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Pricing models and analysis of corporate coupon-bonds and credit default swaptions

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Pricing Models and Analysis of Corporate Coupon-Bonds and Credit Default Swaptions

by

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ABSTRACT

In this work, pricing models of corporate coupon-bonds and credit default swaptions are derived and analyzed. Corporate coupon-bonds are priced incorporating both intensity models and structural models, and also jumps introduced by seasonal effects. In deriving the models, we form portfolios to hedge the risk incurred by the instruments, then derive PDE equations using the arbitrage principle and the Ito Lemma for jump processes. The mathematical models are the parabolic-type PDE equations with terminal conditions and boundary conditions. These PDE problems are analyzed and solved by various transformations and incorporation with probabilistic properties. Either a unique solution in the exponential form is obtained, or a particular solution in the separation form is acquired. Further, the pricing model of credit default swaptions is derived using the pricing of corporate coupon-bonds in the similar manner. The main idea of deriving the price of credit default swaptions is to use the price of existing products, i.e., corporate bonds, as opposed to the existing models, which use non-existing forward credit default swap price of the reference entity. The prices of corporate coupon-bonds and credit default swaptions with unexpected default, obtained from these models, are compared to the actual market prices and analyzed.

Chapter 1

Introduction

1.1 Background and Motivation

When a company needs financing, there are only two ways to raise money. Either to borrow, or to find someone to co-own the company. The latter is done by issuing new stocks; the former basically takes two forms: borrowings from banks or other institutions, and issuing bonds.

Financial institutions, such as banks, securities houses, insurance companies and so on, have been traditionally largest participants in the bond market. They invest in bonds not only because they constitute a part of their investment portfolios, but also because they want to maintain the relationship with bond issuers. If that is the case, often the time, they are expected to hold the bonds until their maturities, and it could be a nuisance to the investing institution for the following reasons. First, most financial institutions have internal and external guidelines about the maximum amount of financial products they can invest. So holding bonds until their maturities prevent them from diversifying their portfolios. Second, it incurs a risk of unreasonable loss in case of default. Therefore, it is important for institutional investors to assess the price of the bonds in the portfolio and also to avoid the risk inherent in the bonds without selling them.

Finding a pricing model of defaultable bonds is one of the main purposes of this paper. There are two fundamental approaches to bond pricing: (i) intensity models, and (ii) structural models (also, known as Merton's model). Each type of models has pros and cons; but the emphasis of both models is on finding or estimating the default probability. Intensity models consider default as an exogenous event and can be found in Jarrow & Turnbul (1995), Duffie & Singleton (1999), Hughston & Turnbull (2001), etc. In structural models, default is considered to be an endogenous event, that is, default originates from within the firm structure. In Merton's model (1974), a firm's total assets comprise of one zero-coupon bond and one stock, which follows a geometric Brownian motion. The firm defaults if its assets fall below the value of its outstanding bond. Black & Cox (1976) extended this framework by bringing a certain threshold as a barrier.

Most existing models adopt either one approach or the other. O, et al. (2005) combined both approaches and came up with a new model on corporate zero-coupon bonds. However, most corporate bonds bear coupons, and coupon-bond issuers, especially small-size companies, are exposed to the risk of default as interest payment dates approach. In Chapters 2 and 4 of this paper, the idea of O is extended to corporate coupon-bonds and we take into consideration the default risk arisen by coupon payment.

However, finding a better pricing model does not result in avoiding the default risk incurred by holding bonds. New financial derivatives, called "credit derivatives", were introduced to the financial market in the early 1990s, which made it possible for bondholders to get rid of default risk without selling bonds. A credit derivative is a derivative security whose payoff is conditioned on the occurrence of credit events such as bankruptcy of a certain bond-issuer. Two thirds of the credit derivatives in the current market are credit default swaps. A pricing model on credit default swaps is given by Schonbucher (1997, 2003b). Some pricing models of credit default swaption, which is an option on forward credit default swap are given by Schonbucher (2000) and Schmidt (2004). In their papers, they assume that they know the dynamics of the value of underlying forward credit default swap, and they hedge the swaption against the forward credit default swap. However, the dynamics of the value of forward credit default swap is not easy to obtain. In Chapters 3 and 5 of this paper, we will try to find the pricing model to hedge the swaption by the bonds issued by the forward default swap's reference entity.

The rest of this paper is organized as follows. In the following section, basic mathematical and financial concepts are introduced. Chapter 2 introduces the pricing models of corporate coupon-bonds, which incorporate the jump terms caused by coupon payments. In Chapter 3, we will use the dynamics of the value of the corporate coupon-bonds to find the price of the credit default swaption. In Chapters 2 and 3, we assume that the short-term, risk-free interest rate is constant. In Chapter 4, we re-examine the pricing of corporate coupon-bonds, however, this time with a stochastic short-term risk-free interest rate. We remodel the price of a credit default swaption using the bond pricing model established in Chapter 5. Chapter 6 devotes to the data analysis of the bond-price and the credit default swaption price with constant risk-free rate. Finally, in Chapter 7, we summarize the result of this paper and discuss the directions of the research in the future.

1.2 Definition and Basic Concepts

1.2.1 Mathematical Concepts

Definition 1.2.1 (Martingale) *A stochastic process $\{X(t), t \geq 0\}$ adapted to a filtration $\mathbb{F}$ is called a martingale if for any $s < t$, it is integrable, i.e., $E|X(t)| < \infty$, and*

$$E(X(t)|F_s) = X(s)$$

where F_t is the information about the process up to time t, and the equality holds almost surely. □

Definition 1.2.2 (Brownian Motion) *A standard Brownian motion (or a standard Wiener process) is a stochastic process $\{W(t), t \geq 0\}$, defined on a common probability space (Ω, F, P) with the following properties: (1) W(0) = 0. (2) With probability 1, the function $W(t)$ is continuous in t. (3) The increment random variables associated with non-overlapping interval are independent. (4) The increment $W(t+s) - W(s)$ is a normal random variable with mean 0 and variance t for any $s < t$.* □

Throughout this paper, $W(t), W_t$ and W_i are exclusively used to denote a standard Brownian motion, unless stated otherwise.

Definition 1.2.3 (Geometric Brownian Motion) *A geometric Brownian motion is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion. A stochastic process $S(t)$ is said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a Brownian motion, and μ ('drift') and σ ('volatility') are constants. □

The equation has an analytic solution:

$$S_t = S_0 \exp\big((\mu - \sigma^2/2)t + \sigma dW_t\big)$$

for an arbitrary initial value S_0.

