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Longtime dynamics of hyperbolic evolutionary equations in ubounded domains and lattice systems

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Longtime Dynamics of Hyperbolic Evolutionary Equations

in Unbounded Domains and Lattice Systems

by

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A dissertation submitted in partial fulfillment
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Specially dedicated to both my parents who have always been a source of inspiration and encouragement since first grade.

Diaradieuf Yaaybooy.

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Djiby Fall

ABSTRACT

This dissertation is a contribution to the study of longtime dynamics of evolutionary equations in unbounded domains and of lattice systems. It is of particular interest to prove the existence of global attractors for solutions of such equations. To this end, one needs in general some type of asymptotical compactness. In the case that the evolutionary PDE is defined on a bounded domain Ω in space, asymptotical compactness follows from the regularity estimates and the compactness of the Sobolev embeddings and therefore the existence of attractors has been established for most of the dissipative equations of mathematical physics in a bounded domain. The problem is more challenging when Ω is unbounded since the Sobolev embeddings are no longer compact, so that the usual regularity estimates may not be sufficient.

To overcome this obstacle of compactness, A.V. Babin and M.I. Vishik introduced some weighted Sobolev spaces. In their pioneering paper [2], they established the existence of a global attractor for the reaction-diffusion equation

$$u_t - \nu\Delta u + f(u) + \lambda u = g, \quad x \in \mathbb{R}^N \tag{1}$$

Lately, a new technique of "tail estimation" has been introduced by B. Wang [49] to prove the existence of global attractors for the reaction-diffusion equation (1) in the usual Hilbert space $L^2(\mathbb{R}^N)$. In this research we take on the same approach to prove the existence

of attracting sets for some nonlinear wave equations and hyperbolic lattice systems.

The dissertation is organized as follows. In the first part (Chapter 2), we prove the existence of a global attractor in $H_0^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ for the wave equation

$$u_{tt} + \lambda u_t - \Delta u + u + f(u) = g, \quad t > 0, \ x \in \mathbb{R}^N. \tag{2}$$

Removing the coercive mass term u from (2), we achieve the same result for the more challenging equation

$$u_{tt} + \lambda u_t - \Delta u + f(u) = g, \quad t > 0, \ x \in \Omega \tag{3}$$

where Ω is a domain of $\mathbb{R}^N$ bounded only in one direction.

The second part of the dissertation deals with some lattice systems. We establish in Chapter 3 the existence of global attractor for the equation

$$\ddot{u}_i + \lambda \dot{u}_i - (u_{i-1} - 2u_i + u_{i+1}) + f(u_i) = g_i, \qquad i \in \mathbb{Z} \tag{4}$$

which is a spatial discretization of (3).

1 INTRODUCTION AND GENERAL CONCEPTS

Henri Poincaré (1854-1912) is often referred to as the father of "*nonlinear dynamics*". Toward the end of the nineteenth century, he for the first time pointed out that irregular behaviors in mechanics are not at all an unusual feature if the system being studied involves a nonlinear interaction. Very simple systems can have highly complex dynamics. The behavior of the solutions of such systems over longer time is quite irregular and practically cannot be predicted. Since Poincaré the study of nonlinear dynamics started to develop, with major contributions of great mathematicians and physicists such as Lyapunov (1857-1918) and Birkhoff (1884-1944) who developed the concept of dynamical systems as we know it today. Meanwhile the study of nonlinear dynamics extends far beyond mechanics to many fields not only of physics but also of chemistry, biology, economics etc. Before 1950, it primarily focused on the finite dimensional systems usually modeled by ordinary differential equations.

The theory for the evolutionary partial differential equations was slower to emerge. This theory along with the study of differential-delay and lattice systems constitute what is known as infinite dimensional dynamical systems. The literature is extensive on the study of these systems, mainly for partial differential equations in a bounded domain, see for instance J. Hale [23], G. Sell & Y. You [42] or R. Temam [47], and the references therein. It is just recently, since the pioneering work of Babin and Vishik [2], that mathematicians got interested in the dynamics of partial differential equations in unbounded domains. In this direction, several types of evolutionary equations have been investigated with interesting results; yet there are still lots of open problems and large room for mathematical contributions.

One can be interested in different features in the study of the dynamics of a system:

singularity formations, finite-time blow up, existence of chaos, attracting sets, bifurcation theory, invariant manifolds, exponential dichotomies, stability, ...etc. In this work we focus on the longtime behavior of the solutions of evolutionary equations in unbounded domains and of lattice systems. In particular, the topic is the existence of global attractors. The global attractor is a compact invariant set attracting the trajectory bundles of all bounded subsets as time goes to infinity. Therefore, if it exists, a global attractor contains all the essential, permanent dynamics of the system. And very often the global attractor has finite fractal (or Hausdorff) dimension, thus reducing the initial infinite dimensional problem to a finite dimensional one in the long run.

In the current theory of infinite dimensional dynamical systems, the global attractor is a highlighted, core topic. The existence of global attractors for dissipative systems follows in general from some type of asymptotical compactness of the corresponding semiflow. This is proved in case the domain Ω is bounded by *a priori* estimates and the compactness of Sobolev embedings. This method seems not to work when the domain is unbounded since the Sobolev embeddings are no longer compact. It then becomes a difficult task to deal with this compactness issue. Two major techniques seem to work in overcoming this difficulty: working with weighted Sobolev spaces as phase space or using the "tail estimation methods".

In 1990 Babin and Vishik [2] for the first time showed the existence of a global attractor for the reaction-diffusion equation in $\mathbb{R}^N$,

$$u_t - \nu\Delta u + f(u) + \lambda u = g. \tag{1.1}$$

To this end they used the weighted Sobolev space $L^2_\gamma(\mathbb{R}^N)$ as phase space. They proved, for $\gamma < 0$, the existence of a global attractor in the weak topology under certain growth conditions on the nonlinearity f. For $\gamma > 0$, the attractor is in the strong topology and if $g(x)$ decreases sufficiently fast, then the attractors for different γ do not depend on γ.

In 1994 Feireisl, Laurençot, Simondon and Touré [20] considered a similar equation as (1.1) without the mass term λu and they showed that for $\gamma < -\frac{N}{2}$, the attractor is indeed in the strong topology of $L^2_\gamma(\mathbb{R}^N)$.

Some other authors have also considered weighted spaces for different types of equations.

For the wave equation

$$u_{tt} + \delta u_t - \phi(x)\Delta u + \lambda f(u) = \eta(x), \qquad x \in \mathbb{R}^N, \quad t > 0,$$

N.I. Karachalios amd N.M. Stavrakakis [24] established the existence and finite dimensionality of a global attractor in $L^2_g(\mathbb{R}^N)$ where $g(x) = \dfrac{1}{\phi(x)}$.

Working with the weighted spaces has a disadvantage of restraining the choice of the initial data. In 1999, B. Wang [49] came up with the new idea of "*tail estimations*" to prove the asymptotic compactness of the semiflow generated by the reaction-diffusion equation (1.1). This led to the existence of a global attractor in $L^2(\mathbb{R}^N)$ for system (1.1). This method features an approximation of R^N by sufficiently large bounded domains Ω_k and then show the null convergence of the solutions in $\mathbb{R}^n - \Omega_k$. This method has been useful to studying the dynamics of many evolutionary equations in unbounded domains as well as lattice dynamical systems (see for instance [4, 49, 43, 50, 28]).

The dynamics of nonlinear wave equations in unbounded domains have been extensively studied by Eduard Feireisl. In [16] he showed the existence of global attractor for 3D wave equation in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, but for the n-dimensional problem the attractor is only locally compact, see [17]. Similar results have been obtained in [53].

It seems little is known on the existence of global attractor in the traditional Sobolev spaces for nonlinear wave equations in unbounded domains. In Chapter 2, we consider the damped wave equation with mass term,

$$u_{tt} + \lambda u_t - \triangle u + u + f(u) = g(x), \qquad x \in \mathbb{R}^N, \quad t > 0 \tag{1.2}$$

and show the existence of global attractor for the corresponding dynamical system. This is done by applying the generalized "tail end estimation" method introduced in [4] and [49]. Moreover, removing the mass term in (1.2), we achieve the existence result for the equation

$$u_{tt} + \lambda u_t - \triangle u + f(u) = g(x), \qquad x \in \Omega, \quad t > 0 \tag{1.3}$$

where Ω is a domain of $\mathbb{R}^N$, bounded only in one direction. This features the use of the Poincaré inequality to achieve the monotonicity of the linear operator for the corresponding transformed first-order problem.

In Chapter 3 we study the dynamics of the seconder order lattice system, without mass term,

$$\ddot{u}_i + \lambda \dot{u}_i - (u_{i-1} - 2u_i + u_{i+1}) + f(u_i) = g_i, \qquad i \in \mathbb{Z} \tag{1.4}$$

which can be seen as a spatial discretization of (1.3)in one dimension.

In chapter 4 we give some remarks on the dimension f the global attractors and present some new directions with interesting open problems.

Before we get into the details of our work, let us introduce first the basic definitions and results relevant to the general theory of dynamical systems. We present in this introductory chapter the notions of semiflows and attractors along with a brief presentation of the theory of semigroups and its relation to solving abstract nonlinear equations in a Banach space. We also give an example of a sectorial operator and we finish with some inequalities that will be useful in the subsequent chapters.

1.1 Semiflows and Attractors

We will use in this work the definition of semiflows as in Temam [47]. A stronger version can be found in Sell & You [42], where the only difference is the continuity property.

Definition 1.1.1 *Let (H, d) be a complete metric space. A family of operators $\{S(t)\}_{t\geq 0}$ is called a* ***semiflow*** *on H, if it satisfies the following properties:*

1. *$S(0) = I$ (identity in H), i.e. $S(0)u = u \quad \forall u \in H$,*
2. *$S(t)S(s) = S(t+s), \qquad \forall s, t \in \mathbb{R}^+$,*
3. *The mapping $\quad S(t) : H \to H \quad$ is continuous for every $t \geq 0$.*

We introduce now the concepts of invariant sets and attractors of a semiflow

Definition 1.1.2 *Let $S(t)$ be a semiflow on H and $K \subset H$. We say that K is* ***positively invariant*** *if $S(t)K \subset K$, for all $t \geq 0$. K is* ***invariant*** *if $S(t)K = K$, for all $t \geq 0$.*

To define attractors we will need the following asymmetric Hausdorff pseudodistance:

$$\sup_{a\in A}\inf_{b\in B} d(A,b) \tag{1.5}$$

where $A,\ B$ are bounded sets in H.

We say that A **attracts** B if

$$h(S(t)B,A)\to 0, \qquad \text{as } t\to\infty, \tag{1.6}$$

that is: for every $\varepsilon>0$, there exists $T\geq 0$ such that $d(S(t)u,A)\leq\varepsilon$, for all $t\geq T$ and $u\in B$.

Definition 1.1.3 *A subset $\mathcal{A}$ of H is called an* ***attractor*** *for the semiflow $S(t)$ provided that*

1. *$\mathcal{A}$ is a compact, invariant set in H, and*
2. *there is a neighborhood U of $\mathcal{A}$ in H such that $\mathcal{A}$ attracts every bounded set in U.*

An attractor $\mathcal{A}$ that attracts every bounded set in H is called a **global attractor**.

The existence of global attractor is in general related to what some authors call the "dissipativity" of the dynamical system. This is equivalent to the existence of absorbing sets.

Definition 1.1.4 *Let $\mathcal{B}$ be a subset of H and $\mathcal{U}$ an open set containing $\mathcal{B}$. We say that $\mathcal{B}$ is an* ***absorbing set*** *in $\mathcal{U}$ if the orbit of any bounded set in $\mathcal{U}$ enters into $\mathcal{B}$ after a finite time (which may depend on the set):*

$$\begin{cases} \forall\mathcal{B}_0\subset\mathcal{U}, \quad \mathcal{B}_0 \textit{ bounded} \\ \exists t_1(\mathcal{B}_0) \ \textit{ such that } S(t)\mathcal{B}_0\subset\mathcal{B}, \quad \forall t\geq t_1(\mathcal{B}_0). \end{cases}$$

We also say that $\mathcal{B}$ attracts the bounded sets of $\mathcal{U}$.

We have also the related concept of asymptotical compactness.

Definition 1.1.5 *A semiflow $\{S(t)\}_{t\geq 0}$ is said to be* ***asymptotically compact*** *on $\mathcal{U}$ if for every bounded sequence $\{u_n\}$ in $\mathcal{U}$ and $t_n\to\infty$, $\{S(t_n)u_n\}_{t\geq 0}$ is precompact in H.*

We are now ready to present a standard result on the existence of global attractors which can be found in [23, 47].

Theorem 1.1.1 *Let $\{S(t)\}_{t\geq 0}$ be a semiflow in X. If $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set and is asymptotically compact in H, then $\{S(t)\}_{t\geq 0}$ possesses a global attractor which is a compact invariant set that attracts every bounded set in H.*

1.2 Evolutionary Equations and Semigroup Theory

In practice semiflows are generated by the solutions of differential equations. We will consider abstract nonlinear ODEs of the form

$$\frac{du}{dt} + Au = F(u, t) \tag{1.7}$$

in a Banach space X, where A is an unbounded linear operator in X and $F : X \times \mathbb{R} \to X$ is a nonlinear functional. In this section we will present the general existence theory for equations such as (1.7). This will apply directly to a wide range of evolutionary partial differential equations. We will first give some basic notions on semigroup theory which is related to solving the corresponding linear problem

$$\frac{du}{dt} + Au = 0. \tag{1.8}$$

1.2.1 Semigroups of Linear Operators

In the remainder of this section, X denotes a Banach space with norm $\|\cdot\|_X$ and $\mathcal{L}(X)$ is the space of bounded linear operators on X.

Definition 1.2.1 *We will say that a family of operators $\{T(t)\}_{t\geq 0}$ is a* **C_0*-semigroup of linear operators*** *on X, if $T(t) \in \mathcal{L}(X)$ for all $t \in [0, +\infty)$ and the following hold:*

(i) $T(0) = I$ *(identity in X)*

(ii) $T(t)T(s) = T(t+s), \qquad s, t \in [0, +\infty)$

(iii) $\lim_{t\to 0^+} T(t)x = x, \qquad$ *for all* $x \in X$.

We see that a C_0-semigroup is a typical example of a semiflow on X.

