). But in this case, $\phi$ and its inverse are defined in the entire plane, so we get a total estimate (rather than just in the upper half plane) of at most $n + 3$ solutions of the lensing equation. We summarize this as

Corollary 4.1.6 There are at most $n + 3$ images lensed by a collinear configuration of point masses when the projection of the source onto the lens plane is also collinear.
The $n + 1$ solutions located on the real axis are not surprising. Indeed, the $n$ masses divide the real line into $n + 1$ intervals. Consider for instance a finite interval between two masses. If a ray from the source lands too close to the left endpoint it will be deflected too sharply in that direction and miss our telescope. If the ray lands too close to the right endpoint, it will be deflected too sharply in the other direction. We expect an intermediate value where the ray is properly deflected. Also not surprising is the symmetry of the two images that occur off the real axis. But what is not physically obvious is that there should only be two such images.

For an example, we consider the following lensing equation

$$ \bar{z} = \frac{3}{z - 1} + \frac{4}{z + 4} + \frac{1}{z + 1}. \quad (4.1.4) $$

Here we have masses of 3, 4, and 1 at the points 1, −4, and −1, respectively and the observer, origin of the lens plane and the source are collinear. As mentioned above, it is expected to have four images on the real axis, which are separated by the masses. In this case, the approximate locations of these images are 2.63, −0.18, −1.51, and −4.97. We also have the two symmetric images off the real axis, which are located at approximately 0.78 ± i2.01. See Figure 4.2 for a depiction of this example.
We now return to the task of finding a Blaschke product, \( B \), such that \( B(z) = \overline{z} \) has exactly \( n + 1 \) solutions in \( \mathbb{C} \). To do this, we examine another lensing equation, namely

\[
\overline{z} = \frac{1}{z - 1} + \frac{10}{z + 1}.
\]

(4.1.5)

Rearranging (3.2), we have

\[
z(10 - |z|^2) = 9 - 2\text{Re}(z)
\]

(4.1.6)

Notice that solutions to (3.3) must be real or lie on the circle \( \{ z : |z| = \sqrt{10} \} \). Since \( 0 = 9 - 2\sqrt{10}\cos \theta \) has no solutions, we may conclude that (3.3) has no solutions on \( \{ z : |z| = \sqrt{10} \} \). Therefore, any solutions to (3.3) must be real. From this we may conclude that (3.2) has exactly 3 solutions in \( \mathbb{C} \). After conjugating by the appropriate Möbius transformation, we obtain a Blaschke product, \( B \), such that \( B(z) = \overline{z} \) has 3 solutions.

**Remarks:**
1. If the hypothesis of Corollary 4.1.5 could be weakened, it would have potential as a tool for locally analyzing lensing maps in regions which are sent to themselves.
2. Aside from a bound of \( n^2 \) provided by an argument of Wilmshurst [106] along with Bezout’s Theorem, little progress has been made bounding zeros of harmonic polynomials \( p(z) + \overline{q(z)} \) when \( \deg(p(z)) > \deg(q(z)) > 1 \) (The question regarding a bound on the number of zeros of harmonic polynomials was raised by T. Sheil-Small [98]). Perhaps bounding solutions of \( B(z) = \overline{A(z)} \), with \( B(z) \) and \( A(z) \) Blaschke products could prove less stubborn.
3. It is of interest to ask how sensitive images are to small perturbations of the position of one of the lensing masses. This corresponds to a gravitational lens which includes a star possessing a planet. In practice, microlensing techniques search for a change in the brightness of a lensed image in order to detect a planet. Interestingly, in the collinear systems considered in this paper, the position of the symmetric pair of images can be especially sensitive to perturbations of the masses. In order to
deem this an alternative technique, an informed investigation using realistic mass and distance scales would be necessary.

4.2 Transcendental Harmonic Mappings and Gravitational Lensing by Isothermal Galaxies

This section is taken from the joint paper [58] with Dmitry Khavinson published in Complex Analysis and Operator Theory.