Definition 1.2.4 (Ito Integral) *Let $(\Omega, F, \mathbb{P})$ be a probability space on which a standard Brownian motion $W(t)$ is defined. Let $X(t)$ be a simple process which is adapted to the same filtration as $W(t)$ and given by*

$$X(t) = c_0 I_0(t) + \sum_{i=0}^{n-1} c_i I_{(t_i, t_{i+1}]}(t),$$

Then, Ito integral $\int_0^T X(t)dW(t)$ for a simple process X(t), is defined as

$$\int_0^T X(t)dW(t) = \sum_{i=0}^{n-1} c_i\big(W(t_{i+1}) - W(t_i)\big).$$

Let $X^n(t)$ be a sequence of simple processes convergent in probability to the process $X(t)$ satisfying $\lim\limits_{n\to\infty} \int_0^\infty E\big(X(t) - X^n(t)\big)^2 dt = 0$. Then Ito integral with general process $X(t)$ is defined as

$$\int_0^T X(t)dW(t) = \lim_n \int_0^T X_n(t)dW(t). \quad \square$$

Lemma 1.2.5 (Ito Lemma for Jump Processes) Let $X(t)$ be a right-continuous stochastic process with left limit, and it has at most finite number of jumps over finite time intervals a.s. For every path of the process, we define:

$$\begin{aligned}
&X(t-) := \lim_{h\to 0} X(t-h), \quad X(0-) := X(0),\\
&\Delta X(t) := X(t) - X(t-),\\
&X^d(t) := \sum_{s\le t} \Delta X(s),\\
&X^c(t) := X(t) - X^d(t).
\end{aligned}$$

Alternately, let $X = (X^1, X^2, \cdots, X^n)$ be an n-dimensional semi-martingale with a finite number of jumps, and f a twice continuously differentiable function on $\mathbb{R}^d$. Then $f(X)$ is also a semi-martingale, and it follows that:

$$f(X(t)) - f(X(0)) = \sum_{i=1}^{n} \int_0^t \frac{\partial f(X(s-))}{\partial x_i} dX^{c,i} + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f(X(s-))}{\partial x_i \partial x_j} d < X^{c,i}, X^{c,j} > (s) + \sum_{s \le t} \Delta f(X(s)),$$

where the integral means a stochastic integral (*cf.* Kigima (2003)) and $< X^{c,i}, X^{c,j} >$ stands for the quadratic covariance of two stochastic processes (*cf.* Kigima(2003)). Its proof can be found in Jacod and Shiryaev (1988).

1.2.2 Financial Concepts

Definition 1.2.6 (Bonds) *The bond is a debt instrument issued for a period of time in purpose of raising money. It promises to repay the principal amount on a specified day ("maturity" date). Some bonds bear coupons, which are promissory notes for interest; so they are called "coupon (bearing) bonds". Others do not pay interest; instead they are sold at a deep discounted price, so they are called "zero-coupon bonds" or "deep discounted bond".* □

Bond prices fluctuate in accordance to two factors: (1) changes in interest rates and (2) change in credit quality. The interest rate considered is what is known as a "risk-free, short-term interest rate", $r(t)$. In Chapters 2 and 3, we consider $r(t)$ to be constant; in chapters 4 and 5, we consider $r(t)$ to follow the Vasicek model (1977). Here we introduce several interest rate models:

One of the earliest short-rate models introduced by Black (1976) and Rendleman and Bartter (1980) is lognormally distributed and given by

$$dr = \mu r dt + \sigma r dW, \qquad \mu, \sigma : \text{constant},$$

where $W = \{W(t)\}_{t \geq 0}$ is a standard Brownian motion. (Hereinafter, all W's used in Stochastic differential equations are standard Brownian motions.) However, this model does not capture mean-reverting property of interest rate; Vasicek (1977) introduced the following normal mean-reverting process with constant parameters, i.e.,

$$dr = \theta(a - r)dt + \sigma dW, \quad \theta, a, \sigma : \text{constant}.$$

The drawback of this model is that the short term rate can assume a negative number. In 1985, Cox, Ingross, and Ross (CIR) added square-root diffusion term to the Vasicek model, which makes $r(t)$ chi-square distributed and given by

$$dr = \theta(a - r)dt + \sigma\sqrt{r}dW, \quad \theta, a, \sigma : \text{constant.}$$

Hull and White (1990) extended the Vasicek model to fit both the current structure and volatilities of interest rates. In their model, the short-term rate follows a normal mean-reverting process with time dependent parameters and is given by

$$dr = (\theta(t) - \alpha r)dt + \sigma(t)dW, \quad \alpha : \text{constant.}$$

If we take α to be time-dependent, the model is know as extended Vasicek model. We can write the model as

$$dr = (\theta(t) - \alpha(t)r)dt + \sigma(t)dW,$$

which can also be written as

$$dr = \theta(\nu(t) - r(t))dt + s_r(t)dW, \quad \theta : \text{constant}$$

to better see the attributes of the short term rate. Here the variable s_r is the same as σ, and is used so that it aligns with the rest of the paper. In this extended Vasicek model, ν gives the mean, θ gives how fast the rate fluctuates, and s_r gives the volatility of the interest rate. The last three models are widely accepted and popular in practice because of their closed form solutions. In Chapter 4 and 5, we shall use this extended Vasicek model for risk-free, short-term rate.

The coupon bearing bond consists of the principal and predetermined number of coupons attached to the principal. The principal is paid at the face value on the maturity date unless there are some other conditions stated otherwise. Each coupon pays predetermined amount of money, normally expressed as a certain percentage (coupon rate) of the face value of the principal, or as a dollar amount, on a predetermined date. Coupons are detachable from the principal and transferable by themselves.

Therefore, the coupon bearing bond can be considered as a portfolio of zero-coupon bearing bonds: one zero-coupon bearing bond with a principal being the same as the original bond and n zero-coupon bonds with a principal being the same amount as the original bond's coupon, each maturing on original bond's i-th coupon date, where n is the number of coupon payments and $i = 1, 2, \cdots, n$ (La Grandvill, 2001). So if we let the value of corporate coupon-bond at time t to be $G = G(r, t; T)$, the value of zero-coupon bond to be $C(r, t; T)$, where T is the maturity of the bond, and c_i to be i-th coupon rate, then we have

$$G(r, t; T) = C(r, t; T) + \sum_{t_i \geq t} c_i C(r, t; t_i)$$

Often the time, the coupons are separated from the body of the bond (which pays only the principal amount) and they are traded separately. Thus separated bond is called a stripped bond.

Each coupon entitles the coupon holder to be paid a certain percentage of the face value of the bond. In most papers and documents (e.g. Hanke, 2003), coupons are treated as paying a certain percentage of the bond price, which is incorrect. To correct this kind of treating, throughout this paper, the coupon bearing bond is considered as a portfolio of zero-coupon bearing bonds.

Definition 1.2.7 (Option) *The option is a financial derivative which gives the holder the right (but not the obligation) to buy ("call" option) or to sell ("put" option) a particular asset such as stocks or bonds at a specified time or time period in the future for previously determined price ("strike price" or "exercise price"). If exercise is permitted only at expiry, the options are called "European" options, and if exercise is allowed at any time before expiry, they are called "American" options.* □

In the chapters followed, the options are assumed to be European. The payoff functions of call option and put option are given by

$$\max(S(T) - E, 0) \quad \text{and} \quad \max(E - S(T), 0)$$

respectively, where E is the exercise price and $S(T)$ is the price of the underlying asset at the expiration date.