Definition 1.2.2 *Let $T(t)$ be a C_0-semigroup on X, its* ***infinitesimal generator*** *is the linear operator A on X defined as follows*

- *The domain of A is:*

$$D(A) = \{x \in X \ : \ \lim_{h \to 0^+} \frac{T(h) - I}{h} x \ \textit{ exists in } X\}$$

- *for $x \in D(A)$ we set:*

$$Ax = \lim_{h \to 0^+} \frac{T(h) - I}{h} x = \frac{d^+(T(t)x)}{dt}|_{t=0}.$$

Next we will give a necessary and sufficient condition for an operator to be the infinitesimal generator of a C_0-semigroup in a Hilbert space H. We need to introduce first some concepts. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A linear operator $A : D(A)(\subset H) \to H$ is said to be **accretive** if

$$\mathrm{Re}\,\langle Ax, x\rangle \geq 0, \qquad \forall x \in D(A).$$

If in addition we have $R(I + A) = H$ (range of $I + A$ is equal to H) then we say that A is **maximal accretive**.

A C_0-semigroup is said to be **nonexpansive** if $\|T(t)\| \leq 1$ for every $t \geq 0$.

Theorem 1.2.1 (Lumer-Phillips) *Let H be a Hilbert space. Then a linear operator $-A : D(A)(\subset H) \to H$ is the infinitesimal generator of a nonexpansive C_0-semigroup e^{-At} on H if and only if both the following condtions are satisfied:*

(1) A is a closed linear operator and $D(A)$ is dense in H, and

(2) A is a maximal accretive operator.

This is a classical result on semigroups and their generators. The proof can be found in [38, 42]. However this result applies only to Hilbert spaces; there is a more general one on Banach spaces, namely the **Hille-Yosida** theorem.

As we mentioned in the beginning of this section, the semigroup theory will allow us to solve the linear problem (1.8). Indeed let A be the generator of a C_0-semigroup $T(t)$ on X; it is shown that for every $x_0 \in D(A)$, $T(t)x_0$ is a solution of equation (1.8) in the classical sense. For $x_0 \in X$ we call $x(t) = T(t)x_0$ a **mild solution** of (1.8). The existence of solutions in the classical sense is related to the differentiability of the C_0-semigroup and there is a particular class of differentiable semigroups called **analytic semigroups**.

Definition 1.2.3 *We will say that $T(t)$ is an **analytic semigroup** in X if there is an extension of it to a mapping $T(z)$ defined on some sector $\Delta_\delta \cup \{0\}$ such that:*

(1) $T(z_1 + z_2) = T(z_1)T(z_2)$ for all z_1 and z_2 in $\Delta_\delta \cup \{0\}$,

(2) for each $x \in X$, one has $T(z)x \to x$ as $z \to 0$ in $\Delta_\delta \cup \{0\}$,

(3) for each $x \in X$, the function $z \to T(z)x$ is an analytic mapping from Δ_δ into X

where the sector Δ_δ is defined as

$$\Delta_\delta = \{z \in \mathbb{C} \ : \ |arg\, z| < \delta, z \neq 0\}, \qquad \text{for } \delta \in (0, \pi).$$

A related concept is that of sectorial operators.

Definition 1.2.4 *A linear operator $A : D(A) \subset X \to X$ is said to be a **sectorial operator** on X if it satisfies the following:*

(1) A is densely defined and closed,

(2) there exist real numbers $a \in \mathbb{R}, \sigma \in (0, \frac{\pi}{2})$ and $M \geq 1$ such that one has $\Sigma_\sigma(a) \subset \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - a|}, \qquad \text{for all } \lambda \in \Sigma_\sigma(a) \tag{1.9}$$

where $\rho(A)$ is the resolvent set of A, $R(\lambda, A)$ the resolvent operator and $\Sigma_\sigma(a)$ defined as:

$$\Sigma_\sigma(a) = \{z \in \mathbb{C} \ : \ |arg(z - a)| > \sigma, \ \ z \neq a\}.$$

The next theorem which can be found in [38, 42] gives the relation between analytic semigroups and sectorial operators

Theorem 1.2.2 *Let $T(t)$ be a C_0-semigroup on X with infinitesimal generator A and let $M \geq 1$, $a \in \mathbb{R}$ be chosen so that $\|T(t)\| \leq Me^{-at}$, for all $t \geq 0$. Then the following statements are equivalent:*

(1) $T(t)$ is an analytic semigroup and there is an analytic extension semigroup $T(z)$ defined on some sector $\Delta_\delta \cup \{0\}$ with $0 < \delta < \frac{\pi}{2}$, and a constant $M_1 \geq M$ such that $\|T(z)\| \leq M_1 e^{-aR_e z}$ for $z \in \Delta_\delta$.

(2) A is a sectorial operator and one has

$$\|R(\lambda, A)\| \leq \frac{M_2}{|\lambda - a|}, \qquad \text{for all } \lambda \in \Sigma_\xi(a), \tag{1.10}$$

for appropriate constants $M_2 \geq 1$ and $\xi \in (0, \frac{\pi}{2})$.

Moreover, $T(t)$ is a differentiable semigroup.

For many partial differential equations, particularly the parabolic ones, the corresponding linear operator is sectorial. In the follwing example we present some elliptic operators that turn out to be sectorial. This will be used in the next chapter to show that the transformed linear operator for the nonlinear wave equation is maximal accretive.

Example 1.2.1 *Let Ω be either $\mathbb{R}^n$ or an open bounded subset of $\mathbb{R}^n$ with uniformly C^2 boundary $\partial\Omega$.*

We consider a second order differential operator

$$\mathcal{A}(x, D) = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} + \sum_{i=1}^{n} b_i(x) D_i + c(x) I$$

with real uniformly continuous and bounded coefficients a_{ij}, b_i, c.
We assume that the matrix $[a_{ij}]$ is symmetric and that it satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq \mu |\xi|^2, \quad x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{1.11}$$

for some $\mu > 0$. Moreover if $\Omega \neq \mathbb{R}^n$, we consider a first order differential operator acting on the boundary

$$\mathcal{B}(x, D) = \sum_{i=1}^{n} \beta_i(x) D_i + \gamma(x) I. \tag{1.12}$$

We assume that β_i, γ, *belong to* $UC^1(\bar{\Omega})$, *and that the uniform nontangentiality condition*

$$\inf_{x\in\partial\Omega} |\sum_{i=1}^{n} \beta_i(x)\nu(x)| > 0 \tag{1.13}$$

holds, with $\nu(x)$ *being the exterior unit normal vector to* $\partial\Omega$ *at* $x \in \partial\Omega$.

Let $X = L^p(\Omega)$, $1 < p < \infty$, *be endowed with the usual norm* $\|\cdot\|_p$. *Let* $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ *denote the usual Sobolev spaces on* Ω *with the usual norms.*

If $\Omega = \mathbb{R}^n$ *we set*

$$D(A) = W^{2,p}(\mathbb{R}^n), \qquad Au = \mathcal{A}(\cdot, D)u \qquad \textit{for } u \in D(A)$$

If $\Omega \neq \mathbb{R}^n$, *we set*

$$D(A_0) = W^{2,p}(\Omega) \cap W_0^{2,p}(\Omega), \quad A_0u = \mathcal{A}(\cdot, D)u \quad \textit{for } u \in D(A_0)$$

$$D(A_1) = \{u \in W^{2,p}(\Omega) : \mathcal{B}(\cdot, D)u = 0 \textit{ in } \partial\Omega\}, \quad A_1u = \mathcal{A}(\cdot, D)u \quad \textit{for } u \in D(A_1)$$

Theorems 3.1.2, 3.1.3 in [29] state that the operators A, A_0, A_1 *are sectorial operators in* X *and that there exists* ω, ω_1, $\omega_2 \in \mathbb{R}$ *such that*

$$\rho(A) \supset \{\lambda \in \mathbb{C} \,:\, R_e\lambda \geq \omega\},\ \ \rho(A_0) \supset \{\lambda \in \mathbb{C} \,:\, R_e\lambda \geq \omega_0\},\ \ \rho(A_1) \supset \{\lambda \in \mathbb{C} \,:\, R_e\lambda \geq \omega_1\}.$$

This example of sectorial operator will be useful in Chapter 2 when we work with the wave equations.

1.2.2 Nonlinear Evolution Equations

In this section, we will briefly present the existence theory for abstract nonlinear evolutionary equations in a Banach space X. There exists a vast literature on this issue, but we will just give some basic results, see [38, 42, 47].

We consider the following initial value problem in the Banach space X:

$$\begin{cases} \dfrac{du}{dt} + Au = F(u) \\ u(t_0) = u_0 \in X, \qquad t \geq t_0 \geq 0. \end{cases} \tag{1.14}$$

Assume that the nonlinearity F belongs to $F \in C_{Lip} = C_{Lip}(X, X)$, the collection of all continuous functions $G : X \to X$ that are Lipschitz continuous on every bounded set B in X. We suppose also that $-A$ generates a C_0-semigroup $T(t)$ on X.

At first, we give different notions of solution for problem (1.14) and then present some existence results for such types of solutions.

Definition 1.2.5 *Let $I = [t_0, t_0 + \tau)$ be an interval in $\mathbb{R}^+$, where $\tau > 0$. A strongly continuous mapping $u : I \to X$ is said to be a **mild solution** of (1.14) in X if it solves the following integral equation*

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s)F(u(s))\, ds, \qquad t \in I. \tag{1.15}$$

If u is differentiable almost everywhere in I with $u_t, Au \in L^1_{loc}(I, X)$, and satisfies the differential equation

$$\frac{du}{dt} + Au =^{a.e.} F(u), \text{ on } (t_0, t_0 + \tau), \quad \text{and } u(t_0) = u_0, \tag{1.16}$$

*then u is called a **strong solution** of (1.14). If in addition, one has $u_t \in C(I, X)$ and the differential equation in (1.16) is satisfied for $t_0 < t < t_0 + \tau$, then u is called a **classical solution** of (1.14) on I.*

We have the following result which is a particular case of Theorems 46.1 & 46.2 in G. Sell & Y. You [42].

Theorem 1.2.3 *Let $-A$ generate a C_0-semigroup $T(t)$ on X and $F \in C_{Lip} = C_{Lip}(X, X)$. Then for every $u_0 \in X$ and $t_0 > 0$, the Initial Value Problem (1.14) has a unique mild solution u in X on some interval $I = [t_0, t_0 + \tau)$, for some $\tau > 0$.*

Assume that $X = H$ is a Hilbert space or a reflexive Banach space. If $u_0 \in D(A)$ or $T(t)$ is a differentiable semigroup, then the mild solution is a strong one.

Remark 1.2.1 *The solution u in Theorem 1.2.3 can be extended to a maximum possible interval I. Indeed u is maximally deifined if either $\tau = +\infty$ or $\lim_{t \to \tau^-} \|u(t)\|_X = +\infty$.*

1.3 Some Useful Inequalities

We present in this section some inequalities that will be used in the consequent chapters. The most used inequality throughout our work is the Gronwall inequality which comes in different forms. We present here some variants of it.

Lemma 1.3.1 (The Gronwall inequality) *Suppose that a and b are nonnegative constants and $u(t)$ a nonnegative integrable function. Suppose that the following inequality holds for $0 \leq t \leq T$:*

$$u(t) \leq a + b \int_0^t u(s)\, ds. \tag{1.17}$$

Then for $0 \leq t \leq T$, we have

$$u(t) \leq a e^{bt}. \tag{1.18}$$

Lemma 1.3.2 (The uniform Gronwall inequality) *Let g, h, y be nonnegative functions in $L^1_{loc}[0, T; \mathbb{R})$, where $0 < T \leq \infty$. Assume that y is absolutely continuous on $(0, T)$ and that*

$$\frac{dy}{dt} \leq gy + h \qquad \text{almost everywhere on } (0, T). \tag{1.19}$$

Then $y \in L^\infty_{loc}(0, T; \mathbb{R})$ and one has

$$y(t) \leq y(t_0) exp\Big(\int_{t_0}^t g(s)\, ds\Big) + \int_{t_0}^t exp\Big(\int_s^t g(r)\, dr\Big) h(s)\, ds, \tag{1.20}$$

for $0 < t_0 < t < T$. If in addition one has $y \in C[0, T; \mathbb{R})$, then inequality (1.20) is valid at $t_0 = 0$.

The following theorem is concerned with the Poincaré inequality.

Theorem 1.3.1 (Poincaré inequality) *Let Ω be a domain of $\mathbb{R}^N$ bounded only in one direction and let $u \in H^1_0(\Omega)$. Then there is a positive constant C depending only on Ω and n such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{(L^2(\Omega))^N}, \qquad \forall u \in H^1_0(\Omega). \tag{1.21}$$

Remark 1.3.1 *The Poincaré inequality is usually presented for bounded domains but the proof requires only the boundedness in one direction* x_i.

2 ATTRACTORS FOR DAMPED WAVE EQUATIONS

We study in this chapter the existence of a global attractor for the following two damped nonlinear wave equations in an unbounded domain of $\mathbb{R}^N$:

$$u_{tt} + \lambda u_t - \triangle u + u + f(u) = g(x), \quad t > 0 \tag{2.1}$$

and

$$u_{tt} + \lambda u_t - \triangle u + f(u) = g(x), \quad t > 0 \tag{2.2}$$

where λ is a positive constant, g is a given function and f is a nonlinear term satisfying some growth conditions to be specified later. The long-time behavior of solutions of such equations in a bounded domain was studied by many authors, for instance in [47], [42] and the references therein.

In the unbounded domain case, there also exists an extensive literature. In 1994, E. Feireisl [16] showed that the more challenging equation (2.2) admits a global attractor in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ when $N = 3$. For arbitrary n, he obtain in [17] the same result in the phase space $H^1_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N)$.

In 2001, S.V. Zelik [53] considered the nonautonomous case for equation (2.1), in which the forcing term g depends on time. He obtained the existence of locally compact global attractor and the upper and lower bounds for their Kolomogorov's ε-entropy. Some other authors have also considered different types of wave equations in unbounded domains ([24], [25]) in weighted spaces.

In this chapter we establish the existence of global attractors in the usual Hilbert spaces $H^1 \times L^2$, for equations (2.1) and (2.2) in unbounded domains of $\mathbb{R}^N$. To this end we cannot apply the same procedure as for bounded domains, since the Sobolev embeddings are no longer compact. We will apply the "tail estimation" method, introduced for the first order lattice systems and the reaction diffusion equations ([4, 49, 50]). It features an "approximation" of R^N by sufficiently large bounded domains Ω_k, then using the compactness of the embeddings in Ω_k and showing the uniform null convergence of the solutions on $\mathbb{R}^N - \Omega_k$, we finally arrive to get the asymptotical compactness of the semiflow.