Using the Schwarz function of an ellipse, it was recently shown that galaxies with density constant on confocal ellipses can produce at most four “bright” images of a single source. The more physically interesting example of an isothermal galaxy has density that is constant on homothetic ellipses. In that case bright images can be seen to correspond to zeros of a certain transcendental harmonic mapping.

4.2.1 A simple problem in complex analysis with a direct application

We will use complex dynamics to give an upper bound on the total number of solutions of the equation

\[
\arcsin \left( \frac{k}{\bar{z} + w} \right) = z, \quad (4.2.7)
\]

where \( w \) is a complex parameter, and \( k \) is a real parameter.

Our motivation for doing so is that solutions of (4.2.7) in fact correspond to virtual images observed when the light from a distant source passes near an isothermal, ellipsoidal galaxy. Indeed, using the complex formulation of the thin-lens approximation ([99]), the lensing equation is calculated by finding the Cauchy transform of the mass distribution projected to the “lens plane”. This was carried out in [38] with the following result for the lensing equation of an isothermal galaxy.

\[
C \arcsin \left( \frac{c}{\zeta} \right) + \omega = \zeta \quad (4.2.8)
\]
We sketch the derivation of Eq. (4.2.8) at the end of this section for the reader’s convenience. Here, we take the principal branch of arcsin, $C$ and $c$ are real constants depending on the elliptical projection of the galaxy onto the lens plane, and $\omega$ is the position of the source (projected to the lens plane). Values of $\zeta$ which satisfy (4.2.8) give positions of the observed images. Changing variables to $z = \frac{\zeta - \omega}{c}$, $w = \frac{\omega}{C}$, and $k = \frac{c}{C}$ puts (4.2.8) into the form of equation (4.2.7) while preserving the number of solutions.

We should mention that the anti-analytic potential in the lensing equation considered here (and also in [18] and [38]) differs from the potential in the lensing equation in the model often used by astrophysicists (see [54] and the references therein), where the projected mass density is supported in the entire complex plane. Both models use the “isothermal” density proportional to $1/t$ on ellipses $\{x^2/a^2 + y^2/b^2 = t^2\}$ ($a$ and $b$ fixed). The model considered here that yields equation (4.2.8) assumes that the density is zero for all $t$ greater than some value (see end of this section). Letting the density have infinite support assumes that the galaxy has infinite mass and fills the universe, yet it is the simplest way to avoid giving the galaxy a “sharp edge”, and astronomers have found that the model behaves reasonably in the region where the lensed images occur. We consider the model with physically realistic compact support but less realistic “sharp edge” for a mathematical reason: in that setting, lensed images described by solutions of equation (4.2.7) correspond to zeros of a harmonic function. (We note that models with “sharp edges” have been considered by astrophysicists as well, cf. the recent preprints [83] and [84].)

For gravitational lenses consisting of $n$ point masses, Mao, Petters, and Witt [82] suggested that the bound for the number of images was linear in $n$. (Bezout’s theorem provides a bound quadratic in $n$.) Rhie refined this in 2001, conjecturing that a gravitational lens consisting of $n$ point masses cannot create more than $5n - 5$ images of a given source [87]. In 2003, she constructed point-mass configurations for which these bounds are attained [88]. D. Khavinson and G. Neumann [60] settled her conjecture by giving a bound of $5n - 5$ zeros for harmonic mappings of the form.
$r(z) - \bar{z}$, where $r(z)$ is rational of degree $n > 1$. (See [60] for the exposition and further details.) Solutions of (4.2.7) are zeros of a transcendental harmonic function, so extending the techniques used in [60] will require some care (a priori, it is not even clear that the number of zeros is finite, cf. [13], [69]). Still, our approach draws on the same two main results: (i) the argument principle generalized to harmonic mappings and (ii) the Fatou theorem from complex dynamics regarding the attraction of critical points. In the next section, we will formulate (i). (ii) will have to be modified for our purposes, so ideas from complex dynamics are worked from scratch into the proof of Lemma 4.2.5.