Definition 1.2.8 (Put-Call Parity) *If C, P and S are the prices of a call option, a put option and their underlying asset at time t, respectively, and T is the expiration of the options, then they satisfy the following equation:*

$$C - P = S - Ee^{-r(T-t)},$$

where r is the risk free interest rate. This relationship is called "put-call parity".□

This relationship is useful since once we find the price of the call option, it enables us to find the price of put option easily and vice versa.

Definition 1.2.9 (Forward) *A forward is a contract obligating one party to buy and the other party to sell a financial instrument, such as stock, bond, commodity or currency at a specific future date.* □

Definition 1.2.10 (Credit Default Swap) *The Credit Default Swap ("CDS") is a bilateral financial contract in which one counterparty (the "Protection Buyer") pays a periodic fee, paid on the notional amount and the other counterparty (the "Protection Seller") pays a predetermined amount in case a credit event with respect to a reference entity occurs.* □

The scheme is shown in Figure 1.

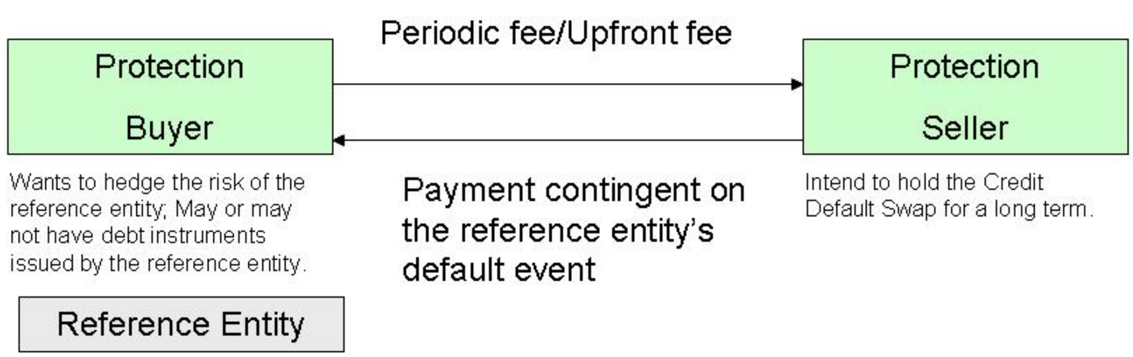

Figure 1.: Credit Default Swap

The reference entity is the issuer of the bonds, whose credit event triggers the protection seller's obligations. Credit events are precisely defined in each contract; they normally include:

- Bankruptcy,
- Failure to pay interest or principal,
- Obligation default

(This is where the reference entity's obligation becomes due as a result of any covenant breach under the relative obligation contract),

- Obligation acceleration

  (This is where the bond-holders demand immediate repayment in full as a result of any covenant breach of other obligations of the reference entity),

- Reconstruction

  (Reconstruction includes events such as a reduction in the principal amount or interest payable under the obligation, a postponement of payment, a change in ranking in priority of payment or any other composition of payment.)

The protection buyer does not necessarily need to hold the bonds issued by a reference entity (as in the case where the buyer deals CDSs for speculation). Credit default swaps are the most traded derivative on the market of all credit derivatives.

Definition 1.2.11 (Credit default swaption) *Credit default swaption is an option on credit default swap. The underlying credit default swap does not exist during the life of credit default swaption; it is initiated only upon the exercise of the swaption.* □

Example. Let $t \geq 0$ be the trading date of the call option, $T_0 \geq t$ be the date at which the forward credit default swap becomes effective, and $T_N \geq T_0$ be the maturity date of the forward credit default swap. The expiration date of the option falls on T_0. The swaption holder is entitled to enter the credit default swap (therefore, becomes a protection buyer) at time T_0 at the predetermined periodic fee, say s^*. Let the price of the swaption be $X(t)$ (or $\hat{X}(t), 0 \leq t \leq T_0$) as modeled later. Let $s(t)$ be the periodic fee of a forward credit default swap at time t, with the same reference entity and the same contract duration as the underlying credit default swap for the swaption; so that $s(T_0)$ is the price of the credit default swap at $t = T_0$. An investor buys the swaption only if he/she anticipates that the credit of the reference company deteriorates and the market fee at the time T_0, i.e., $s(T_0)$ exceeds s^*. This way, if the swaption holder actually holds the bonds issued by the reference company, it can acquire the credit default swap for less cost. If the swaption holder does not own the bonds, then it can lock the profit $s(T_0) - s^*$ (ignoring the initial cost) by selling the protection on the bonds issued by the reference company. Table 1 shows the summary of the cashflow in three scenarios. (Note that $1\% = 1p = 100bp$ (basis points).)

Table 1: Cashflow of the Protection Buyer with or without Credit Default Swaption

	With Swaption	Without Swaption	
Case	-	Case 1 The reference entity's financial condition deteriorates	Case 2 The reference entity's financial condition improves
Initial cost	X (or $\hat{X}$)	0	0
Periodic fee	s^*=50bp	100bp	25bp

Definition 1.2.12 ((No) Arbitrage Principle) *Loosely stated, the principle asserts that "there is no such thing as a free lunch." Formally,*

Principle 1: If the value of two portfolios are $\Pi_1(t)$ and $\Pi_2(t)$ at time t, then

$$\Pi_1(t) \leq \Pi_2(t) \quad \textit{if} \quad \Pi_1(T) \leq \Pi_2(T) \quad \textit{a.s.}, \quad t < T$$

and

$$\Pi_1(t) = \Pi_2(t) \quad \textit{if} \quad \Pi_1(T) = \Pi_2(T) \quad \textit{a.s.}, \quad t < T$$

Principle 2: Suppose Π is the value of a risk-free portfolio, and $d\Pi$ is its price increment during a small period of time dt. Then,

$$\frac{d\Pi}{\Pi} = rdt,$$

where r is the risk-free interest rate. □

Definition 1.2.13 (Risk neutral measure) *Given a probability space (Ω, F, P) with filtration $\{F_n; n = 0, ..., N\}$, a probability measure Q is said to be risk-neutral if*

(1) Q is equivalent to P, i.e. $P(A) > 0$ if and only if $Q(A) > 0$ for all $A \in F$, and

(2) the discounted price process $\bar{S} = (\bar{S}_n)_{0 \leq n \leq N}$ is a martingale with respect to (Ω, F, Q) with filtration $\{F_n; n = 0, ..., N\}$. □

Under a risk neutral measure the current price of each security in the economy is equal to the present value of the discounted expected value of its future payoffs given a risk-free interest rate. Under a viable market, i.e., under the market where there is no arbitrage opportunities, the Fundamental

Theorem of Asset Pricing guarantees the existence and the uniqueness of such risk neutral measure. Its proof can be found in Harrison & Pliska (1981).