2.1 The Wave Equation with Mass Term

We consider in this section, the nonlinear wave equation with mass term,

$$u_{tt} + \lambda u_t - \triangle u + u + f(u) = g(x), \quad t > 0$$

in $\mathbb{R}^N$. We shall establish first the existence and boundedness of solutions, then we shall prove the asymptotic compactness of the corresponding semiflow to obtain the global attractor.

2.1.1 Existence of Solutions and Absorbing Set

We start by transforming our problem into an abstract ODE in the space $L^2 \times H^1$ and prove that the new operator is maximal accretive. This will allow us to show the existence of solutions and the uniform boundedness of such solutions.

We consider the system

$$u_{tt} + \lambda u_t - \triangle u + u + f(u) = g(x), \qquad x \in \mathbb{R}^N, \quad t > 0 \tag{2.3}$$

with initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \qquad x \in \mathbb{R}^N \tag{2.4}$$

where $\lambda > 0, \quad g \in L^2(\mathbb{R}^N)$, and $f \in C^1(\mathbb{R},\ \mathbb{R})$ satisfies the following condition:

$$f(0) = 0, \qquad f(s)s \geq \nu F(s) \geq 0, \qquad \forall s \in \mathbb{R} \tag{2.5}$$

where ν is a positive constant and $F(s) = \int_0^s f(t)\,dt$. In addition we assume that

$$0 \leq \limsup_{s\to\infty} \frac{f(s)}{s} < \infty \tag{2.6}$$

Now set $H = L^2(\mathbb{R}^N)$, $V = H^1(\mathbb{R}^N)$, and $X = V \times H$ with the usual norms and scalar products. We define the operator G in X by:

$$D(G) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$$

$$Gw = \begin{pmatrix} \delta u - v \\ \\ -\Delta u + (\lambda - \delta)v + (\delta^2 - \delta\lambda + 1)u \end{pmatrix} \tag{2.7}$$

for $w = (u, v) \in D(G)$.

Then (2.3), (2.4) are equivalent to the initial value problem in X:

$$\begin{cases} w_t + Gw = R(w), \qquad t > 0, \quad w \in X \\ \\ w(0) = w_0 = (u_0,\, v_0 + \delta u) \end{cases} \tag{2.8}$$

where

$$R(w) = \begin{pmatrix} 0 \\ \\ -f(u) + g \end{pmatrix}$$

The next result establishes the maximal accretivity of the the operator G in X.

Lemma 2.1.1 *For a suitable δ chosen to be $\delta = \dfrac{\lambda}{\lambda^2 + 4}$, the operator G defined previously is maximal accretive in X, and there exists a constant $C(\delta) > 0$ depending on δ such that*

$$\langle Gw,\, w\rangle_X \ \geq C(\delta)\|w\|_X^2, \qquad \forall w \in D(G) \tag{2.9}$$

Proof: We first prove the positivity. Let $w = (u,\, v) \in D(G)$, then

$$\begin{aligned}
\langle Gw\,,\, w\rangle_X &= \langle \delta u - v, u\rangle_V + \langle -\Delta u + (\lambda-\delta)v + (\delta^2 - \delta\lambda + 1)u,\, v\rangle_H \\
&= \delta\|u\|_V^2 - \textstyle\int_{\mathbb{R}^N} \nabla u \cdot \nabla v\, dx - \langle u,\, v\rangle_H + \textstyle\int_{\mathbb{R}^N} \nabla u \cdot \nabla v\, dx \\
&\quad +(\lambda-\delta)\|v\|_H + (\delta^2 - \delta\lambda + 1)\langle u,\, v\rangle_H \\
&= \delta\|u\|_V^2 + (\lambda-\delta)\|v\|_H^2 + (\delta^2 - \delta\lambda)\langle u,\, v\rangle_H
\end{aligned}$$

Then setting

$$\sigma = \frac{\lambda}{\sqrt{\lambda^2+4}(\lambda + \sqrt{\lambda^2+4})}, \tag{2.10}$$

we have

$$\begin{aligned}
\langle G(w)\,,\, w\rangle_X - \sigma(\|u\|_V^2 + \|v\|_H^2) - \tfrac{\lambda}{2}\|v\|_H^2 &\geq (\delta-\sigma)\|u\|_V^2 + (\tfrac{\lambda}{2} - \delta - \sigma)\|v\|_H^2 \\
&\quad -\delta\lambda\|u\|_V\|v\|_H \\
&\geq 2\sqrt{(\delta-\sigma)(\tfrac{\lambda}{2} - \delta - \sigma)}\|u\|_V\|v\|_H \\
&\quad -\delta\lambda\|u\|_V\|v\|_H.
\end{aligned}$$

We can check that $4(\delta-\sigma)(\frac{\lambda}{2} - \delta - \sigma) = \lambda^2\delta^2$ so that

$$\langle G(w)\,,\, w\rangle_X - \sigma\|w\|_X^2 - \frac{\lambda}{2}\|v\|^2 \geq 0.$$

It suffices to take $C(\delta) = \sigma$

Now we prove that the range of $G + I$ equals X. Let $f = (h, g) \in X$; the question is whether there exists a $w = (u, v) \in D(G)$ such that:

$$Gw + w = f \quad ?$$

$$\text{i.e.}\quad \begin{cases} \delta u - v + u = h \\ -\Delta u + (\lambda - \delta)v + (\delta^2 - \delta\lambda + 1)u = g \end{cases}$$

$$\text{i.e.}\quad \begin{cases} v = (\delta + 1)u - h \\ -\Delta u + (\lambda - \delta)[(\delta + 1)u - h] + (\delta^2 - \delta\lambda + 1)u = g \end{cases}$$

$$\text{i.e.}\quad \begin{cases} v = (\delta + 1)u - h \\ -\Delta u + (\lambda - \delta + 1)u = g + (\lambda - \delta)h \end{cases}$$

Note that the operator $Au = -\Delta u$ in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$ is a sectorial operator and there exists $\omega \in \mathbb{R}$ such that $\varrho(A) \supset \{\lambda \in \mathbb{C} : Re\,\lambda \geq \omega\}$ (this is a particular case in Example 1.2.1). So the equation

$$-\Delta u + (\lambda - \delta + 1)u = g + (\lambda - \delta)h$$

has a unique solution $u \in H^2(\mathbb{R}^N)$, thus letting $v = (\delta + 1)u - h$ and $w = (u, v)$, we get a unique $w \in D(G)$ such that $Gw + w = f$. So the range of $G + I$ equals X. This, with (2.9), shows that G is maximal accretive and finishes the proof of lemma 2.1.1.

Lemma 2.1.1 together with the Lumer-Phillips Theorem 1.2.1 imply that $-G$ generates a nonexpansive C_0-semigroup e^{-Gt} on X. Furthermore since f verifies (2.6), the operator $R : X \to X$ is locally Lipschitz continuous. By the standard theory of evolutionary equations (see G. R. Sell & Y. You [42], Theorem 46.1) this leads to the existence and uniqueness of local solutions as stated in the next lemma.

Lemma 2.1.2 *If $g \in L^2(\mathbb{R}^N)$ and f satisfies (2.6), then for any initial data $w_0 = (u_0, v_0) \in X$, there exists a unique local solution $w(t) = (u(t), v(t))$ of (2.8) such that $w \in C^1((-T_0, T_0), E)$ for some $T_0 = T_0(w_0) > 0$.*

In fact we will show that the local solution $w(t)$ of (2.8) is bounded and exists globally.

Lemma 2.1.3 *Assume that (2.5) and (2.6) are satisfied and that $g \in H$. Then any solution*

$w(t)$ of problem (2.8) satisfies

$$\|w(t)\|_X \leq M, \qquad t \geq T_1 \tag{2.11}$$

where M is a constant depending only on $(\lambda,\ g)$ and T_1 depending on the data $(\lambda,\ g,\ R)$ when $\|w_0\|_X \leq R$.

Proof: Let $w_0 \in D(G)$ be the initial condition in (2.8). Taking the inner-product of (2.8) with w in X we find that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|w\|_X^2 &= -\langle Gw\,,\, w\rangle_X + \langle R(w),\, w\rangle_E \\
&= -\langle Gw\,,\, w\rangle_X + \langle g\,,\, v\rangle_H - \langle f\,,\, v\rangle_H \\
&\leq -C(\delta)\|w\|_X^2 + \|g\|_H\|v\|_H - \delta\langle f(u)\,,\, u\rangle_H - \langle f(u)\,,\, u_t\rangle_H,
\end{aligned}$$

by (2.5) we have

$$-\delta\langle f(u)\,,\, u\rangle_H \leq -\delta\nu\int_{\mathbb{R}^N} F(u)\,dx$$

and

$$-\langle f(u)\,,\, u_t\rangle_H = -\frac{d}{dt}\int_{\mathbb{R}^N} F(u)\,dx.$$

Then using the Young inequality, it follows for any $\alpha > 0$ that

$$\frac{1}{2}\frac{d}{dt}\|w\|_X^2 \leq -C(\delta)\|w\|_X^2 + \frac{\alpha}{2}\|v\|_H^2 + \frac{1}{2\alpha}\|g\|_H^2 - \delta\nu\int_{\mathbb{R}^N} F(u)\,dx - \frac{d}{dt}\int_{\mathbb{R}^N} F(u)\,dx$$

which implies that

$$\frac{d}{dt}\left[\|w\|_X^2 + 2\int_{\mathbb{R}^N} F(u)\,dx\right] \leq 2\left(\alpha - C(\delta)\right)\|w\|_X^2 - 2\delta\nu\int_{\mathbb{R}^N} F(u)\,dx + \frac{1}{\alpha}\|g\|_H^2$$

Now we can choose α small enough so that $\alpha - C(\delta) < 0$ and taking

$\mu = \min\{-2(\alpha - C(\delta))\ ,\ \delta\nu\} > 0$ we have

$$\frac{d}{dt}\left[\|w\|_X^2 + 2\int_{\mathbb{R}^N} F(u)\,dx\right] \le -\mu\left[\|w\|_X^2 + 2\int_{\mathbb{R}^N} F(u)\,dx\right] + \frac{1}{\alpha}\|g\|_H^2 \tag{2.12}$$

and then by Uniform Gronwall inequality we get

$$\|w\|_X^2 + 2\int_{\mathbb{R}^N} F(u)\,dx \le e^{-\mu t}\left(\|w_0\|_X^2 + 2\int_{\mathbb{R}^N} F(u_0)\,dx\right) + (1 - e^{-\mu t})\frac{1}{\mu\alpha}\|g\|_H^2$$

which yields

$$\|w\|_X^2 \le e^{-\mu t}\left(\|w_0\|_X^2 + 2\int_{\mathbb{R}^N} F(u_0)\,dx\right) + \frac{1}{\mu\alpha}\|g\|_H^2. \tag{2.13}$$

Now by (2.5) we have

$$\int_{\mathbb{R}^N} F(u_0)\,dx \le \frac{1}{\nu}\int_{\mathbb{R}^N} f(u_0)u_0\,dx \le \frac{C}{\nu}\int_{\mathbb{R}^N} u_0^2(x)\,dx.$$

Then we deduce from (2.13) that for every $w_0 \in D(G)$,

$$\|w\|_X^2 \le e^{-\mu t}\left(\|w_0\|_X^2 + \frac{C}{\nu}\|u_0\|_H^2\right) + \frac{1}{\mu\alpha}\|g\|_H^2. \tag{2.14}$$

And by density of $D(G)$ in X and the continuity of the solution of (2.8) in $X \times (0\,, T(w_0))$ we see that (2.13) holds for every $w_0 \in X$.

Now let $R > 0$ and $\|w_0\|_X \le R$, then $\|u_0\|_H \le R$ and

$$\|w\|_X^2 \le e^{-\mu t}\left(R^2 + \frac{CR^2}{\nu}\right) + \frac{1}{\mu\alpha}\|g\|_H^2 \tag{2.15}$$

which yields

$$\|w\|_X^2 \le \frac{2}{\mu\alpha}\|g\|_H^2, \qquad \text{for } t \ge T_1 = \frac{1}{\mu}\ln\left\{\frac{\mu\lambda(R^2 + \frac{CR^2}{\nu})}{\|g\|_H^2}\right\} \tag{2.16}$$

and (2.11) follows with $M = \dfrac{2}{\mu\alpha}\|g\|_H^2$ and the proof is complete.

By (2.15), We have also the following result.

Lemma 2.1.4 *Let $g \in H$. Then for any given $T > 0$, every solution w of (2.8) satisfies*

$$\|w\|_X \leq L, \qquad 0 \leq t \leq T \tag{2.17}$$

where L depends on $(\lambda, \delta, \|g\|_H)$, T and $\|w_0\|_X$.

Lemma 2.1.3 implies that the solution $w(t)$ exists globally, that is $T(w_0) = +\infty$, which implies that the system (2.8) generates a continuous semiflow $\{S(t)\}_{t\geq 0}$ on X. Denote by O the ball

$$O = \{w \in X \,:\, \|w\|_X \leq M\} \tag{2.18}$$

where M is the constant in (2.11). Then it follows from (2.11) that O is an absorbing set for $S(t)$ in X and that for every bounded set B in X there exists a constant $T(B)$ depending only on (λ, g) and B such that

$$S(t)B \subseteq O, \quad t \geq T(B). \tag{2.19}$$

In particular there exists a constant T_0 depending only on (λ, g) and O such that

$$S(t)O \subseteq O, \quad t \geq T_0. \tag{2.20}$$

2.1.2 Global Attractor

The existence of an absorbing set is the first step toward the existence of a global attractor. We need now to prove the asymptotic compactness of $S(t)$. The key idea lies in establishing uniform estimates on "Tail Ends" of solutions, that is, the norm of the solutions $w(t)$ are uniformly small with respect to t outside a sufficiently large ball.