4.2.2 Preliminaries: The Argument Principle

In order to state the generalized argument principle (see [27] for a complete exposition and proof), we need to define the order of a zero or pole of a harmonic function. A harmonic function $h = f + \bar{g}$, where $f$ and $g$ are analytic functions, is called sense-preserving at $z_0$ if the Jacobian $Jh(z) = |f'(z)|^2 - |g'(z)|^2 > 0$ for every $z$ in some punctured neighborhood of $z_0$. We also say that $h$ is sense-reversing if $\bar{h}$ is sense-preserving at $z_0$. If $h$ is neither sense-preserving nor sense-reversing at $z_0$, then $z_0$ is called singular and necessarily (but not sufficiently) $Jh(z_0) = 0$, cf. [27], Ch. 2.

The notation $\Delta_C \text{arg} h(z)$ denotes the increment in the argument of $h(z)$ along a curve $C$. The order of a non-singular zero is given by $\frac{1}{2\pi} \Delta_C \text{arg} h(z)$, where $C$ is a sufficiently small circle around the zero. The order is positive if $h$ is sense-preserving at the zero and negative if $h$ is sense-reversing. Suppose $h$ is harmonic in a punctured neighborhood of $z_0$. We will refer to $z_0$ as a pole of $h$ if $h(z) \to \infty$ as $z \to z_0$. Following [102], the order of a pole of $h$ is given by $-\frac{1}{2\pi} \Delta_C \text{arg} h(z)$, where $C$ is a sufficiently small circle around the pole. We note that if $h$ is sense-reversing in some punctured neighborhood of the pole, then the order of the pole will be negative. We will use the following version of the argument principle which is taken from [102]:

**Theorem 4.2.1** Let $F$ be harmonic, except for a finite number of poles, in a Jordan domain $D$. Let $C$ be a closed Jordan curve contained in $D$ not passing through a pole
or a zero, and let $R$ be the open, bounded region surrounded by $C$. Suppose $F$ has no singular zeros in $R$ and let $N$ be the sum of the orders of the zeros of $F$ in $R$. Let $P$ be the sum of the orders of the poles of $F$ in $R$. Then $\Delta_C \arg F(z) = 2\pi(N - P)$.

### 4.2.3 An Upper Bound for the Number of Images

**Lemma 4.2.2** The solutions of equation (4.2.7) are all contained in the rectangle $R := \{|\Re z| \leq \pi/2, |\Im z| \leq M\}$, where $M$ is sufficiently large.

*Proof.* The requirement that $|\Re z| \leq \pi/2$ is immediate since this strip is the image of $\mathbb{C}$ under the principal branch of arcsin. To see that there exists an $M$ such that solutions of (4.2.7) satisfy $|\Im z| \leq M$, take sin of both sides. This leads to

$$k \sin(z) = z + w.$$  \hspace{1cm} (4.2.9)

We consider the modulus of each side of (4.2.9) for $z = x + iy$ with large values of $|y|$. Recall, $\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$. As $y \to \pm \infty$, $|\frac{k \sin(x + iy)}{\sin(z)}| = k/\sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} \to 0$, uniformly in $x$. On the other hand, $|z + w| \to \infty$.

**Remark:** By this lemma, we can bound the number of solutions of (4.2.7) by bounding the number of zeros of $F(z) := z + w - \frac{k}{\sin(z)}$ in the rectangle, $R$. Let us calculate the increment of the argument of $F(z)$ when $\partial R$ is traced counterclockwise. If $\partial R$ passes through a zero of $F$ then we can instead consider the boundary of $R_\varepsilon := \{|\Re z| \leq \pi/2 + \varepsilon, |\Im z| \leq M\}$ in the following Lemma and in the rest of this section. With a small choice of $\varepsilon$, the calculation in the following proof does not change.