If the payoff price of a financial product at time T is $H(T)$, where $H(T)$ is a random variable on the probability space describing the market, and the discount factor from time $t = 0$ to $t = T$ is $P(0, T)$, then the fair price of the product at time $t = 0$ is given by

$$H(0) = P(0, T)E_Q[H(T)],$$

where the risk-neutral measure is denoted by Q. If the real world probability measure of $H(T)$ is given by P, then $H(0)$ can be also given as

$$H(0) = E_P\left[\frac{dQ}{dP}H(T)\right],$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P.

The valuation of credit default swaps requires estimating both expected default time and the expected loss of the reference entity. The risk neutral default probability can be estimated either from the reference entity's total asset or from debt market. When we use the total asset, we actually consider the company's financial structure; hence default is considered to be an endogenous event, and the models are called "structural models". On the other hand, when we use the data from debt market to estimate the default probability, we completely ignore the company's financial structure and consider default is an exogenous event. These models are called "intensity" or "reduced" models. Both structural models and intensity models have pros and cons; we will examine both models briefly below.

Definition 1.2.14 (Structural models) *Let the market values of the firm asset, the equity, and the debt of a company at time t be $V(t), S(t)$ and $C(t)$, respectively. Then, we have the relationship $V(t) = S(t)+C(t)$, which is known as the accounting equation. The left-hand side of this equation explains how the firm's money was invested, in a nutshell. The right-hand side of this equation represents the source of the firm's assets, that is, how the firm's money was raised. It should be noted that while $V(t)$ and $C(t)$ are non-negative, $S(t)$ can assume negative value. In these models, we assume that $V(t)$ follows a geometric Brownian motion. Then, the payoff $R(t)$ of the debt at it* *maturity T is given by*

$$R(T) = \min\big(C(T), V(T)\big).$$

Therefore, under these models, we conclude that the company has default when the firm asset $V(T)$ falls below $C(T)$, or other predetermined barrier (as we will see in Subsection 2.2.1). $\square$

The merits of using structural models are:

- They intuitively make an economic sense, and default is considered as an endogenous event.
- The time of default is not random (as opposed to the intensity model).
- We can assess the value of defaultable debt.

However, these models have the following demerits:

- They are unwieldy to implement.
- They are incoherent with the historical data.

Definition 1.2.15 (Intensity models) *In intensity models the time of default is the first jump of an exogenously given jump process. The parameter governing the default intensity are inferred from the relative market. The default time is modeled as the first jump of a Poisson process. Let us assume that the default intensity follows a stochastic process of the form*

$$dp(t) = a(t)dt + s(t)dW(t)$$

where $W(t)$ is a Brownian motion. If the default time is τ, then, the survival probability at time t is given by

$$\mathbb{P}[\tau > t] = \exp\Big[-\int_0^t p(s)ds\Big].$$

The default intensity $p(t)$ is the instantaneous rate of default. Let $P(t,T)$ be the conditional proba-

bility of survival at time T as seen from time $t < T$. Then the intensity is obtained by

$$\begin{aligned} p(t) &= \lim_{\Delta t \to 0} \frac{P(t,T) - P(t+\Delta t, T)}{\Delta t \cdot P(t,T)} \\ &= \frac{-1}{P(t,T)} \lim_{\Delta t \to 0} \frac{P(t+\Delta t, T) - P(t,T)}{\Delta t} \\ &= -\frac{\frac{\partial}{\partial t} P(t,T)}{P(t,T)} \quad \square \end{aligned}$$

The advantages of these models are:

- They are easy to implement;
- Bond prices derived using these models fit the spread between default-free and defaultable bonds.

However, the major disadvantage of these models is the fact that there is no direct relationship between the intensity and the asset value. In these models, the recovery rate will be also exogenously specified.

Definition 1.2.16 (Frictionless Market) *A frictionless market is a market where (i) there are no transaction costs, no bid-ask spread, no restrictions on trade such as margin requirements or short sale restrictions; (ii) there are no taxes; (iii) borrowing and lending are done at the same risk-free interest rate, and (iv) asset shares (stocks or bonds) are divisible, i.e., an investor can buy any fraction of one stock or one bond certificate.* □

Throughout this paper, we assume that all the markets involved are frictionless.

Definition 1.2.17 (Credit Rating) *A credit rating assesses the borrowing capacity of an individual or company.* □

Credit ratings are calculated from financial history and current assets and liabilities. Typically, a credit rating tells a borrower or investor the probability of the borrower being able to pay back a loan. Ratings can be assigned not only to short-term, long-term debt obligations but also to securities or bank borrowing. A poor credit rating indicates a high risk of defaulting on a loan, and thus leads to high interest rates.

Moody's, and Standard and Poor's (S&P's) are considered to be world-wide top credit-rating agencies. Each of them intends to provide a rating system to help investors determine the risk associated with investing in a specific company, in an investing instrument or in a market.

Table 2 summarizes the different ratings symbols that Moody's and Standard and Poor's issue for long-term debt obligations *(Source: Heakal, R, "What Is A Corporate Credit Rating?", Wikipedia)*.

Table 2: Long-Term Bond Rating System

Moody's	Standard Poor's	Risk
Investment Grade		
Aaa	AAA	The best quality companies, reliable and stable
Aa	AA	Quality companies, a bit higher risk than AAA
A	A	Economic situation can affect finance
Baa	BBB	Medium class companies, which are satisfactory at the moment
Non-Investment/Speculative Grade (also known as junk bonds)		
Ba	BB	More prone to changes in the economy
B	B	Financial situation varies noticeably
Caa	CCC	Currently vulnerable and dependent on favorable economic conditions to meet its commitments
Ca	CC	Highly vulnerable, very speculative bonds
C	C	Highly vulnerable, perhaps in bankruptcy or in arrears but still continuing to pay out on obligations

Ratings can be assigned to short-term and long-term debt obligations as well as securities, loans, preferred stock and insurance companies. Long-term credit ratings tend to be more indicative of a country's investment surroundings and/or a company's ability to honor its debt responsibilities.

1.3 Black-Scholes Formulas

The derivation of the corporate coupon-bond in this paper has many similarities to the derivation of the Black-Sholes formulas. The Black-Scholes theory has its origin in the seminal paper "The Pricing of Options and Corporate Liabilities," by Black & Scholes (1973) and has been most widely cited in option pricing. The Black-Scholes Formula gives the price of a European call option $C(S_t, T)$ with exercise price K on a stock currently trading at price S_t, i.e., the right to buy a share of the stock at price K after T years. The constant interest rate is r, and the constant stock volatility is σ. The formula is given by

$$C(S,T) = S_t \Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{1.1}$$

where

$$d_1 = \frac{ln(S_t/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T} \tag{1.2}$$

In the following subsections, we will show two approaches to derive this formula: (i) PDE approach, and (ii) Martingale approach. In derivation, please note that we assume that the market is frictionless and the price of the underlying instrument S_t follows the lognormal model, that is, it satisfies the following stochastic differential equation :

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.3}$$

where $S_0 > 0$, the drift μ and volatility σ are constants, (W_t) is a Brownian motion defined on a filtered probability space $(\Omega, F, (F_t), P)$, where $F_t = \sigma(W_s, s \leq t)$.