Lemma 2.1.5 *If (2.5) and (2.6) hold, $g \in H$ and $w_0 = (u_0, v_0) \in O$, then for every $\varepsilon > 0$, there exists positive constants $T(\varepsilon)$ and $K(\varepsilon)$ such that the solution $w(t) = (u(t), v(t))$ of problem (2.8) satisfies*

$$\int_{|x|\geq k} \left\{|u|^2 + |\nabla u|^2 + |v|^2\right\} dx \leq \varepsilon, \qquad t \geq T(\varepsilon), \quad k \geq K(\varepsilon). \tag{2.21}$$

Proof: Choose a smooth function θ such that $0 \leq \theta(s) \leq 1$ for $s \in \mathbb{R}^+$, and

$$\theta(s) = 0 \quad \text{for } 0 \leq s \leq 1; \qquad \theta(s) = 1 \quad \text{for } s \geq 2.$$

Then there exists a constant $C > 0$ such that $|\theta'(s)| \leq C$ for $s \in \mathbb{R}^+$.
Let $w(t) = (u(t), v(t))$ be the solution of problem (2.8) with initial condition $w_0 = (u_0, v_0) \in O$ then $v(t) = \delta u + u_t$ satisfies the equation

$$v_t - \Delta u + (\lambda - \delta)v + (\delta^2 - \lambda\delta + 1)u = -f(u) + g \tag{2.22}$$

taking inner product of (2.22) with $\theta(\frac{|x|^2}{k^2})v$ in H we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) v v_t \, dx - \int_{\mathbb{R}^N} \Delta u \theta(\frac{|x|^2}{k^2}) v \, dx + (\lambda - \delta) \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) |v|^2 \, dx \\ &+ (\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) uv \, dx = -\int_{\mathbb{R}^N} f(u) \theta(\frac{|x|^2}{k^2}) v \, dx + \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) gv \, dx \end{aligned} \tag{2.23}$$

But

$$\begin{aligned} -\int_{\mathbb{R}^N} \Delta u \theta(\frac{|x|^2}{k^2}) v \, dx &= \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) \nabla u \cdot \nabla v + \frac{2}{k^2} \int_{\mathbb{R}^N} \theta'(\frac{|x|^2}{k^2}) v x \cdot \nabla u \\ &= \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) \left[\delta |\nabla u|^2 + \nabla u \cdot \nabla u_t \right] + \frac{2}{k^2} \int_{\mathbb{R}^N} \theta'(\frac{|x|^2}{k^2}) v x \cdot \nabla u \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) |\nabla u|^2 + \delta \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) |\nabla u|^2 \\ &\quad + \frac{2}{k^2} \int_{\mathbb{R}^N} \theta'(\frac{|x|^2}{k^2}) v x \cdot \nabla u, \end{aligned}$$

and

$$(\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) uv \, dx = (\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) (\delta |u|^2 + u u_t)$$

$$= \frac{1}{2} (\delta^2 - \lambda\delta + 1) \frac{d}{dt} \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) |u|^2 + \delta(\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) |u|^2.$$

Then (2.23) becomes

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})\left[(\delta^2-\lambda\delta+1)|u|^2+|\nabla u|^2+|v|^2\right]$$

$$+\delta\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})\left[(\delta^2-\lambda\delta+1)|u|^2+|\nabla u|^2+|v|^2\right]+(\lambda-2\delta)\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|v|^2 \quad (2.24)$$

$$=-\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})f(u)(\delta u+u_t)+\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})gv\,dx-\frac{2}{k^2}\int_{\mathbb{R}^N}\theta'(\frac{|x|^2}{k^2})vx\cdot\nabla u,$$

and since

$$\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})f(u)(\delta u+u_t)\geq\frac{d}{dt}\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})F(u)+\delta\nu\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})F(u),$$

we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})\left[(\delta^2-\lambda\delta+1)|u|^2+|\nabla u|^2+|v|^2+2F(u)\right]$$

$$+\delta\alpha\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})\left[(\delta^2-\lambda\delta+1)|u|^2+|\nabla u|^2+|v|^2+2F(u)\right] \quad (2.25)$$

$$\leq-(\lambda-2\delta)\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|v|^2+\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})gv\,dx-\frac{2}{k^2}\int_{\mathbb{R}^N}\theta'(\frac{|x|^2}{k^2})vx\cdot\nabla u,$$

where $\alpha=\min\{1\ ,\ \nu\}$. Now, there exists a constant $K(\varepsilon)>0$ such that for $k\geq K$, we have

$$-(\lambda-2\delta)\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})|v|^2+\int_{\mathbb{R}^N}\theta(\frac{|x|^2}{k^2})gv\,dx-\frac{2}{k^2}\int_{\mathbb{R}^N}\theta'(\frac{|x|^2}{k^2})vx\cdot\nabla u\leq\frac{\varepsilon}{2},$$

which implies by Uniform Gronwall inequality that

$$\int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 + 2F(u)\right]$$

$$\leq e^{-\delta\alpha t} \int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta + 1)|u_0|^2 + |\nabla u_0|^2 + |v_0|^2 + 2F(u_0)\right] + \varepsilon\frac{1 - e^{-\delta\alpha}}{2\delta\alpha}.$$

Now since $w_0 \in O$, there exist a constant $M > 0$, uniformly chosen for $w_0 \in O$, such that

$$\int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta + 1)|u_0|^2 + |\nabla u_0|^2 + |v_0|^2 + 2F(u_0)\right] \leq M.$$

Then we get for $k \geq K(\varepsilon)$,

$$\int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 + 2F(u)\right] \leq Me^{-\delta\alpha t} + \varepsilon\frac{1 - e^{-\delta\alpha}}{2\delta\alpha}.$$

Choosing $T(\varepsilon) = \frac{1}{\delta} \ln\Big(\frac{2M\delta\alpha}{2\varepsilon\delta\alpha - \varepsilon}\Big)$, we deduce that

$$\int_{\mathbb{R}^N} \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta + 1)|u|^2 + |\nabla u|^2 + |v|^2\right] \leq \varepsilon \qquad \text{for } t \geq T(\varepsilon), \;\; k \geq K(\varepsilon)$$

which yields (2.21) since $0 < \delta^2 - \lambda\delta + 1 < 1$ for the particular choice of δ, and the proof is complete.

By multiplying equation (2.22) with v and integrating we deduce the following energy equation

$$\frac{d}{dt}E(w(t)) + 2\delta E(w(t)) = G(w(t)) \qquad \forall t > 0, \tag{2.26}$$

where $E(w)$ is the quasi-energy functional,

$$E(w) = (\delta^2 - \lambda\delta + 1)\|u\|_H^2 + \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \|v\|_H^2, \tag{2.27}$$

and

$$G(w) = -2(\lambda - 2\delta)\|v\|_H^2 + 2\int_{\mathbb{R}^N} gv\,dx - 2\int_{\mathbb{R}^N} f(u)v\,dx. \tag{2.28}$$

This energy functional E will be used later as an equivalent norm, more suitable in proving the asymptotical compactness.

The following lemma will be also useful in proving the asymptotical compactness.

Lemma 2.1.6 *Let $w_n = (u_n, v_n) \longrightarrow w_0 = (u_0, v_0)$ weakly in X, then for every $T > 0$ we have*

$$S(t)w_n \longrightarrow S(t)w_0 \qquad \textit{weakly in } L^2(0,T;X) \tag{2.29}$$

and

$$S(t)w_n \longrightarrow S(t)w_0 \qquad \textit{weakly in } X, \quad \textit{for } 0 \le t \le T. \tag{2.30}$$

Proof: Since $\{w_n\}_n$ converges weakly in X, then it is bounded in X so that, by lemma 2.1.4 $\{S(t)w_n\}_n$ is bounded in $L^\infty(0,T;X)$. This, with (2.8), implies that

$$\frac{\partial}{\partial t}S(t)v_n \quad \text{is bounded in } L^\infty(0,T;H^{-1}(\mathbb{R}^N)) \tag{2.31}$$

and

$$S(t)v_n \quad \text{is bounded in } L^\infty(0,T;L^2(\mathbb{R}^N)). \tag{2.32}$$

We infer that there exists a subsequence $\{w_{n_j}\}_j$ and $w_\infty = (u_\infty, v_\infty) \in L^\infty(0,T;X)$ such that

$$S(t)w_{n_j} \longrightarrow w_\infty \qquad \text{weakly in } L^2(0,T;X), \tag{2.33}$$

$$\frac{\partial}{\partial t}S(t)v_{n_j} \longrightarrow \frac{\partial}{\partial t}v_\infty \qquad \text{weakly in } L^\infty(0,T;H^{-1}(\mathbb{R}^N)) \tag{2.34}$$

and

$$\frac{\partial}{\partial t}S(t)u_{n_j} \longrightarrow \frac{\partial}{\partial t}u_\infty \qquad \text{weakly in } L^2(0,T;H). \tag{2.35}$$

We can show that w_∞ is a solution of (2.8) with $w_\infty(0) = w_0$. Indeed, we have by the mild solution formula,

$$S(t)w_{n_j} = e^{-Gt}w_{n_j} + \int_0^t e^{-G(t-s)}R(S(s)w_{n_j})\,ds. \tag{2.36}$$

And, since $w_{n_j} \to w_0$ weakly in X, we deduce from (2.33) that

$$e^{-Gt}w_{n_j} + \int_0^t e^{-G(t-s)}R(S(s)w_{n_j})\,ds \longrightarrow e^{-Gt}w_0 + \int_0^t e^{-G(t-s)}R(w_\infty(s))\,ds, \tag{2.37}$$

weakly in X. which implies, by the uniqueness of weak limit that

$$w_\infty(t) = e^{-Gt}w_0 + \int_0^t e^{-G(t-s)}R(w_\infty(s))\,ds. \tag{2.38}$$

That is w_∞ is a solution of (2.8) and by the uniqueness of solutions we have $w_\infty(t) = S(t)w_0$. This shows that any subsequence of $S(t)w_n$ has a weakly convergent subsequence in $L^2(0,T;X)$, therefore we conclude (2.29). A similar argument yields (2.30).

Similar to (2.29) we also have that if $w_n \longrightarrow w$ weakly in X, then for $0 \le s \le T$,

$$S(t)w_n \longrightarrow S(t)w_0 \qquad \text{weakly in } L^2(s,T;X) \tag{2.39}$$

We state here another useful lemma.

Lemma 2.1.7 *Let Ω be a bounded domain in $\mathbb{R}^N$. Suppose $u_n \longrightarrow u$ in $L^2(\Omega)$ and $v_n \longrightarrow v$ weakly in $L^2(\Omega)$, then $\int_\Omega f(u_n)v_n\,dx \longrightarrow \int_\Omega f(u)v\,dx$ in $\mathbb{R}$ (up to a subsequence extraction).*

Proof: By (2.6) we can show, up to a subsequence, that $f(u_n) \longrightarrow f(u)$ in $L^2(\Omega)$. Now define the linear functionals I_n and I on $L^2(\Omega)$ by

$$I_n(v) = \int_\Omega f(u_n)v\,dx, \quad I(v) = \int_\Omega f(u)v\,dx.$$

Then $I_n \longrightarrow I$ in $L^2(\Omega)^*$ (the dual space of $L^2(\Omega)$. Indeed

$$|I_n(v) - I(v)| \;\le\; \int_\Omega |f(u_n) - f(u)||v|\,dx$$

$$\leq \|f(u_n) - f(u)\|_{L^2}\|v\|_{L^2}.$$

which implies that

$$\|I_n - I\|_{L^2} \leq \|f(u_n) - f(u)\|_{L^2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

So $I_n \to I$ in $L^2(\Omega)^*$ and $v_n \to v$ weakly in $L^2(\Omega)$, then it follows, by a classical result in functional analysis that $I_n(v_n) \longrightarrow I(v)$, which proves the lemma.

We are now ready to prove the asymptotic compactness of the semiflow $S(t)$.

Theorem 2.1.1 *The semiflow $S(t)$ generated by the system (2.8) is asymptotically compact in X, that is if $\{w_n\}_n$ is a bounded sequence in X and $t_n \longrightarrow +\infty$, then $\{S(t)w_n\}_{n\geq 1}$ is precompact in X.*

Proof: Let w_n be a bounded sequence in X with $\|w_n\|_X \leq R$ and $t_n \longrightarrow +\infty$ then by (2.19) there exists a constant $T(R) > 0$ depending only on $R > 0$ such that

$$S(t)w_n \in O, \qquad \forall n \geq 1, \quad \forall t \geq T(R). \tag{2.40}$$

Since $t_n \longrightarrow +\infty$, there exists $N_1(R)$ such that $n \geq N_1$ implies $t_n \geq T(R)$ so that

$$S(t_n)w_n \in O, \qquad \forall n \geq N_1(R). \tag{2.41}$$

Then there exists $w \in X$ such that, up to a subsequence

$$S(t_n)w_n \longrightarrow w \qquad \text{weakly in } X. \tag{2.42}$$

Now for every $T > 0$ there exists $N_2(R,T)$ such that for $n \geq N_2(R,T)$ we have $t_n - T \geq T(R)$ so that

$$S(t_n - T)w_n \in O \qquad \forall n \geq N_2(R,T). \tag{2.43}$$

Thus there is a $w_T \in O$ such that

$$S(t_n - T)w_n \longrightarrow w_T \qquad \text{weakly in } X, \tag{2.44}$$

and by the weak continuity (2.30) we must have $w = S(T)w_T$ which implies that

$$\liminf_{n\to\infty} \|S(t_n)w_n\|_X \geq \|w\|_X. \tag{2.45}$$

So we only need to prove that

$$\limsup_{n\to\infty} \|S(t_n)w_n\|_X \leq \|w\|_X. \tag{2.46}$$

By the energy equation (2.26), it follows that any solution $w(t) = S(t)w$ of (2.8) satisfies

$$E(S(t)w) = e^{-2\delta(t-s)}E(S(s)w) + \int_s^t e^{-2\delta(t-r)}G(S(r)w)\,dr\,, \quad t \geq s \geq 0. \tag{2.47}$$

where E and G are given by (2.27) and (2.28), respectively.