**Lemma 4.2.3** $\Delta_{\partial R} \arg F(z) \geq -2\pi$, where $F(z) := z + w - \frac{k}{\sin(z)}$.  

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Proof. Consider the four links $V_{\pm}(t) = \pm \frac{\pi}{2} \pm i(-M + t), 0 \leq t \leq 2M, H_{\pm}(t) = \pm \left( \frac{\pi}{2} - t \right) \pm iM, 0 \leq t \leq \pi$, which trace the right, left, top, and bottom edges, respectively. We need to determine the effect of the term $-\frac{k}{\sin(z)}$. Without this term, $F(z)$ is just the translation $z \mapsto z + w$, and in that case $F(V_{\pm}(t))$ and $F(H_{\pm}(t))$ trace the edges of the translated rectangle.

By choosing $M$ large enough in the previous lemma, we can neglect the term $-\frac{k}{\sin(z)}$ on the top and bottom edges. On the right edge, $\frac{k}{\sin(V_{+}(t))} = \frac{k}{\cosh(-M + t)}$ is pure real and increases monotonically from a small value at $t = 0$ to the value $k$ at $t = M$. On the interval $M \leq t \leq 2M$, $\frac{k}{\cosh(-M + t)}$ decreases monotonically from $k$ at $t = M$ back to the original value at $t = 2M$. Similarly, on the left edge, $\frac{k}{\sin(V_{-}(t))} = -\frac{k}{\cosh(-M + t)} > -k$. Thus, the effect of the term $-\frac{k}{\sin(z)}$ is to bend the left and right sides of the translated rectangle inward, so that they cross each other if and only if $k > \frac{\pi}{2}$ (compare the two images in figure 4.3).

If $0 < k < \pi/2$, then the images of the left and right edges do not intersect, and either $\Delta_{\partial R} \arg F(z) = 2\pi$, or, if $F(\partial R)$ does not surround the origin, $\Delta_{\partial R} \arg F(z) = 0$. See the left image in figure 4.3. The case $k < 0$ is not physical, but we note that it produces the same possibilities as $0 < k < \pi/2$.

If $k > \pi/2$, then the images of the left and right edges intersect exactly twice. In this case, there is a third possibility in which $\Delta_{\partial R} \arg F(z) = -2\pi$. See the right image in figure 4.3.

Define the function $f(z) := \frac{k}{\sin(z)} - \bar{w}$, and notice that fixed points of $\bar{f}(z)$ coincide with the zeros of $F(z)$. Also, define the function $f^\#(z) = \frac{k}{\sin(z)} - w$ so that $f^\#(z) = \bar{f}(z)$, and denote the composition $f^\#(f(z))$ by $g(z)$. Suppose $z_0$ is a zero of $F(z)$. Then $g(z_0) = f^\#(z_0) = \bar{f}(z_0) = z_0$, so that $z_0$ is a fixed point of the analytic function $g(z)$. Moreover, if $z_0$ is a sense-preserving zero then $|f'(z_0)| < 1$ and $g'(z_0) = (f^\#)'(z_0)f'(z_0) = |f'(z_0)|^2 < 1$, so that $z_0$ is an attracting fixed point of $g$ (see [28] and [21]). Finally, if $z_0$ is a singular zero then $g'(z_0) = 1$ (not just in modulus!)
Figure 4.3: The image of $\partial R$ under $F(z)$ with $w = 0$ and two choices for the value of $k$. In the left, $k = 1 < \pi/2$. In this case, $\frac{1}{2\pi} \Delta_{\partial R} \arg F(z) = 1$. If we set $w$ to, say, 1 then we have $\frac{1}{2\pi} \Delta_{\partial R} \arg F(z) = 0$. In the right, $k = 2 > \pi/2$, and $\frac{1}{2\pi} \Delta_{\partial R} \arg F(z) = -1$. If we set $w$ to, say, 1 or $i$ then we have $\frac{1}{2\pi} \Delta_{\partial R} \arg F(z) = 0$ or 1, respectively.

so that $z_0$ is a so-called *parabolic* fixed point of $g$.