1.3.1 PDE Approach

The Black-Scholes PDE was first introduced in "The Pricing of Options and Corporate Liabilities," by Black & Scholes (1973). In this approach, the idea of hedging and arbitrage was used. First we construct a portfolio consisting of one derivative C and φ units of the underlying instrument S. The value of the portfolio is $\Pi = V + \varphi S$. Then the change in price of the portfolio over a small time increment is given by

$$\begin{aligned} d\Pi &= dC + \varphi dS \\ &= \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}\sigma^2 S^2 dt + \varphi dS \\ &\quad \text{By lognormal model (1.3)} \\ &= (\frac{\partial C}{\partial S} + \varphi)\sigma S dW + (\frac{\partial C}{\partial S} + \varphi)\mu S dt + \frac{\partial C}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial s^2} dt \end{aligned} \tag{1.4}$$

By choosing φ so that uncertainty caused by dW-term is eliminated completely, i.e., by setting $\varphi = -\dfrac{\partial C}{\partial S}$, and, applying the arbitrage principle, we obtain the following Black-Scholes PDE:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{1.5}$$

where r is the risk-free interest rate and $0 \leq t \leq T$.

If the derivative in discussion is a call option on some financial instrument with exercise price K and expiration T, then after applying changes of variables and Fourier Transformation, we will find the Black-Scholes Formulas (1.1) and (1.2) as a solution for the Black-Scholes PDE.

1.3.2 Martingale Approach

Harrison & Kreps (1979) and Harrison & Pliska (1981) showed that a natural mathematical framework for analysis of financial markets is martingale theory and stochastic analysis.

Let the process (B_t) be the value of the risk-free account satisfying $dB_t = rB_t dt$ with $B_0 = 1$. Let $\{a, b\}$ be a pair of F_t-adapted process (which is called a *"trading strategy"*), where $a(t)$ and $b(t)$ are numbers of the units of the asset and the risk-free account at time t, respectively. Then, the value of the trading strategy at time t of a portfolio $\{a(t), b(t)\}$ is given by

$$V_t = a(t)S_t + b(t)B_t$$

Lemma 1.3.1 A trading strategy is self-financing (meaning the change of the value of the trading strategy is due to the changes in the assets prices), i.e., $dV_t = a(t)dS_t + b(t)dB_t$ if and only if its discounted wealth process $\tilde{V}_t$ satisfies

$$d\tilde{V}_t = a(t)d\tilde{S}_t. \tag{1.6}$$

Its proof is found in Yan & Ju (1999). Note that we can rewrite (1.3) as

$$d\tilde{S}_t = \tilde{S}_t\big[(\mu - r)dt + \sigma dW_t\big] \tag{1.7}$$

and by putting $\frac{dQ}{dP}|_{F_t} = \exp\{-\frac{\mu-r}{\sigma}W_T - \frac{1}{2}(\frac{\mu-r}{\sigma})^2 T\}$, we have Q-Brownian motion $W_t^* = W_t + \frac{\mu-r}{\sigma}t$ by the Girsanov's theorem, and

$$d\tilde{S}_t = \tilde{S}_t \sigma dW_t^*, \tag{1.8}$$

so that $(\tilde{S}_t)$ is a Q-martingale.

Theorem 1.3.2 Let $\xi = f(S_T)$ be a European contingent claim under Q-martingale. Then there exists an admissible self-financing strategy $\{a, b\}$ replicating ξ such that its value process (V_t) is given by

$$V_t = E^*[e^{-r(T-t)}\xi|F_t] \tag{1.9}$$

or equivalently, the discounted process (V_t) is a Q-martingale.

Its proof can be found in Yan & Ju (1999). Note that we can say that V_t is the fair price at time t of the contingent claim ξ since there is no arbitrage opportunity at this price.

Corollary 1.3.3 Under the assumption of Theorem 1.3.2, we have $V_t = F(t, S_t)$, where

$$F(t, x) = e^{-r(T-t)} \int_{\infty}^{\infty} f\left(xe^{(r-\sigma^2)(T-t)+\sigma y\sqrt{T-t}}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy. \tag{1.10}$$

If we consider a European call option $\xi = (S_T - K)^+$, whose price is $V_t = C(t, S_t)$, its price is given by (1.1) and (1.2).

Chapter 2

Defaultable Corporate Coupon-Bond Pricing with Constant Interest Rate

In this chapter, we will find the price of corporate coupon-bond. In the first section, we consider default as an exogenous event, therefore, default is an unexpected. In the second section, in addition to exogenous cause, we take an endogenous event into consideration as a possible cause for default.

2.1 Corporate Coupon-Bond with Constant Interest Rate - Unexpected Default

2.1.1 Formulation

In this section, we will assume the following. As suggested in the introduction, we consider coupon bearing bonds as a portfolio of zero-coupon bonds consisting of one principal portion due on the maturity date and coupons due on coupon payment days. And throughout Chapters 2 and 3, we will assume that the risk-free short-term rate *r* is constant.

Assumption 1: Default is an exogenous event. Unexpected default probability in $[t, t + dt]$ is $p_t dt$. If coupon is not due on the interval, then the default intensity $p(t) = p_t$ follows

$$dp = a_p(p, t)dt + s_p(p, t)dW_1.$$

where $a_p(p, t)$ and $s_p(p, t)$ are the drift and the volatility of p respectively. On predetermined coupon payment date $t = \tau_j$, where j refers to *j*-th interest payment and $j = 1, 2, \dots, n$ (this means that $\tau_n = T$, where T is the maturity of the bond), and the jump of p_t is given by

$$\Delta p = p_{\tau_j} - p_{\tau_j -} = p_{\tau_j -} U_j$$

where U_j is a jump size at $t = \tau_j$. A sequence $(U_j)_{1\le j\le n}$ is independent, identically distributed random variable taking values in $[x, 0]$ with $-1 < x < 0$. Thus $p_{\tau_j} = p_{\tau_j-}(1 + U_j)$. So we have

$$p_t = p_0 + \int_0^t a_p(p,t)\,dt + \int_0^t s_p(p,t)\,dW + \sum_{j=1}^{N_t} p_{\tau_j-}U_j,$$

where N_t is the number of interest payments up to time t. This is right- continuous, adapted process with finite and predetermined discontinuities.