In the following, T_0 is the constant in (2.20), and for $\varepsilon > 0$, $T(\varepsilon)$ is the constant in (2.21). Let $T_0(\varepsilon)$ be a fixed constant such that $T_0(\varepsilon) \geq \max\{T(\varepsilon)\,,\, T_0\}$. Taking $T \geq T_0(\varepsilon)$, and applying (2.47) to the solution $S(t)(S(t_n - T)w_n)$ with $s = T_0$ and $t = T$, then we get, for $n \geq N_2(R,T)$,

$$\begin{aligned} E(S(t_n)w_n) &= E(S(T)(S(t_n - T)w_n)) \\ &= e^{-2\delta(T-T_0)}E(S(T_0)(S(t_n - T)w_n)) \\ &+ \int_{T_0}^T e^{-2\delta(T-r)}G(S(r)(S(t_n - T)w_n))\,dr. \end{aligned} \tag{2.48}$$

Since $T_0 \geq T_0$ we have $S(T_0)(S(t_n - T)w_n) \in O$ for $n \geq N_2(R,T)$, therefore by the definition of E we find that

$$e^{-2\delta(T-T_0)}E(S(T_0)(S(t_n - T)w_n)) \leq Ce^{-2\delta(T-T_0)}, \quad \forall n \geq N_2(R,T). \tag{2.49}$$

On the other hand, we have

$$\int_{T_0}^T e^{-2\delta(T-r)}G(S(r)(S(t_n - T)w_n))\,dr$$

$$
\begin{aligned}
&= -2(\lambda-\delta)\int_{T_0}^{T} e^{-2\delta(T-r)}\|S(r)S(t_n-T)v_n\|^2\,dr \\
&\quad +2\int_{T_0}^{T} e^{-2\delta(T-r)}\int_{\mathbb{R}^N} gS(r)S(t_n-T)v_n\,dxdr \\
&\quad -2\int_{T_0}^{T} e^{-2\delta(T-r)}\int_{\mathbb{R}^N} f(S(r)S(t_n-T)u_n)S(r)S(t_n-T)v_n\,dxdr
\end{aligned} \tag{2.50}
$$

Let's handle the first and last term of (2.50). Since we have,

$$
e^{-2\delta(T-r)}S(r)S(t_n-T)v_n \longrightarrow e^{-2\delta(T-r)}S(r)v \text{ weakly in } L^2(T_0,T;H),
$$

it follows that:

$$
\liminf_{n\to\infty}\|e^{-2\delta(T-r)}S(r)S(t_n-T)v_n\|_{L^2(T_0,T;H)} \geq \|e^{-2\delta(T-r)}S(r)v\|_{L^2(T_0,T;H)},
$$

which implies that

$$
\begin{aligned}
&\limsup_{n\to\infty} -2(\lambda-\delta)\|e^{-2\delta(T-r)}S(r)S(t_n-T)v_n\|_{L^2(T_0,T;H)} \\
&\qquad \leq -2(\lambda-\delta)\|e^{-2\delta(T-r)}S(r)v\|_{L^2(T_0,T;H)}.
\end{aligned} \tag{2.51}
$$

Also by (2.44) and (2.39) we have

$$
\int_{T_0}^{T} e^{-2\delta(T-r)}\int_{\mathbb{R}^N} gS(r)S(t_n-T)v_n\,dxdr \longrightarrow \int_{T_0}^{T} e^{-2\delta(T-r)}\int_{\mathbb{R}^N} gS(r)v_T\,dxdr \tag{2.52}
$$

Now let's handle the nonlinear term of (2.50). We have

$$
\begin{aligned}
&-2\int_{T_0}^{T} e^{-2\delta(T-r)}\int_{\mathbb{R}^N} f(S(r)S(t_n-T)u_n)S(r)S(t_n-T)v_n\,dxdr \\
&= -2\int_{T_0}^{T} e^{-2\delta(T-r)}\int_{|x|\geq k} f(S(r)S(t_n-T)u_n)S(r)S(t_n-T)v_n\,dxdr \\
&\quad -2\int_{T_0}^{T} e^{-2\delta(T-r)}\int_{|x|\leq k} f(S(r)S(t_n-T)u_n)S(r)S(t_n-T)v_n\,dxdr.
\end{aligned} \tag{2.53}
$$

Handling the first term on the right-hand side of (2.53) gives

$$
\begin{aligned}
&\left|2\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{|x|\geq k} f(S(r)S(t_n-T)u_n)S(r)S(t_n-T)v_n \, dxdr\right| \\
&\leq C\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{|x|\geq k} |S(r)S(t_n-T)u_n||S(r)S(t_n-T)v_n| \\
&\leq C\int_{T_0}^{T} e^{-2\delta(T-r)} \left(\int_{|x|\geq k} |S(r)S(t_n-T)u_n|^2\right)^{\frac{1}{2}} \left(\int_{|x|\geq k} |S(r)S(t_n-T)v_n|^2\right)^{\frac{1}{2}} \\
&\leq \varepsilon^2 C\int_{T_0}^{T} e^{-2\delta(T-r)} \, dr \leq \frac{\varepsilon^2 C}{2\delta}, \qquad n \geq N_2(R,T). \qquad (2.54)
\end{aligned}
$$

We treat now the second term on the right-hand side of (2.53). We want to prove that as $n \to +\infty$,

$$
\begin{aligned}
&\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{|x|\leq k} f(S(r)S(t_n-T)u_n)S(r)S(t_n-T)v_n \, dxdr \\
&\longrightarrow \int_{T_0}^{T} e^{-2\delta(T-r)} \int_{|x|\leq k} f(S(r)u_T)S(r)v_T \, dxdr \qquad (2.55)
\end{aligned}
$$

Set $\Omega_k = \{x \in \mathbb{R}^N : |x| \leq k\}$ and let $r \in [T_0, T]$. Then we have

$$S(r)S(t_n-T)w_n \longrightarrow S(r)w_T, \qquad \text{weakly in } X.$$

By the compactness of the Sobolev embedding $H^1(\Omega_k) \subset L^2(\Omega_k)$, we infer that

$$S(r)S(t_n-T)u_n \longrightarrow S(r)u_T, \qquad \text{strongly in } L^2(\Omega_k) \qquad (2.56)$$

and

$$S(r)S(t_n-T)v_n \longrightarrow S(r)v_T, \qquad \text{weakly in } L^2(\Omega_k) \qquad (2.57)$$

then (2.55) follows from lemma 2.1.7.

By (2.53), (2.54) and (2.55) we find that for $k \geq K(\varepsilon)$,

$$\limsup_{n\to\infty} -2\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} f(S(r)S(t_n-T)u_n)S(r)S(t_n-T)v_n\, dxdr \\ \leq \varepsilon C - 2\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{|x|\leq k} f(S(r)u_T)S(r)v_T\, dxdr.$$

Letting $k \longrightarrow \infty$ we obtain

$$\limsup_{n\to\infty} -2\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} f(S(r)S(t_n-T)u_n)S(r)S(t_n-T)v_n\, dxdr \\ \leq \varepsilon C - 2\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} f(S(r)u_T)S(r)v_T\, dxdr. \tag{2.58}$$

By (2.50), (2.51), (2.52) and (2.58), we finally obtain

$$\limsup_{n\to\infty} \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)(S(t_n-T)w_n))\, dr \\ \leq -2(\lambda-\delta)\int_{T_0}^{T} e^{-2\delta(T-r)} \|S(r)v_T\|^2\, dr \\ +2\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} gS(r)v_T\, dxdr \\ -2\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} f(S(r)u_T)S(r)v_T\, dxdr + \varepsilon C,$$

that is

$$\limsup_{n\to\infty} \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)(S(t_n-T)w_n))\, dr \\ \leq \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)w_T)\, dr + \varepsilon C. \tag{2.59}$$

Taking limit of (2.48), (2.49) and (2.59) we get, as $n \to \infty$,

$$\limsup_{n\to\infty} E(S(t_n)w_n) \leq Ce^{-2\delta(T-T_0)} + \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)w_T)\, dr + \varepsilon C. \tag{2.60}$$

On the other hand, since $w = S(T)w_T$, by (2.47) we also have that

$$E(w) = E(S(T)w_T) = e^{-2\delta(T-T_0)}E(S(T_0)w_T) + \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)w_T)\, dr. \tag{2.61}$$

Hence it follows from (2.60)-(2.61) that

$$\limsup_{n\to\infty} E(S(t_n)w_n) \le E(w) + Ce^{-2\delta(T-T_0)} + \varepsilon C - e^{-2\delta(T-T_0)}E(S(T_0)w_T). \tag{2.62}$$

Now since $w_T \in O$ and $T_0 \ge T(O)$ we find that

$$|e^{-2\delta(T-T_0)}E(S(T_0)w_T)| \le Ce^{-2\delta(T-T_0)}.$$

Then from (2.62) we have

$$\limsup_{n\to\infty} E(S(t_n)w_n) \le E(w) + Ce^{-2\delta(T-T_0)} + \varepsilon C. \tag{2.63}$$

Now taking limit of (2.63) as $T \to \infty$ and then letting $\varepsilon \to 0$, we obtain

$$\limsup_{n\to\infty} E(S(t_n)w_n) \le E(w),$$

that is

$$\begin{aligned}\limsup_{n\to\infty}\ &(\delta^2 - \lambda\delta + 1)\|S(t_n)u_n\|_H^2 + \|\nabla S(t_n)u_n\|_{L^2(\mathbb{R}^N)}^2 + \|S(t_n)v_n\|_H^2 \\ &\le (\delta^2 - \lambda\delta + 1)\|u\|_H^2 + \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|v\|_H^2. \end{aligned}\tag{2.64}$$

Noting that $E(w) = (\delta^2 - \lambda\delta + 1)\|u\|_H^2 + \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 + \|v\|_H^2$ is equivalent to the norm of X, we can assume without loss of generality that the norm of X is defined by it. Then we have

$$\limsup_{n\to\infty} \|S(t_n)w_n\|_X \le \|w\|_X$$

as desired in (2.46). Therefore we get the strong convergence of $S(t_n)w_n$ to w in X. The proof is complete.

Now we state our main result obtained in this section.

Theorem 2.1.2 *Assume that f satisfies (2.5), (2.6) and $g \in L^2(\mathbb{R}^N)$. Then, problem (2.8) possesses a global attractor in $X = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ which is a compact invariant subset that attracts every bounded set of X with respect to the norm topology.*

Proof: Since we have established the existence of an absorbing set in (2.18) and the asymptotic compactness of the semiflow $S(t)$ in X in Theorem 2.1.1, the conclusion follows from Theorem 1.1.1.

2.2 The Wave Equation Without Mass Term

In this section we will study the existence of global attractor for the wave equation without mass term,

$$\begin{cases} u_{tt} + \lambda u_t - \triangle u + f(u) = 0, \qquad x \in \Omega, \quad t > 0, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \end{cases} \tag{2.65}$$

where Ω is a domain of $\mathbb{R}^N$ bounded only in one direction, with smooth boundary. The case $\Omega = \mathbb{R}^N$, for this equation is still an open problem due to some difficulties in getting an inequality such as (2.9) for the operator G in H^1 norm. In our case we will use an equivalent norm (provided by the Poincaré inequality) for which the desired estimate works. We assume the same condtions (2.5) and (2.6) for the nonlinear function f.

We will work in the phase space $X = V \times H$ where $V = H_0^1(\Omega)$, $H = L^2(\Omega)$. H is endowed with the norm and inner product for L^2 and V is endowed with the inner product and norm defined as follows,

$$(u,v)_V = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad u,v \in V \quad \text{ and } \|u\|_V = \|\nabla u\|_{H^n}, \;\; u \in V. \tag{2.66}$$

Now define the following bilinear operator in V:

$$(u,v)_1 = \int_\Omega uv \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx, \quad u,v \in V, \tag{2.67}$$

which is also an inner product in V with induced norm $\|u\|_1 = \Big[\|u\|_H^2 + \|\nabla u\|_{H^n}^2\Big]^{\frac{1}{2}}$. By the Poincaré inequality $\|\cdot\|_V$ and $\|\cdot\|_1$ are equivalent norms in V.That is, there are positive constants C_1 and C_2 such that

$$C_1\|u\|_V \leq \|u\|_1 \leq C_2\|u\|_V, \qquad \forall u \in V. \tag{2.68}$$

Let's make a transformation to write the equation (2.65) as a first order abstract ODE. Choose $\delta = \dfrac{\lambda}{\lambda^2+4}$ and set $v = \delta u + u_t$, $w = \begin{pmatrix} u \\ v \end{pmatrix}$. Then, problem (2.65) is equivalent to

$$\begin{cases} w_t + Gw = R(w), & t > 0, \quad w \in X \\ w(0) = w_0 = (u_0,\, u_1 + \delta u_0) \end{cases} \tag{2.69}$$

where

$$R(w) = \begin{pmatrix} 0 \\ -f(u) \end{pmatrix}$$

and

$$Gw = \begin{pmatrix} \delta u - v \\ -\Delta u + (\lambda - \delta)v + (\delta^2 - \delta\lambda)u \end{pmatrix}$$

for $w = \begin{pmatrix} u \\ v \end{pmatrix} \in D(G) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H^1(\Omega)$.

As in Lemma 2.1.1 we show the positivity of the operator G with a similar estimate.

Lemma 2.2.1 *For $\delta = \dfrac{\lambda}{\lambda^2+4}$, the operator G is maximal accretive in X and verifies the following*

$$(G(w)\,,\, w)_X \geq \sigma\|w\|_X^2 + \frac{\lambda}{2}\|v\|_H^2, \qquad \forall\, w = \begin{pmatrix} u \\ v \end{pmatrix} \in X, \tag{2.70}$$

where

$$\sigma = \frac{\lambda}{\sqrt{\lambda^2+4}(\lambda + \sqrt{\lambda^2+4})}. \tag{2.71}$$

Proof: Let $w = \begin{pmatrix} u \\ v \end{pmatrix} \in X$ then we have:

$$\begin{aligned}
(G(w)\,,\,w)_X &= (\delta u - v\,,\,u)_V + (-\Delta u + (\lambda-\delta)(v-\delta u)\,,\,v)_H \\
&= \delta\|u\|_V^2 - (\nabla u\,,\,\nabla v)_{H^n} + (-\Delta u\,,\,v)_H + (\lambda-\delta)\|v\|_H^2 \\
&\quad -\delta(\lambda-\delta)(u\,,\,v)_H \\
&= \delta\|u\|_V^2 + (\lambda-\delta)\|v\|_H^2 - \delta(\lambda-\delta)(u\,,\,v)_H \\
&\geq \delta\|u\|_V^2 + (\lambda-\delta)\|v\|_H^2 - \delta\lambda\|u\|_H\|v\|_H.
\end{aligned}$$

Then setting $\sigma = \dfrac{\lambda}{\sqrt{\lambda^2+4}(\lambda+\sqrt{\lambda^2+4})}$ as in (2.10), we have

$$\begin{aligned}
(G(w)\,,\,w)_X - \sigma(\|u\|_V^2 + \|v\|_H^2) - \tfrac{\lambda}{2}\|v\|_H^2 &\geq (\delta-\sigma)\|u\|_V^2 + (\tfrac{\lambda}{2}-\delta-\sigma)\|v\|_H^2 \\
&\quad -\delta\lambda\|u\|_V\|v\|_H \\
&\geq 2\sqrt{(\delta-\sigma)(\tfrac{\lambda}{2}-\delta-\sigma)}\|u\|_V\|v\|_H \\
&\quad -\delta\lambda\|u\|_V\|v\|_H
\end{aligned}$$

we can check that $4(\delta-\sigma)(\frac{\lambda}{2}-\delta-\sigma) = \lambda^2\delta^2$ so that

$$(G(w)\,,\,w)_X - \sigma\|w\|_X^2 - \frac{\lambda}{2}\|v\|_H^2 \geq 0.$$

The proof is complete.