We use complex dynamics to bound the number of attracting and parabolic fixed points of $g(z)$. The version of the Fatou theorem found in most textbooks on complex dynamics such as [21] bounds the number of attracting fixed points of a polynomial or rational function by the number of critical points. This falls short, since we are considering here a function $g$ with infinitely many essential singularities and infinitely many critical points. However, the more updated version of the Fatou Theorem provided by the following Lemma found in [11] (we formulate a special case of Lemma 10 in that paper) is perfectly suited to our situation. For an exposition of the extensions of the Fatou theorem leading up to this modern formulation, see the survey [17].

**Lemma 4.2.4** Suppose $g$ is meromorphic outside an at most countably infinite, compact set (considered as a subset of the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$) of essential
singularities. Then the basin of attraction of an attracting or parabolic fixed point $z_0$ contains the forward orbit of some singular point of $g^{-1}$.

**Lemma 4.2.5** The number of sense-preserving zeros, $n_+$, of $F$ plus the number of singular zeros, $n_0$, is at most 3.

**Proof.** Suppose $z_0$ is a sense-preserving or singular zero of $F$. Then, by the discussion above, $z_0$ is an attracting or parabolic fixed point of $g(z) = f^\#(f(z))$. We note that $g$ is meromorphic except at $z = \infty$ and at the countably many zeros of $f(z)$ converging to $\infty$, so that $g$ satisfies Lemma 4.2.4.

For the set of singular points of $g^{-1}$, $\text{Sing}(g^{-1})$, we have

$$\text{Sing}(g^{-1}) = \text{Sing}((f^\#)^{-1}) \cup f^\#(\text{Sing}(f^{-1})).$$

We can find explicitly, $(f^\#)^{-1}(\zeta) = \arcsin \left( \frac{k}{\zeta + w} \right)$, so that

$$\text{Sing}((f^\#)^{-1}) = \{-w, -w \pm k\}.$$

Similarly, by writing $f^{-1}$ explicitly we see that

$$f^\#(\text{Sing}(f^{-1})) = \{f^\#(-\bar{w}), f^\#(-\bar{w} \pm k)\},$$

so $\text{Sing}(g^{-1})$ is the set of at most six points $\{-w, -w \pm k, f^\#(-\bar{w}), f^\#(-\bar{w} \pm k)\}$. By Lemma 4.2.4, $z_0$ attracts at least one of these points, giving us the bound $n_0 + n_+ \leq 6$. The following observation improves this estimate.

Let $z_c$ be one of the three points $-w, -w \pm k$ and suppose $z_0$ attracts $z_c$. We claim that $z_0$ also attracts $f^\#(z_c)$. Indeed,

$$\lim_{n \to \infty} g^n(f^\#(z_c)) = \lim_{n \to \infty} (f^\# \circ f)^n(f^\#(z_c)) = \lim_{n \to \infty} f(g^n(z_c)) = f(z_0) = z_0,$$

so that $g^n(f^\#(z_c))$ converges to $z_0$. Thus, each sense-preserving or singular zero of $F$ attracts, under iteration of $g$, one of the three points $f^\#(-\bar{w}), f^\#(-\bar{w} \pm k)$. So
\[ n_0 + n_+ \leq 3. \]