Hence, for any interval $[t, t + dt]$,

$$dp = a_p(p,t)dt + s_p(p,t)dW_1 + p_{\tau_j-}U_j I_{\{\tau_j\in(t,t+dt]\}},$$

where $I_{\{\tau_j\in(t,t+dt]\}}$ is an indicator function taking 1 when $\tau_j \in (t, t + dt]$ and 0 otherwise.

Assumption 2: Default recovery, i.e., the recoverable amount of a defaulted bond, is given either in the form of face value exogenous recovery ($R \cdot e^{-r(T-t)}$ where R is constant with $0 \le R \le 1$, and T is the maturity of the bond) or in the form of market price exogenous recovery ($R\times$ bond price at default time).

Assumption 3: The defaultable corporate coupon-bond price is given by the function $\hat{G} = \hat{G}(p,t)$, which constitutes of $\hat{C} = \hat{C}(p,t)$, the value at time t of the principal portion only, $c_i\hat{C}(p,t;\tau_i)$ which is the value at time t of i-th coupon due on τ_i. Therefore, we have

$$\hat{G}(p,t) = \hat{C}(p,t) + \sum_{\tau_i\ge t} c_i\hat{C}(p,t;\tau_i). \tag{2.1}$$

Problem: Under these assumptions, find the price of defaultable corporate coupon-bond.

2.1.2 Derivation of the model

By the variation of Ito Formula for jump diffusion, if there is no default (with probability $1 - p_t dt$), we have

$$\begin{aligned} d\hat{C} &= \frac{\partial \hat{C}}{\partial t}dt + \frac{\partial \hat{C}}{\partial p}dp + \frac{1}{2}\frac{\partial^2 \hat{C}}{\partial p^2}(dp)^2 + \{\hat{C}(p_{\tau_j}, t) - \hat{C}(p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}} \\ &= \frac{\partial \hat{C}}{\partial t}dt + \frac{\partial \hat{C}}{\partial p}dp + \frac{1}{2}\frac{\partial^2 \hat{C}}{\partial p^2}(a_p dt + s_p dW)^2 + \{\hat{C}(p_{\tau_j}, t) - \hat{C}(p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}} \\ &= \frac{\partial \hat{C}}{\partial t}dt + \frac{\partial \hat{C}}{\partial p}dp + \frac{1}{2}s_p^2\frac{\partial^2 \hat{C}}{\partial p^2}dt + \{\hat{C}(p_{\tau_j}, t) - \hat{C}(p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}}. \end{aligned}$$

In case there is a default (with probability $p_t dt$), the change in price will be given by

$$d\hat{C} = R - \hat{C}$$

where R is the default recovery, which, later in the computation, we shall use its face value exogenous recovery. Note that in the event of a default, the bonds shall not be traded and the buyer shall be entitled to receive its recovery amount.

We construct a hedged portfolio by hedging one bond with another bond with different maturity. Let us denote the prices of these bonds by $\hat{C}_i = \hat{C}_i(p, t; T_i),\ i = 1, 2$. Here, T_i is the maturity of each bond. Let R_i be the recovery rate for each. Assume that coupon payment dates for both bonds are the same. Now construct a portfolio:

$$\Pi = \hat{C}_1 - \Lambda \hat{C}_2.$$

The change of value in this portfolio over a small time increment $[t, t + dt]$ is given by

$$d\Pi = d\hat{C}_1 - \Lambda d\hat{C}_2.$$

At the same time, by the Arbitrage Principle, we must have $d\Pi = r\Pi dt$.

If there is no default, we have

$$\begin{aligned} d\Pi &= \left(\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}_1}{\partial p^2}\right)dt + \frac{\partial \hat{C}_1}{\partial p}dp + \{\hat{C}_1(p_{\tau_j}, t) - \hat{C}_1(p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}} \\ &\quad -\Lambda\left[\left(\frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}_2}{\partial p^2}\right)dt + \frac{\partial \hat{C}_2}{\partial p}dp + \{\hat{C}_2(p_{\tau_j}, t) - \hat{C}_2(p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}}\right]. \end{aligned}$$

To get rid of uncertainty caused by dp term, we choose

$$\Lambda = \frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1}.$$

Then,

$$d\Pi = \left[\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}_1}{\partial p^2} - \frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1}\left(\frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}_2}{\partial p^2}\right)\right]dt$$
$$+(\{\hat{C}_1(p_{\tau_j},t) - \hat{C}_1(p_{\tau_j -},t)\}I_{\{\tau_j \in (t,t+dt]\}}$$
$$-\frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1}\{\hat{C}_2(p_{\tau_j},t) - \hat{C}_2(p_{\tau_j -},t)\}I_{\{\tau_j \in (t,t+dt]\}}).$$

If there is a default, the price change in the portfolio is

$$d\Pi = (R_1 - \hat{C}_1) - \frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1}(R_2 - \hat{C}_2).$$

Taking the expectation of $d\Pi = d\hat{C}_1 - \Lambda \hat{D}_2 = r\Pi dt$ and neglecting the higher order of infinitesimal of dt-term, we have

$$\left(\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}_1}{\partial p^2} - \frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1}\left(\frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}_2}{\partial p^2}\right)\right)dt$$
$$+\left(\left(\hat{C}_1(p_{\tau_j},t) - \hat{C}_1(p_{\tau_j -},t)\right)I_{\{\tau_j \in (t,t+dt]\}}\right.$$
$$\left.-\frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1}\left(\hat{C}_2(p_{\tau_j},t) - \hat{C}_2(p_{\tau_j -},t)\right)I_{\{\tau_j \in (t,t+dt]\}}\right)(1 - p_t dt)$$
$$+\left((R_1 - \hat{C}_1) - \frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1}(R_2 - \hat{C}_2)\right)p_t dt$$
$$= r\left(\hat{C}_1 - \frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1} C_2\right)dt,$$

which yields

$$\left[\left(\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}_1}{\partial p^2} - \left(\{\hat{C}_1(p_{\tau_j},t) - \hat{C}_1(p_{\tau_j -},t)\}I_{\{\tau_j \in (t,t+dt]\}} - (R_1 - \hat{C}_1)\right)p_t - r\hat{C}_1\right)dt\right.$$
$$\left.+\{\hat{C}_1(p_{\tau_j},t) - \hat{C}_1(p_{\tau_j -},t)\}I_{\{\tau_j \in (t,t+dt]\}}\right]\left(\frac{\partial \hat{C}_1}{\partial p}\right)^{-1}$$
$$= \left[\left(\frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}_2}{\partial p^2} - \left(\{\hat{C}_2(p_{\tau_j},t) - \hat{C}_2(p_{\tau_j -},t)\}I_{\{\tau_j \in (t,t+dt]\}} - (R_2 - \hat{C}_2)\right)p_t - r\hat{C}_2\right)dt\right.$$
$$\left.+\{\hat{C}_2(p_{\tau_j},t) - \hat{C}_2(p_{\tau_j -},t)\}I_{\{\tau_j \in (t,t+dt]\}}\right]\left(\frac{\partial \hat{C}_2}{\partial p}\right)^{-1}.$$