The existence of solution for (2.69) follows in the same approach as for equation (2.8). Similarly, we can prove an analogous result as in lemma 2.1.3 and we have shown that there also exists a bounded absorbing set O in X.

Now let's establish the tail ends estimates for equation (2.69).

Lemma 2.2.2 *If (2.5), (2.6) hold, $g \in H$ and $w_0 = (u_0, v_0) \in O$, then for every $\varepsilon > 0$, there exists $T(\varepsilon)$ and $K(\varepsilon)$ such that the solution $w(t) = (u(t), v(t))$ of problem (2.69) satisfies*

$$\int_{\Omega\cap\{|x|\geq k\}} \left[|u(t)|^2 + |\nabla u(t)|^2 + |v(t)|^2\right] dx \leq \varepsilon, \qquad t \geq T(\varepsilon), \quad k \geq K(\varepsilon). \tag{2.72}$$

Proof:

The proof works basically like that for equation (2.8). Any solution $w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ satisfies:

$$v_t - \Delta u + (\lambda - \delta)v + (\delta^2 - \lambda\delta)u = -f(u) + g \tag{2.73}$$

and

$$u_t + \delta u = v. \tag{2.74}$$

We choose the same cut-off function θ.

Now take inner product in H of $\theta(\frac{|x|^2}{k^2})v(x)$ with (2.73) to get

$$\begin{aligned}
&\int_\Omega \theta(\frac{|x|^2}{k^2})vv_t\,dx - \int_\Omega \Delta u\theta(\frac{|x|^2}{k^2})v\,dx + (\lambda-\delta)\int_\Omega \theta(\frac{|x|^2}{k^2})|v|^2\,dx \\
&+(\delta^2-\lambda\delta)\int_\Omega \theta(\frac{|x|^2}{k^2})uv\,dx = -\int_\Omega f(u)\theta(\frac{|x|^2}{k^2})v\,dx + \int_\Omega \theta(\frac{|x|^2}{k^2})gv\,dx.
\end{aligned} \tag{2.75}$$

But

$$\begin{aligned}
-\int_\Omega \Delta u\theta(\frac{|x|^2}{k^2})v\,dx &= \int_\Omega \theta(\frac{|x|^2}{k^2})\nabla u\cdot\nabla v + \frac{2}{k^2}\int_\Omega \theta'(\frac{|x|^2}{k^2})vx\cdot\nabla u \\
&= \int_\Omega \theta(\frac{|x|^2}{k^2})\left[\delta|\nabla u|^2 + \nabla u\cdot\nabla u_t\right] + \frac{2}{k^2}\int_\Omega \theta'(\frac{|x|^2}{k^2})vx\cdot\nabla u \\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega \theta(\frac{|x|^2}{k^2})|\nabla u|^2 + \delta\int_\Omega \theta(\frac{|x|^2}{k^2})|\nabla u|^2 \\
&\quad + \frac{2}{k^2}\int_\Omega \theta'(\frac{|x|^2}{k^2})vx\cdot\nabla u,
\end{aligned}$$

and

$$(\delta^2 - \lambda\delta) \int_\Omega \theta(\frac{|x|^2}{k^2}) uv \, dx = (\delta^2 - \lambda\delta + 1) \int_\Omega \theta(\frac{|x|^2}{k^2})(\delta|u|^2 + uu_t)$$

$$= \frac{1}{2}(\delta^2 - \lambda\delta + 1)\frac{d}{dt} \int_\Omega \theta(\frac{|x|^2}{k^2})|u|^2 + \delta(\delta^2 - \lambda\delta + 1) \int_\Omega \theta(\frac{|x|^2}{k^2})|u|^2.$$

Then (2.73) becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_\Omega \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta)|u|^2 + |\nabla u|^2 + |v|^2\right] \\ &+\delta \int_\Omega \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta)|u|^2 + |\nabla u|^2 + |v|^2\right] + (\lambda - 2\delta) \int_\Omega \theta(\frac{|x|^2}{k^2})|v|^2 \\ &= - \int_\Omega \theta(\frac{|x|^2}{k^2}) f(u)(\delta u + u_t) + \int_\Omega \theta(\frac{|x|^2}{k^2}) gv \, dx - \frac{2}{k^2} \int_\Omega \theta'(\frac{|x|^2}{k^2}) vx \cdot \nabla u. \end{aligned} \tag{2.76}$$

But $\delta^2 - \lambda\delta$ could be negative for certain values of λ. Since $\delta^2 - \lambda\delta + 1 > 0$, let's introduce another equation to get a more desirable identity.

Taking inner product of $\theta(\frac{|x|^2}{k^2})u(x)$ with (2.74), we get

$$\frac{1}{2}\frac{d}{dt} \int_\Omega \theta(\frac{|x|^2}{k^2})|u|^2 \, dx + \int_\Omega \theta(\frac{|x|^2}{k^2})|u|^2 \, dx = \int_\Omega \theta(\frac{|x|^2}{k^2}) uv \, dx.$$

And adding the above and (2.76) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_\Omega \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta + 1)|u|^2 + |\nabla u|^2 + |v|^2\right] \\ &+\delta \int_\Omega \theta(\frac{|x|^2}{k^2}) \left[(\delta^2 - \lambda\delta + 1)|u|^2 + |\nabla u|^2 + |v|^2\right] + (\lambda - 3\delta) \int_\Omega \theta(\frac{|x|^2}{k^2})|v|^2 \\ &= - \int_\Omega \theta(\frac{|x|^2}{k^2}) f(u)(\delta u + u_t) + \int_\Omega \theta(\frac{|x|^2}{k^2}) gv \, dx - \frac{2}{k^2} \int_\Omega \theta'(\frac{|x|^2}{k^2}) vx \cdot \nabla u. \end{aligned} \tag{2.77}$$

Then the conclusion follows the same way as in the proof of lemma 2.1.5.

Similarly, we have the following energy equation for the solution of (2.69),

$$\frac{d}{dt}E(w(t)) + 2\delta E(w(t)) = G(w(t)) \qquad \forall t > 0, \tag{2.78}$$

where

$$E(w) = (\delta^2 - \lambda\delta + 1)\|u\|_H^2 + \|\nabla u\|_{H^N}^2 + \|v\|_H^2, \tag{2.79}$$

and

$$G(w) = -2(\lambda - 3\delta)\|v\|_H^2 + 2\int_\Omega gv\,dx - 2\int_\Omega f(u)v\,dx. \tag{2.80}$$

The rest of the proof of existence of a global attractor is again similar to the case with mass term. We get the main result in this section

Theorem 2.2.1 *Let Ω be a domain of R^N bounded in only one direction. Assume that f satisfies (2.5), (2.6) and $g \in L^2(\Omega)$. Then, problem (2.69) possesses a global attractor in $X = H_0^1(\Omega) \times L^2(\Omega)$ which is a compact invariant subset that attracts every bounded set of X with respect to the norm topology.*

3 DYNAMICS OF SECOND ORDER LATTICE SYSTEMS

In this chapter we take on the longtime dynamics of a second order lattice differential equation (LDE). Broadly speaking an LDE is an infinite system of ordinary differential equations with a discrete structure in the phase space. They often come from a spatial dicretization of an evolutionary PDE. However, many LDE's occur as models in their own right and are not approximations to the continuum limit. Lattice systems occur in many applications such as electric circuit theory, neural networks, material science, theory of chemical reactions, image processing, and biology. The mathematical study of LDE's is quite recent: the literature goes back to about 1987, with the full mathematical development starting in 1990's.

We consider in this chapter the following lattice system

$$\ddot{u}_i + \lambda \dot{u}_i - (u_{i-1} - 2u_i + u_{i+1}) + f(u_i) = g_i, \qquad i \in \mathbb{Z} \tag{3.1}$$

where $\dot{u}$ and $\ddot{u}$ represent respectively the first and second derivatives of u with respect to time t, f is a nonlinear function satisfying some growth conditions and $g = (g_i)_{i\in\mathbb{Z}} \in \ell^2$.

Equation (3.1) can be viewed as a spatial discretization of the one-dimensional damped nonlinear wave equation,

$$u_{tt} + \lambda u_t - u_{xx} + f(u) = g, \quad x \in \mathbb{R}. \tag{3.2}$$

In this chapter we will show the existence of global attractor for the semiflow generated by (3.1). Here again the key lies in a variant of the "tail end estimates" used in the previous chapter for the nonlinear wave equation.

3.1 The Existence and Boundedness of Solutions

In this section we prove the existence and uniqueness of solutions of the following second order lattice system, for all time $t \geq 0$. We also show the uniform boundedness of solutions.

Consider the system

$$\ddot{u}_i + \lambda \dot{u}_i - (u_{i-1} - 2u_i + u_{i+1}) + f(u_i) = g_i, \qquad i \in \mathbb{Z} \tag{3.3}$$

with initial conditions

$$u_i(0) = u_{i,0}, \qquad \dot{u}_i(0) = u_{i,1}, \qquad i \in \mathbb{Z}. \tag{3.4}$$

Here $\lambda > 0$ is a constant and $f \in C^1(\mathbb{R},\ \mathbb{R})$ satisfies $f(0) = 0$ and the following condition:

$$f(s)s \geq \nu F(s) \geq 0, \qquad \forall s \in \mathbb{R} \tag{3.5}$$

where ν is a positive constant and $F(s) = \int_0^s f(t)\,dt$. We remark that condition (3.5) is satisfied if f is a nondecreasing function satisfying $f(s)s \geq o$; for instance, if f is a polynomial with positive coefficients and odd degree monomials.

We will consider the space $\ell^2 = \{u = (u_i)_{i\in\mathbb{Z}} \mid \sum_{i\in\mathbb{Z}} u_i^2 < \infty\}$ which is a Hilbert space with the usual inner product $(u\,,\, v) = \sum_{i\in\mathbb{Z}} u_i v_i$ and norm $\|u\| = (\sum_{i\in\mathbb{Z}} u_i^2)^{\frac{1}{2}}$.

Introduce two linear operators B, $\bar{B}$ and A from ℓ^2 to ℓ^2 as follows. For $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, define

$$(Bu)_i = u_{i+1} - u_i, \quad (\bar{B}u)_i = u_{i-1} - u_i, \text{ and } (Au)_i = -(u_{i-1} - 2u_i + u_{i+1}). \tag{3.6}$$

Then we see that

$$A = \bar{B}B = B\bar{B} = -(B + \bar{B}) \tag{3.7}$$

$$(Bu\,,\, v) = (u\,,\, \bar{B}v), \quad \text{and} \quad (Au\,,\, v) = (Bu\,,\, Bv), \quad \forall u, v \in \ell^2. \tag{3.8}$$

The bilinear form $(u\,,\,v)_1 = (Bu\,,\,Bv)$ defines also an inner product in ℓ^2 with induced norm $\|u\|_1 = \|Bu\|$. We let

$$H = (\ell^2,\ (\cdot,\cdot),\ \|\cdot\|), \qquad V = (\ell^2,\ (\cdot,\cdot)_1,\ \|\cdot\|_1) \qquad \text{which are Hilbert spaces.}$$

The norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent norms in ℓ^2. In fact we have

$$\|u\|^2 \le \|u\|_1^2 = \sum_{i\in\mathbb{Z}} |u_{i+1} - u_i|^2 \le 4\|u\|^2 \qquad \forall u \in \ell^2. \tag{3.9}$$

3.1.1 The Existence and Uniqueness of Solutions

In the remaining analysis our phase space will be $X = V \times H$ equipped with the product topology, which makes X a Hilbert space. The inner product and norm in X are as follows: for $\varphi^j = \begin{pmatrix} u^j \\ v^j \end{pmatrix} \in X \quad j = 1, 2$ we have

$$(\varphi^1\,,\,\varphi^2)_X = (u^1\,,\,u^2)_1 + (v^1\,,\,v^2),$$

$$\|\varphi\|_X^2 = (\varphi\,,\,,\varphi)_X \qquad \forall\, \varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in X.$$

Let $v = \delta u + \dot{u}$, where $\delta > 0$ is a positive parameter chosen as

$$\delta = \frac{\lambda}{\lambda^2 + 4}. \tag{3.10}$$

Then the initial value problem (3.3), (3.4) can be reformulated as a first order abstract ODE in X as follows

$$\dot{\varphi} + G(\varphi) = R(\varphi)\,, \qquad \varphi(0) = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \tag{3.11}$$

where G and R are defined on X as follows:

$$G(\varphi) = \begin{pmatrix} \delta u - v \\ \\ Au + (\lambda - \delta)(v - \delta u) \end{pmatrix} \quad \text{and} \quad R(\varphi) = \begin{pmatrix} 0 \\ -f(u) + g \end{pmatrix}.$$

We make the abusive notations $f(u) = (f(u_i))_{i\in\mathbb{Z}}, \quad F(u) = (F(u_i))_{i\in\mathbb{Z}}$.

Now let $u \in \ell^2$ then since $f(0) = 0$ we have

$$\|f(u)\|^2 = \sum_{i\in\mathbb{Z}} |f'(\theta_i u_i)||u_i|^2,$$

where $\theta_i \in (0,1)$. By $|\theta_i u_i| \leq |u_i| \leq \|u\|$, we get

$$\|f(u)\| \leq \|u\| \max_{-\|u\|\leq s\leq \|u\|} f'(s). \tag{3.12}$$

It follows from (3.12) that $f(u) \in \ell^2$. Thus R maps X into itself. Next we prove the existence and uniqueness of the solution of (3.11) as stated in the next lemma.

Lemma 3.1.1 *For every initial data* $\varphi(0) = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in X$*, there is a unique local solution* $\varphi(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ *of (3.11) such that* $\varphi \in C^1\left[-T(\varphi_0)\ ,\ T(\varphi_0)\right]$ *for some* $T(\varphi_0) > 0$.