**Theorem 4.2.6** The number of solutions to (4.2.7) is bounded by 8.

**Proof.** By the remark following Lemma 4.2.2, the total number of solutions to (4.2.7) equals the total number of zeros of \( F(z) \) in \( R \). Recall that \( F(z) \) is called "regular" if it is free of singular zeros (see [66] and [28]). Suppose for the moment that \( F(z) \) is regular so that Theorem 4.2.3 applies. Then, the total number of zeros of \( F(z) \) in \( R \) is \( n_+ + n_- \), where \( n_+ \) and \( n_- \) count, respectively, the sense-preserving and sense-reversing zeros of \( F(z) \) in \( R \). By Lemma 4.2.3 and Theorem 4.2.1, \(-1 \leq N - P\). \( F(z) \) has one sense-reversing pole in \( R \) of order \(-1\) (and all non-singular zeros are of order \( \pm1 \)). By Lemma 4.2.5 \(-1 \leq 3 - n_- + 1\), so that \( n_- \leq 5 \). Thus, \( n_+ + n_- \leq 8 \).

Fix \( k \). There is a dense set of parameters \( w \) for which \( F(z) \) is regular. Indeed, consider the image of \( \{ z : \left| \frac{d}{dz} \left( \frac{k}{\sin(z)} \right) \right| = 1 \} \) under \( z - \frac{k}{\sin(z)} \). This set has empty interior, and if \( w \) is in its complement, \( F(z) \) is free of singular zeros.

Now suppose \( F(z) \) is not regular. Then Lemma 4.2.5 still applies so that \( n_0 + n_+ \leq 3 \), but the previous argument for bounding \( n_- \) does not. If \( F(z) \) is perturbed by a sufficiently small constant to obtain \( F_\varepsilon(z) \), the number of sense-reversing zeros does not decrease by continuity of the argument principle in a sense-reversing region. The zeros simply move in a small neighborhood of each sense-reversing zero. By the preceding, we can choose a perturbation \( F_\varepsilon(z) \) that is regular and therefore has at most five sense-reversing zeros. This gives \( n_- \leq 5 \), and \( n_0 + n_+ + n_- \leq 8 \).

**4.2.4 Remarks**

So far, astronomers have only observed up to 5 images (4 bright + 1 dim) produced by an elliptical lens (see figure 4.4). In [54] there have been constructed explicit models (depending on the semiaxes of the ellipse) having 9 images (8 bright + 1 dim) but only in the presence of a shear, i.e. a (linear) gravitational pull from infinity (a term \( \gamma \bar{z} \) added to equation (4.2.7)). So far, we have not been able to obtain a universal bound
in the presence of a shear that is similar to Theorem 4.2.6. It seemed, based on NASA observations, natural to conjecture that, in the absence of shear, there can be at most 4 bright images. Yet, recently W. Bergweiler and A. Eremenko improved Theorem 4.2.6 by showing that there are at most 6 bright images, and they generated an example with 6 bright images [18]. With their kind permission, we include their example (see figure 4.5). For the case with shear, we conjecture the following.

**Conjecture 4.2.7** The number of bright images lensed by an isothermal elliptical galaxy (with compactly supported mass density) with shear is at most 8.

We caution the reader that in [54] the mass density was assumed to be extended all the way to infinity, so the lensing potential in [54] was different from the one we consider here (and in [38], [18]).

### 4.2.5 Derivation of the complex lensing equation for the isothermal elliptical galaxy

Suppose that light from a distant source star is distorted as it passes by an intermediate, continuous distribution of mass which does not deviate too far from being contained in a common plane (the “lens plane”) perpendicular to our line of sight. Let \( \mu(z) \) denote the projected mass density. Then basic results from General Relativity combined with Geometric Optics (see [99]) lead to the following lensing equation
Figure 4.5: Equation (4.2.7) has 6 solutions when $k = 1.92$ and $w = -0.67i$. Choosing $a = 1, b = 0.041,$ and $M = 2$ (see below) leads to $k = 1.92$ and gives the picture of the six images shown here in the $\zeta$-plane along with the galaxy’s elliptical silhouette and the source (plotted as box).

relating the position of the source (projected to the lens plane) $w$ to the positions of lensed images $z$. 