The left hand side of this equation is a function of T_1 but not T_2, and the right hand side of this equation is a function of T_2 but not T_1, so both sides must be functions independent of their maturity date, say $-a_p(p,t)dt$. Therefore, we have the equation for corporate coupon bond with default intensity p_t, taking the recovery assumption into consideration:

$$\Big[\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}s^2\frac{\partial^2 \hat{C}}{\partial p^2} + a_p\frac{\partial \hat{C}}{\partial p} - \big(\{\hat{C}(p_{\tau_j},t) - \hat{C}(p_{\tau_j -},t)\}I_{\tau_j\in(t,t+dt]} - (R\cdot e^{-r(T-t)} - \hat{C})\big)p_t - r\hat{C}\Big]dt + \Big(\hat{C}(p_{\tau_j},t) - \hat{C}(p_{\tau_j -},t)\Big)I_{\tau_j\in(t,t+dt]} = 0. \quad (2.2)$$

Here a_p is a risk neutral drift of p_t. We can write a_p in the form $a_p(p,t) = a(p,t) - s(p,t)\lambda(p,t)$, where $\lambda(p,t) = \frac{a(p,t)-a_p(p,t)}{s(p,t)}$ is called a market price risk of p_t and measures an extra compensation per unit of risk for taking on the risk incurred by p_t. In the computation below, we assume that $\lambda(p,t) = 0$, so that $a_p(p,t) = a(p,t)$.

Note that the probability of the survival of the bond at time T, given it was not defaulted at time $t(< T)$, denoted by $P(t,T)$ is given by

$$P(t,T) = e^{-\int_t^T p(s)\ ds} \quad (2.3)$$

and since $p_t = p_0 + \int_0^t a_p(p,t)\ dt + \int_0^t s_p(p,t)\ dW + \sum_{j=1}^{N_t} p_{\tau_j -}U_j$, we have

$$P(\tau_{j-},\tau_j) = e^{-p_{\tau_j -}U_j}. \quad (2.4)$$

Now, in addition to the previous assumptions, let us also assume the following.

Assumption 4: The defaultable coupon bond price at time $t = \tau_{j-}$ is the expectation of the price at time $t = \tau_j$ with respect to the survival probability measure P, i.e.,

$$\begin{aligned} \hat{C}(p,\tau_{j-}) &= E(\hat{C}(p,\tau_j)) \\ &= \hat{C}(p,\tau_j)\cdot P(\tau_{j-},\tau_j)) + R\cdot e^{-r(T-\tau_j)}(1 - P(\tau_{j-},\tau_j)) \\ &= \hat{C}(p,\tau_j)e^{-p_{\tau_j -}U_j} + R\cdot e^{-r(T-\tau_j)}(1 - e^{-p_{\tau_j -}U_j}). \end{aligned} \quad (2.5)$$

Especially, if $t = T- = \tau_{n-}$, we have

$$\begin{aligned} \hat{C}(p, T-) &= 1 \cdot P(T-, T) + R \cdot (1 - P(T-, T)) \\ &= e^{-p_{T-}U_n} + R \cdot (1 - e^{-p_{T-}U_n}) \end{aligned}$$

and $C(p, T) = 1$ if there is not default until $t = T$. $\square$

In addition, since as the default intensity increases, the company is more likely to get defaulted we have the following boundary condition:

$$\lim_{p \to \infty} \hat{C}(p, t) = R.$$

If $p = 0$, then it implies that the bond is default-free. Therefore we also have the following boundary condition.

$$\hat{C}(p = 0, t) = e^{-r(T-t)}.$$

Since these boundary conditions are not used in solving the equation, they will not be repeated below.

Then the equation (2.2) on each time interval $[\tau_{j-1}, \tau_j)$ becomes the following

$$\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}s^2 \frac{\partial^2 \hat{C}}{\partial p^2} + a \frac{\partial \hat{C}}{\partial p} + (R \cdot e^{-r(T-t)} - \hat{C})p_t - r\hat{C} = 0$$

or

$$\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}s^2 \frac{\partial^2 \hat{C}}{\partial p^2} + a \frac{\partial \hat{C}}{\partial p} - (r + p_t)\hat{C} + R \cdot e^{-r(T-t)} p_t = 0$$

with the terminal condition:

$$\hat{C}(p, \tau_{j-}) = \hat{C}(p, \tau_j) e^{-p_{\tau_{j-}} U_j} + R \cdot e^{-r(T-\tau_j)}(1 - e^{-p_{\tau_{j-}} U_j}). \tag{2.6}$$

This is a nonhomogeneous parabolic equation with variable coefficients and a terminal condition. First we will solve this on the time interval $[\tau_{n-1}, T)$, i.e.,

$$\begin{cases} \dfrac{\partial \hat{C}}{\partial t} + \dfrac{1}{2}s^2 \dfrac{\partial^2 \hat{C}}{\partial p^2} + a \dfrac{\partial \hat{C}}{\partial p} - (r + p_t)\hat{C} + R \cdot e^{-r(T-t)} p_t = 0 & (\tau_{n-1} \le t < T, p > 0), \\ \hat{C}(p, T-) = \hat{C}(p, T) \cdot e^{-p_{T-}U_n} + R \cdot (1 - e^{-p_{T-}U_n}) & (p > 0). \end{cases}$$

By letting

$$\hat{C} = ue^{-r(T-t)}, \tag{2.7}$$

we have

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}s^2\dfrac{\partial^2 u}{\partial p^2} + a\dfrac{\partial u}{\partial p} - p_t(u - R) = 0 & (\tau_{n-1} \le t < T, p > 0), \\ u(p, T-) = e^{-p_T - U_n} + R \cdot (1 - e^{-p_T - U_n}) = e^{-p_T - U_n}(1 - R) + R & (p > 0), \end{cases} \tag{2.8}$$

since $\hat{C}(p, T) = 1$.

Using the change of unknown function

$$\hat{u} = u - R, \tag{2.9}$$

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} + \dfrac{1}{2}s^2\dfrac{\partial^2 \hat{u}}{\partial p^2} + a\dfrac{\partial \hat{u}}{\partial p} - p_t\hat{u} = 0 & (\tau_{n-1} \le t < T, p > 0), \\ \hat{u}(p, T-) = u - R = [e^{-p_T - U_n}(1 - R) + R] - R = e^{-p_T - U_n}(1 - R) & (p > 0). \end{cases}$$

By a further change of the unknown function to $W = W(p, t)$ so that

$$\hat{u} = We^{-p_T - U_n}(1 - R), \tag{2.10}$$

We obtain the terminal value problem on the unknown W:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}s^2\dfrac{\partial^2 W}{\partial p^2} + a\dfrac{\partial W}{\partial p} - p_tW = 0 & (\tau_{n-1} \le t < T, p > 0), \\ W(p, T-) = 1 & (p > 0). \end{cases} \tag{2.11}$$

From (2.7), (2.9), and (2.10), we can express the price of defaultable coupon bonds for $\tau_{n-1} \le t < T$ in terms of W as follows:

$$\begin{aligned} \hat{C}(p, t) &= ue^{-r(T-t)} = (\hat{u} + R)e^{-r(T-t)} \\ &= \left(We^{-p_T - U_n}(1 - R) + R\right)e^{-r(T-t)} \\ &= (We^{-p_T - U_n})e^{-r(T-t)} + (1 - We^{-p_T - U_n})R \cdot e^{-r(T-t)}. \end{aligned}$$

$\hat{C}(p, t)$ is considered to be the expectation of the bond price at time t . So $We^{-p_{T-}}$ can be regarded as the probability of survival at time t and $1 - We^{-p_{T-}}$ as probability of default at time t.