Proof: We just need to prove that $\varphi \mapsto R(\varphi) - G(\varphi)$ is locally Lispchitz from X into itself. Let B be a bounded subset of X and $\varphi_1, \varphi_2 \in B$, then similar to (3.12), there exists a constant $L(B)$ depending on B such that

$$\begin{aligned} \|R(\varphi_1) - R(\varphi_2)\|_X^2 &= \|f(u^1) - f(u^2)\|^2 \\ &= \sum_{i\in\mathbb{Z}} |f'(u_i^1 + \theta_i(u_i^2 - u_i^1))|^2 |u_i^1 - u_i^2|^2 \\ &\leq L(B)\|\varphi_1 - \varphi_2\|_X^2. \end{aligned}$$

Therefore R is locally Lipschitz. On the other hand, it is easy to see that G is a bounded linear operator so that $R(\varphi) - G(\varphi)$ is locally Lipschitz from X to X. The conclusion of the lemma follows from the standard theory of abstract ordinary differential equations in Banach spaces.

3.1.2 The Boundedness of Solutions

We start with presenting a positivity estimate for the linear operator G, which is crucial toward proving the existence of absorbing set. In fact it is a key estimate used in this work.

Lemma 3.1.2 *The operator G verifies:*

$$(G(\varphi)\,,\,\varphi)_X \geq \sigma\|\varphi\|_X^2 + \frac{\lambda}{2}\|v\|^2, \qquad \forall\,\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in X, \tag{3.13}$$

where

$$\sigma = \frac{\lambda}{\sqrt{\lambda^2+4}(\lambda + \sqrt{\lambda^2+4})} \tag{3.14}$$

Proof: Let $\varphi = \begin{pmatrix} u \\ v \end{pmatrix} \in X$ then we have:

$$\begin{aligned}
(G(\varphi)\,,\,\varphi)_X &= (\delta u - v\,,\,u)_1 + (Au + (\lambda-\delta)(v-\delta u)\,,\,v) \\
&= \delta\|u\|_1^2 - (Bu\,,\,Bv) + (Au\,,\,v) + (\lambda-\delta)\|v\|^2 \\
&\quad -\delta(\lambda-\delta)(u\,,\,v) \\
&= \delta\|u\|_1^2 + (\lambda-\delta)\|v\|^2 - \delta(\lambda-\delta)(u\,,\,v) \\
&\geq \delta\|u\|_1^2 + (\lambda-\delta)\|v\|^2 - \delta\lambda\|u\|\|v\|.
\end{aligned}$$

Then

$$
\begin{aligned}
(G(\varphi)\,,\,\varphi)_X - \sigma\|\varphi\|_X^2 - \tfrac{\lambda}{2}\|v\|^2 &\geq (\delta-\sigma)\|u\|_1^2 + (\tfrac{\lambda}{2}-\delta-\sigma)\|v\|^2 - \delta\lambda\|u\|_1\|v\| \\
&\geq 2\sqrt{(\delta-\sigma)(\tfrac{\lambda}{2}-\delta-\sigma)}\|u\|_1\|v\| - \delta\lambda\|u\|_1\|v\|.
\end{aligned}
$$

We can check that $4(\delta-\sigma)(\frac{\lambda}{2}-\delta-\sigma) = \lambda^2\delta^2$, so that

$$
(G(\varphi)\,,\,\varphi)_X - \sigma\|\varphi\|_X^2 - \frac{\lambda}{2}\|v\|^2 \geq 0.
$$

The proof is completed.

We already established in lemma 3.1.1 the existence of local solutions for the system (3.11). Now we will show that the solution exists globally, which is a direct consequence of the boundedness.

Lemma 3.1.3 *Assume that the nonlinearity f verifies (3.5), then any solution $\varphi(t)$ of system (3.11) exists globally for all $t \geq 0$ and satisfies*

$$
\|\varphi\|_X^2 \leq M^2 = \frac{2}{\lambda\mu}\|g\|^2, \qquad \textit{for } t \geq T_1 \tag{3.15}
$$

for some constants μ and $T_1 = T_1(R, \lambda, g)$ where $\|\varphi_0\| \leq R$.

Proof: Let $\varphi(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in X$ be any solution of system (3.11) with $v(t) = \delta u(t) + \dot{u}(t)$. Taking the inner product of (3.11) with $\varphi(t)$ in X, we get

$$
\frac{1}{2}\frac{d}{dt}\|\varphi\|_X^2 + (G(\varphi)\,,\,\varphi) + (f(u), \dot{u}) + (f(u), \delta u) = (g, v). \tag{3.16}
$$

By (3.5) we have

$$
(f(u), \dot{u}) = \sum_{i\in\mathbb{Z}} f(u_i)\dot{u}_i = \frac{d}{dt}\sum_{i\in\mathbb{Z}} F(u_i), \tag{3.17}
$$

$$
(f(u), u) = \sum_{i\in\mathbb{Z}} f(u_i)u_i \geq \nu \sum_{i\in\mathbb{Z}} F(u_i). \tag{3.18}
$$

Since $(g,v) \leq \frac{1}{2\lambda}\|g\|^2 + \frac{\lambda}{2}\|v\|^2$, it follows from (3.13) that

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_X^2 + 2\sigma\|\varphi\|_X^2 + \frac{\lambda}{2}\|v\|^2 + \frac{d}{dt}\sum_{i\in\mathbb{Z}} F(u_i) + \nu\delta\sum_{i\in\mathbb{Z}} F(u_i) \leq \frac{1}{2\lambda}\|g\|^2 + \frac{\lambda}{2}\|v\|^2,$$

that is

$$\frac{d}{dt}\left[\|\varphi\|_X^2 + 2\sum_{i\in\mathbb{Z}} F(u_i)\right] + 2\sigma\|\varphi\|_X^2 + 2\nu\delta\sum_{i\in\mathbb{Z}} F(u_i) \leq \frac{1}{\lambda}\|g\|^2. \tag{3.19}$$

And taking $\mu = \inf\{2\sigma\,,\,\nu\delta\}$, we get

$$\frac{d}{dt}\left[\|\varphi\|_X^2 + 2\sum_{i\in\mathbb{Z}} F(u_i)\right] + \mu\left[\|\varphi\|_X^2 + 2\sum_{i\in\mathbb{Z}} F(u_i)\right] \leq \frac{1}{\lambda}\|g\|^2. \tag{3.20}$$

Using Gronwall's inequality we have

$$\|\varphi\|_X^2 + 2\sum_{i\in\mathbb{Z}} F(u_i) \leq e^{-\mu t}\left[\|\varphi(0)\|_X^2 + 2\sum_{i\in\mathbb{Z}} F(u_{i,0})\right] + \frac{1}{\lambda\mu}\|g\|^2(1-e^{-\mu t})$$

which implies

$$\|\varphi\|_X^2 \leq e^{-\mu t}\left[\|\varphi(0)\|_X^2 + \frac{2}{\nu}\max_{-\|u^0\|\leq s\leq\|u^0\|}|f'(s)|\|u^0\|^2\right] + \frac{1}{\lambda\mu}\|g\|^2(1-e^{-\mu t}). \tag{3.21}$$

This yields $\lim\limits_{t\to T(\varphi_0)}\|\varphi\|_X < \infty$, so that the solution $\varphi(t)$ exists globally for all $t>0$. Now let $R>0$, $\|\varphi_0\|_X \leq R$ and $C_R = \max\limits_{-R\leq s\leq R}|f'(s)|$ then

$$|\varphi\|_X^2 \leq e^{-\mu t}\left(R^2 + \frac{C_R R^2}{\nu}\right) + \frac{1}{\lambda\mu}\|g\|^2. \tag{3.22}$$

Thus (3.15) follows with $T_1 = \frac{1}{\mu}\ln\left\{\frac{\lambda\mu(R^2+\frac{C_R R^2}{\nu})}{\|g\|^2}\right\}$ and the proof is complete.

The previous Lemma 3.1.3 implies that equation (3.11) generates a continuous semiflow $\{S(t)\}_{t\geq 0}$ on X which posses a bounded absorbing set

$$O = \{w\in X\,:\,\|w\|_X \leq M\}\,. \tag{3.23}$$

That is: for every bounded set $\mathcal{B} \subset X$, there a constant $T(\mathcal{B}) > 0$ such that

$$S(t)\mathcal{B} \subseteq O, \quad t \geq T(\mathcal{B}). \tag{3.24}$$

In particular there exists a constant T_0 depending only on (λ, g) and O such that

$$S(t)O \subseteq O, \quad t \geq T_0. \tag{3.25}$$

3.2 Global Attractor

Now that we have established the existence of absorbing set, it only remains to prove that the semiflow $S(t)$ is asymptotically compact to conclude the existence of global attractor. First we present some type of "tail estimate" which will be useful toward proving the asymptotic compactness.

Lemma 3.2.1 *Let* $\varphi(0) = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in O$*, then for every* $\varepsilon > 0$ *there exist positive constants* $T(\varepsilon)$ *and* $K(\varepsilon)$ *such that the solution* $\varphi(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in X$ *of system (3.11) satisfies*

$$\sum_{|i| \geq K(\varepsilon)} \left[|(Bu(t))_i|^2 + |v_i(t)|^2 \right] \leq \varepsilon, \qquad t \geq T(\varepsilon). \tag{3.26}$$

Proof: Choose a smooth function $\theta \in C^1(R^+, R)$ such that $0 \leq \theta(s) \leq 1$ for $s \in \mathbb{R}^+$, and

$$\theta(s) = 0 \quad \text{for } 0 \leq s \leq 1; \qquad \theta(s) = 1 \quad \text{for } s \geq 2.$$

Then there exists a constant $C > 0$ such that $|\theta'(s)| \leq C$ for $s \in \mathbb{R}^+$.

Let k be a fixed positive integer. Set $w_i = \theta(\frac{|i|}{k})u_i, \;\; z_i = \theta(\frac{|i|}{k})u_i, \;\; y = \begin{pmatrix} w \\ z \end{pmatrix} \in X.$ Take inner product of (3.11) with y in X to get

$$(\dot{\varphi}, y)_X + (G(\varphi), y)_X = (R(\varphi), y)_X. \tag{3.27}$$

We can check that

$$(\dot{\varphi}, y)_X = \frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})|\varphi_i|_X^2, \tag{3.28}$$

where

$$|\varphi_i|_X^2 = |(Bu)_i|^2 + |v_i|^2 = |u_{i+1} - u_i|^2 + |v_i|^2. \tag{3.29}$$

Now

$$(G(\varphi), y) = \delta(Bu, Bw) - (Bv, Bw) + (Bu, Bz) + (\lambda - \delta)(v - \delta u, z). \tag{3.30}$$

Let's estimate the terms in (3.30) one by one.

$$\begin{aligned}
(Bu, Bw) &= \sum_{i\in\mathbb{Z}} \Big\{ \Big[\theta(\frac{|i+1|}{k}) - \theta(\frac{|i|}{k})\Big](u_{i+1} - u_i)u_{i+1} \\
&\quad + \theta(\frac{|i|}{k})(u_{i+1} - u_i)^2 \Big\} \\
&\geq -\frac{4C_0 r_0^2}{k} + \sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})(u_{i+1} - u_i)^2, \qquad \forall t \geq T_0, \\
(Bv, Bw) &= \sum_{i\in\mathbb{Z}} \Big[\theta(\frac{|i+1|}{k})(v_{i+1} - v_i)u_{i+1} - \theta(\frac{|i|}{k})(v_{i+1} - v_i)u_i\Big], \\
(Bu, Bz) &= \sum_{i\in\mathbb{Z}} \Big[\theta(\frac{|i+1|}{k})(u_{i+1} - u_i)v_{i+1} - \theta(\frac{|i|}{k})(u_{i+1} - u_i)v_i\Big], \\
(Bu, Bz) - (Bv, Bw) &= \Big[\theta(\frac{|i+1|}{k}) - \theta(\frac{|i|}{k})\Big](u_{i+1}v_i - u_i v_{i+1}) \\
&\geq -\sum_{i\in\mathbb{Z}} \frac{|\theta'(\tau_i)|}{k}|u_{i+1}v_i - u_i v_{i+1}| \\
&\geq -\frac{4C_0 r_0^2}{k}, \qquad \forall t \geq T_0, \\
(\lambda - \delta)(v - \delta u, z) &= (\lambda - \delta)\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})v_i^2 - \delta(\lambda - \delta)\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})u_i v_i \\
&\geq (\lambda - \delta)\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})v_i^2 - \delta\lambda\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})u_i v_i.
\end{aligned}$$

Thus, since $\delta < 1$, we get that

$$(G(\varphi), y) \geq -\frac{8C_0 r_0^2}{k} + \delta\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})|(Bu)_i|^2 + (\lambda - \delta)\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})v_i^2$$

$$-\lambda\delta\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})u_iv_i \qquad \forall t\geq 0.$$

And following the same arguments as in the proof of (3.13) we can get

$$(G(\varphi),y)\geq -\frac{8C_0r_0^2}{k}+\delta\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})\Big[\sigma|\varphi_i|_X^2+\frac{\lambda}{2}|v_i|^2\Big], \qquad \forall t\geq T_0. \tag{3.31}$$

Now we estimate the right-hand side of (3.28):

$$\begin{aligned}
(R(\varphi),y)_X &= -(f(u),z)+(g,z),\\
(f(u),z) &= \sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})f(u_i)\dot{u}_i+\delta\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})f(u_i)u_i\\
&\geq \frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})F(u_i)+\delta\nu\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})G(u_i), && (3.32)\\
(g,z) &= \sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})g_iv_i=\sum_{|i|\geq k}\theta(\frac{|i|}{k})g_iv_i\\
&\leq \frac{\lambda}{2}\sum_{|i|\geq k}\theta(\frac{|i|}{k})v_i^2+\frac{1}{2\alpha}\sum_{|i|\geq k}g_i^2.\\
(g,z) &\leq \frac{\lambda}{2}\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})v_i^2+\frac{1}{2\alpha}\sum_{|i|\geq k}g_i^2. && (3.33)
\end{aligned}$$

Substituting inequalities (3.28), (3.31)-(3.33) into (3.27), we obtain

$$\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})[|\varphi_i|_X^2+2F(u_i)]+\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})[2\sigma|\varphi_i|_X^2+2\delta\nu F(u_i)]\\
\leq\ &\frac{8C_0r_0^2}{k}+\frac{1}{\alpha}\sum_{|i|\geq k}g_i^2.
\end{aligned}$$

Since $g\in\ell^2$, for every $\varepsilon>0$, there exists a constant $K(\varepsilon)>0$ such that

$$\frac{8C_0r_0^2}{k}+\frac{1}{\alpha}\sum_{|i|\geq k}g_i^2\leq\varepsilon.$$

Then for $t\geq T_0,\ \ k\geq K(\varepsilon)$, we have

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})[|\varphi_i|_X^2+2F(u_i)]+\mu\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{k})[|\varphi_i|_X^2+2F(u_i)]\leq\varepsilon,$$

where $\mu = \inf\{2\sigma\,,\, \delta\nu\}$. By Gronwall's inequality,

$$\begin{aligned} & \sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})[|\varphi_i|_X^2 + 2F(u_i)] \\ \leq \;& e^{-\mu(t-T_0)} \sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})[|\varphi_i(T_0)|_X^2 + 2F(u_i(T_0))] + \frac{\varepsilon}{\mu} \\ \leq \;& e^{-\mu(t-T_0)} r_0^2 \Big(1 + \frac{2}{\nu} M_0\Big) + \frac{\varepsilon}{\mu}, \qquad \forall t \geq T_0. \end{aligned}$$

where $M_0 = \max\limits_{-r_0/\nu \leq s \leq r_0/\nu} |f'(s)|$. Taking

$$T(\varepsilon) = \max\Big\{T_0\,,\, T_0 + \frac{1}{\mu}\ln\frac{\mu}{\varepsilon}(1 + \frac{2}{\nu}M_0) r_0^2\Big\},$$

then for $t \geq T(\varepsilon)$ and $k \geq K(\varepsilon)$ we have

$$\sum_{|i|\geq k} |\varphi_i|_X^2 \leq \sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{k})|\varphi_i|_X^2 \leq \frac{2\varepsilon}{\mu}, \tag{3.34}$$

which implies Lemma 3.1. The proof is completed.