$$z = \int_{\Omega} \frac{\mu(\zeta) dA(\zeta)}{\zeta - \bar{z}} + w$$  \hspace{1cm} (4.2.10)

Consider, first, the case when the projected density $\mu(z) = D$ is constant and supported on $\Omega := \{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, a > b > 0 \}$, an ellipse. Then equation (4.2.10) becomes

$$z = \int_{\Omega} \frac{D dA(\zeta)}{\zeta - \bar{z}} + w.$$ 

By the complex Green’s formula, for $z$ outside $\Omega$ (i.e., for “bright” images), this becomes

$$z = \frac{D}{2i} \int_{\partial\Omega} \frac{\zeta d\bar{\zeta}}{\zeta - \bar{z}} + w.$$
The Schwarz function (by definition, analytic and = \bar{\zeta} on \partial \Omega) for the ellipse equals \((c^2 = a^2 - b^2)\):

\[
S(\zeta) = \frac{a^2 + b^2}{c^2} \zeta - \frac{2ab}{c^2} (\sqrt{\zeta^2 - c^2})
\]

\[
= \frac{a^2 + b^2 - 2ab}{c^2} \zeta + \frac{2ab}{c^2} (\zeta - \sqrt{\zeta^2 - c^2})
\]

\[= S_1(\zeta) + S_2(\zeta)\]

where \(S_1\) is analytic in \(\bar{\Omega}\), and \(S_2\) is analytic outside \(\Omega\) and \(S_2(\infty) = 0\). Since \(z\) is outside \(\Omega\), combining this with Cauchy’s formula gives

\[
\frac{1}{2i} \int_{\partial \Omega} \frac{S(\zeta) d\bar{\zeta}}{\bar{\zeta} - \bar{z}} + w = \pi \frac{2ab}{c^2} D(\bar{z} - \sqrt{\bar{z}^2 - c^2}) + w
\]

for the right-hand-side of the lensing equation.

Next consider the case of “isothermal” density supported on \(\Omega\), \(\mu = M/t\) on \(\partial \Omega\), \(\Omega_t := t\Omega = \{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq t^2 \}\), \(t < 1\), and \(M\) a constant.

Then the Cauchy potential term in the lensing equation (4.2.10) becomes

\[
\int_{\Omega} \frac{\mu(z)}{\zeta - \bar{z}} dA(\zeta) = \int_0^1 \frac{M}{t} \left[ \frac{d}{dt} \int_{\Omega_t} \frac{dA(\zeta)}{\zeta - \bar{z}} \right] dt \tag{4.2.11}
\]

For the inside integral, we see that \(\int_{\Omega_t} \frac{dA(\zeta)}{\zeta - \bar{z}} = t^2 \int_{\Omega} \frac{dA(\zeta)}{\zeta - \bar{z}} = t \int_{\Omega} \frac{dA(\zeta)}{\zeta - \bar{z}/t}\) which according to our previous calculation is \(C_0(\bar{z} - \sqrt{\bar{z}^2 - c^2 t^2})\), where the constant \(C_0\) depends only on \(\Omega\). Now the \(t\)-derivative of this is \(C_0 \frac{t}{\sqrt{\bar{z}^2 - c^2 t^2}}\). Thus (4.2.11) becomes

\[MC_0 \int_0^1 \frac{dt}{\sqrt{\bar{z}^2 - c^2 t^2}}.\]

Finally, we arrive at (4.2.8), the lensing equation for the isothermal elliptical galaxy,

\[z = C \arcsin \left( \frac{c}{\bar{z}} \right) + w,\]

where \(C = \frac{2\pi ab}{c} M\).
REFERENCES


[53] L. Karp, Construction of quadrature domains in $\mathbb{R}^n$ from quadrature domains in $\mathbb{R}^2$, Complex Var. Elliptic Eq., 17 (1992), 179-188.


[90] W. Ross, *private communication*