Lemma 3.2.2 *The semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically compact in X, namely, if $\{\varphi_n\}_n$ is bounded in X and $t_n \to \infty$ then $\{S(t_n)\varphi_n\}_n$ is precompact in X.*

Proof: Assume that $\|\varphi_n\|_X \leq r, \quad n \geq 1$ for some positive constant r. By (3.24) there exists T, such that

$$S(t)\{\varphi_n\} \subset O, \quad \forall t \geq T \tag{3.35}$$

where O is the absorbing set in (3.23). Now since $t_n \to +\infty$, there exists $N_1(r)$ such that $t_n \geq T$ if $n \geq N(r)$ which implies that

$$S(t)\{\varphi_n\} \subset O, \quad \forall n \geq N_1(r) \tag{3.36}$$

so that there exists $\varphi_0 \in X$ and a subsequence of $\{S(t_n)\varphi_n\}_n$ (denoted still by $\{S(t_n)\varphi_n\}_n$) such that

$$S(t_n)\varphi_n \to \varphi_0 \qquad \text{weakly in } X. \tag{3.37}$$

We want to show that this convergence is the strong sense. Indeed let $\varepsilon > 0$, by Lemma 3.2.1 and (3.35) there exists $K_1(\varepsilon),\ T(\varepsilon) > 0$ such that

$$\sum_{|i|\geq K_1(\varepsilon)} \|(S(t)(S(T_r)\varphi_n))_i\|_X^2 \leq \frac{\varepsilon^2}{8}, \quad t \geq T(\varepsilon).$$

By $t_n \to +\infty$, there exists $N_2(r,\varepsilon)$ such that $t_n \geq T_r + T(\varepsilon)$ if $n \geq N_2(r,\varepsilon)$. Hence,

$$\sum_{|i|\geq K_1(\varepsilon)} \|(S(T_n)\varphi_n)_i\|_X^2 = \sum_{|i|\geq K_1(\varepsilon)} \|(S(t_n - T_r)(S(T_r)\varphi_n))_i\|_X^2 \leq \frac{\varepsilon^2}{8}. \tag{3.38}$$

Again, since $\varphi_0 \in X$, there exists $K_2(\varepsilon)$ such that

$$\sum_{|i|\geq K_2(\varepsilon)} \|\varphi_0\|_X^2 \leq \frac{\varepsilon^2}{8}.$$

Let $K(\varepsilon) = \max\{K_1(\varepsilon), K_2(\varepsilon)\}$ then by (3.37) we have

$$((S(T_n)\varphi_n)_i)_{|i|\leq K(\varepsilon)} \to ((\varphi_0)_i)_{|i|\leq K(\varepsilon)} \quad \text{strongly in } \mathbb{R}^{2K(\varepsilon)+1}$$

as $n \to +\infty$, that is there exists $N_3(\varepsilon)$ such that

$$\sum_{|i|\leq K(\varepsilon)} \|(S(T_n)\varphi_n)_i - (\varphi_0)_i\|_X^2 \leq \frac{\varepsilon^2}{2}, \qquad \forall n \geq N_3(\varepsilon). \tag{3.39}$$

Setting $N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon), N_3(\varepsilon)\}$, we conclude for $n \geq N(\varepsilon)$ that

$$\begin{aligned} \|S(T_n)\varphi_n - \varphi_0\|_X^2 &= \sum_{|i|\leq K(\varepsilon)} \|(S(T_n)\varphi_n)_i - (\varphi_0)_i\|_X^2 \\ &+ \sum_{|i|> K(\varepsilon)} \|(S(T_n)\varphi_n)_i - (\varphi_0)_i\|_X^2 \\ &\leq \frac{\varepsilon^2}{2} + 2\sum_{|i|> K(\varepsilon)} \|(S(T_n)\varphi_n)_i\|_X^2 - \|\varphi_0)_i\|_X^2 \\ &\leq \varepsilon^2. \end{aligned}$$

The proof is completed.

Now we state the main result of this chapter as follows.

Theorem 3.2.1 *Assume that f satisfies (3.5) and $g \in \ell^2$. Then, the dynamical system generated by equation (3.11) possesses a global attractor in $X = V \times H$ which is a compact invariant subset that attracts every bounded set of X with respect to the norm topology.*

Proof: The conclusion follows from Theorem 1.1.1 since, by (3.24), there exists a bounded absorbing set and the semiflow is asymptotically compact by Lemma 3.2.2.

4 FINAL REMARKS

We finish this work by presenting some final remarks on the dynamics of evolutionary equations in unbounded domains. We also describe some open problems and new perspectives in this area.

Finite Dimensionality and Exponential Attractors

One major feature of global attractors is that they usually have finite dimension. This reduces the number of degrees of freedom of the system which hopefully will give it a simpler description. There are two concepts of dimensions that are mostly used: the Hausdorff and fractal dimensions.

However, the global attractor has two major drawbacks: on the one hand the rate of attraction can be arbitrarily slow and on the other hand it is in general only upper semicontinuous with respect to perturbations so that the global attractor can change very drastically under very small perturbations in the structure of the original dynamical system. This leads to essential difficulties in numerical simulations of the global attractor and even makes it, in some sense, unobservable.

In view of these drawbacks, the concept of *exponential attractor* has been suggested by Eden, Foias, Nicolaenko and Teman in [7]. It is a compact, positively invariant set with finite fractal dimension, which attracts the bounded sets at an exponential rate. It is therefore more robust than the global attractor but it is not unique.

We will introduce the concepts of fractal and Hausdorff dimensions for general sets in a Banach space X.

Definition 4.0.1 *Let A be a subset of a Banach space X, $d > 0, \varepsilon > 0$. Then set*

$$\mu_{d,\varepsilon}(A) = \inf \Big\{ \sum_{i=1}^{k} r_i^d \ : \ r_i \leq \varepsilon \ \ \text{and} \ \ A \subset \bigcup_{i=1}^{k} B_{r_i} \Big\}, \tag{4.1}$$

where B_{r_i} denotes a ball of radius r_i in X.

It can be shown that $\mu_{d,\varepsilon}(A)$ increases as ε decreases.

Definition 4.0.2 *We define the d-dimensional Hausdorff measure of A as:*

$$\mu_d = \sup_{\varepsilon>0} \mu_{d,\varepsilon}(A) = \lim_{\varepsilon \to 0^+} \mu_{d,\varepsilon}(A). \tag{4.2}$$

*And the **Hausdorff dimesion** of A is defined by:*

$$d_H(A) = \inf\{d > 0 \ : \ \mu_d(A) = 0\}. \tag{4.3}$$

A stronger measure of dimension is furnished by the fractal dimension.

Definition 4.0.3 *Let A be a subset of the Banach space X. Let*

$N_\varepsilon(A) =$ *the minimum number of balls of radii $\leq \varepsilon$ that are necessary to cover A.*

*Then the **fractal dimension** of A, $d_F(A)$ is defined by:*

$$d_F(A) = \limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(A)}{\log(\frac{1}{\varepsilon})}. \tag{4.4}$$

Another characterization of the fractal dimension is

$$d_F(A) = \inf\{d > 0 \ : \ \mu_{d,F} = \limsup_{\varepsilon \to 0} \varepsilon^d N_\varepsilon(A) = 0\}. \tag{4.5}$$

Next, we give the definition of an exponential attractor for a semiflow $\{S(t)\}$ defined on a Banach space X.

Definition 4.0.4 *A compact set $\mathcal{M}$ is called an **exponential attractor** or **inertial set** for the semiflow $\{S(t)\}_{t \geq 0}$ on X if*

(i) $S\mathcal{M} \subset \mathcal{M}$,

(ii) $\mathcal{M}$ *has finite fractal dimension,* $d_F(\mathcal{M})$*,*

(iii) there are positive constants c_0 *and* c_1 *such that*

$$h(S^n B, \mathcal{M}) \leq c_0 e^{-c_1 t}, \qquad \forall n \geq 1. \tag{4.6}$$

Here $h(\cdot,\cdot)$ *is the Hausdorff pseudometric defined in (1.5).*

For evolutionary PDEs in an unbounded domain, there have been results on the finite dimensionality of the global attractor as well as the existence of an exponential attractor, see for instance [2], [9], [25]. However, there have been counterexamples on infinite dimensionality of the global attractor (see [2], [14], [54]), this implies automatically the nonexistence of an exponential attractor. That is why the concept of Kolmogorov's ε-entropy is exploited to obtain some qualitative and quantitative information on such infinite dimensional attractors. It is defined as follows.

Definition 4.0.5 *Let* K *be a precompact set in a metric space* M *and* $\varepsilon > 0$*. Let* $N_\varepsilon(K, M)$ *be the minimal number of* ε*-balls that cover* K*. Then the* ***Kolomogorov's*** ε***-entropy*** *of* K *in* M *is the following number:*

$$\mathbb{H}_\varepsilon(K, M) := \ln N_\varepsilon(K, M). \tag{4.7}$$

It is proved in [15] and [53, 54] that for a large class of equations of mathematical physics in unbounded domains, the ε-entropy of the restrictions $\mathcal{A}|_{\Omega\cap B^R_{x_0}} := \{u_0|_{\Omega\cap B^R_{x_0}},\ u_0 \in \mathcal{A}\}$ of the corresponding global attractor to bounded subdomains $\Omega\cap\mathcal{C}^R_{x_0}$, where $\mathcal{C}^R_{x_0} := x_0 + [-\frac{R}{2}, \frac{R}{2}]^N$ is the R-cube of $\mathbb{R}^N$ centered at x_0, satisfies

$$\mathbb{H}_\varepsilon(\mathcal{A}|_{\Omega\cap\mathcal{C}^R_{x_0}}, L^\infty(\mathcal{C}^R_{x_0})) \leq C vol(\Omega \cap \mathcal{C}^{R+K\ln_+ R'/\varepsilon}_{x_0}) \ln_+ \frac{R'}{\varepsilon}, \tag{4.8}$$

where $\ln_+ z := \max\{0, \ln z\}$ and the constants C, K, and R' are independent of ε, R and x_0.

This type of estimates led to the introduction of **infinite dimensional exponential attractors** by Effendiev, Miranville and Zelik in [10], by modifying the classical definition of exponential attractors and replacing the condition of finite fractal dimensionality by the ε-entropy estimates (4.8). They prove in [10] that for certain reaction-diffusion equations the corresponding semiflow admits an infinite dimensional exponential attractor.

It would be interestting to investigate the existence of infinite dimensional exponential attractors for wave equations in unbounded domains.

Wave Equations in Exterior Domains

In this work we have considered wave equations in unbounded domains. An interesting class of such equations are the wave equations in an exterior domain,

$$\begin{cases} u_{tt} - \Delta u + \rho(x, u_t) = f(u) \quad \text{in } \Omega \times [0, +\infty) \\ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \text{and } u|_{\partial\Omega} = 0, \end{cases} \tag{4.9}$$

where Ω is an exterior domain, that is $\Omega = \mathbb{R}^N - K$ for a compact connected subset K. This type of equations have been extensively studied by M. Nakao [32, 33, 34, 35, 36, 37]. In [35], he showed the total energy decay for the corresponding linear equation which is applied to obtain the global existence of finite energy solutions for the nonlinear equation. Furthermore he derived the energy decay of the nonlinear equation in [32]. Some other authors have also studied similar equations for instance, Tébou [46] and E. Zuazua [57] among others.

There are interesting questions related to equation (4.9). The energy decay established in [32], [34], [36] implies the dissipativity of the system. The question is whether there exists a global attractor for such systems or not? Some other open problems have been mentioned by M. Nakao in [36], for instance, to derive some decay rate of local energy for solutions of (4.9) in the particular case where $\rho(x,t) = a(x)|u_t|^r u_t$ or $\rho(x,t) = a(x)(u_t + |u_t|^r u_t)$ with $a(x)$ a localized function on some part ω of the boundary $\partial\Omega$.

Other Perspectives

Tail estimation

The tail estimation method has been crucial throughout our work to prove the asymptotical compactness in the case of wave equations in unbounded domains or for the lattice systems. In [50], B. Wang proved that this tail estimation (or asymptotical nullness) along with the existence of a bounded absorbing set are necessary and sufficient for the existence of a global attractor for lattice systems. It would of much interests to investigate such a feature for evolutionary PDEs in unbounded domains.

Wave equation without mass term

We obtained the existence of a global attractor for the wave equation without mass term (2.65) only for domains that are bounded in one direction. What happens for general unbounded domains or $\Omega = \mathbb{R}^N$ is still an open question.

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