In solving (2.11), we will follow Wilmott (1998) and O et al (2005) and restrict $a(p,t)$ and $s(p,t)$ to the following cases.

Assumption 5: $a(p,t)$ and $s^2(p,t)$ are linear in p,

$$a(p,t) = b(t) - c(t)p, \tag{2.12a}$$

$$s^2(p,t) = d(t) + e(t)p. \tag{2.12b}$$

Assume that the solution of (2.11) is given in the form:

$$W(p,t) = e^{A(t,T-)-B(t,T-)p}.$$

Then, since

$$\frac{\partial W}{\partial t} = (A' - B'p)W \quad \frac{\partial W}{\partial p} = -WB \quad \frac{\partial^2 W}{\partial p^2} = WB^2$$

where A' and B' are derivatives of A and B with respect to t, respectively; substituting these in (2.11) gives

$$A' + \frac{1}{2}d(t)B^2 - b(t)B + p\left(-B' + \frac{1}{2}e(t)B^2 + c(t)B - 1\right) = 0.$$

This holds for any value of p, so we must have

$$\begin{cases} A' + \frac{1}{2}d(t)B^2 - b(t)B = 0, \\ -B' + \frac{1}{2}e(t)B^2 + c(t)B - 1 = 0. \end{cases} \tag{2.13}$$

Note that since $W(p,T-) = 1$ from (2.11), we have $A(T-,T-) = B(T-,T-) = 0$. Once we solve for B in the second equation in (2.13), we can find A from the first equation as follows:

$$A(t,T-) = -\int_t^{T-} \left(b(s)B(s,T-) - \frac{1}{2}d(s)B^2(s,T-)\right) ds.$$

In solving (2.13), we will restrict (2.12) to the following cases: (i) $c(t) \equiv c$(constant); $e(t) \equiv 0$, (ii) $c(t) \equiv 0$; $e(t) \equiv K > 0$(constant).

Case (i): $c(t) \equiv c$(constant); $e(t) \equiv 0$. Then, we have

$$dp = (b(t) - c \cdot p)dt + \sqrt{d(t)} \cdot dW_1.$$

This case covers Vasicek model (b, c, d are constant; $e = 0$), Ho-Lee model ($c = 0, e = 0, d$ is constant) and Hull-White model (c, d are constant $e = 0$). Then, the second equation in (2.13) becomes

$$B' = cB - 1,$$

and the solution is

$$B(t, T-) = \begin{cases} \dfrac{1 - e^{-c(T-t)}}{c}, & c \neq 0, \\ T - t, & c = 0. \end{cases}$$

Case (ii): $c(t) \equiv 0; e(t) \equiv K > 0$ (constant). Then we have

$$dp = b(t)dt + \sqrt{d(t) + K \cdot p} \cdot dW_1.$$

This case covers Merton model (b, d are constant; $c = 0, e = K = 0$). Then the second equation in (2.13) becomes

$$B' - \frac{K}{2}B^2 + 1 = 0.$$

Letting $x = \sqrt{\frac{K}{2}}B$,

$$\frac{dx}{1 - x^2} = -\sqrt{\frac{K}{2}}dt.$$

Integrating both sides, we obtain

$$\frac{1}{2} \cdot \ln\left|\frac{1+x}{1-x}\right| + k = -\sqrt{\frac{K}{2}}t.$$

Using the terminal condition $x(T-, T-) = \sqrt{\frac{K}{2}}B(T-, T-) = 0$, $k = -\sqrt{\frac{K}{2}}T$. By substituting B back in x and solving for B, we obtain the following:

$$B(t, T-) = -\sqrt{\frac{2}{c}} \cdot \frac{exp(\sqrt{\frac{c}{2}}(T-t)) - exp(-\sqrt{\frac{c}{2}}(T-t))}{exp(\sqrt{\frac{c}{2}}(T-t)) + exp(-\sqrt{\frac{c}{2}}(T-t))}.$$

Therefore, assuming that the price of defaultable coupon bond at time $t = T- = \tau_{n-}$ is $\hat{C}(p, T-) = \hat{C}(p, \tau_{n-}) = e^{-p_{T-}U_n} + R \cdot (1 - e^{-p_{T-}U_n})$, the price on defaultable coupon bond for $\tau_{n-1} \leq t < T$ is given by

$$\hat{C}(p, t) = e^{A(t,T-)-B(t,T-)p_t - p_{T-}U_n - r(T-t)} + (1 - e^{A(t,T-)-B(t,T-)p_t - p_{T-}U_n})R \cdot e^{-r(T-t)}$$

where

$$A(t,T-) = -\int_t^{T-} \left(b(s)B(s,T-) - \frac{1}{2}d(s)B^2(s,T-) \right) \, ds \tag{2.14}$$

and

$$B(t,T-) = \begin{cases} \frac{1-e^{-c(T-t)}}{c}, & dp = (b(t) - c\cdot p)dt + \sqrt{d(t)}\cdot dW_1, c \neq 0 \\ \tau_n - t, & dp = (b(t) - c\cdot p)dt + \sqrt{d(t)}\cdot dW_1, c = 0 \\ -\sqrt{\frac{2}{c}}\cdot \frac{exp(\sqrt{\frac{c}{2}}(T-t))-exp(-\sqrt{\frac{c}{2}}(T-t))}{exp(\sqrt{\frac{c}{2}}(T-t))+exp(-\sqrt{\frac{c}{2}}(T-t))}, & dp = b(t)dt + \sqrt{d(t)+K\cdot p}\cdot dW_1. \end{cases} \tag{2.15}$$

Letting $t = \tau_{n-1}$, we have the value for $\hat{C}(p, \tau_{n-1})$,

$$\hat{C}(p,\tau_{n-1}) = e^{A(\tau_{n-1},T-)-B(\tau_{n-1},T-)p_{\tau_{n-1}}-p_{T-}U_n-r(T-\tau_{n-1})} + (1-e^{A(\tau_{n-1},T-)-B(\tau_{n-1},T-)p_{\tau_{n-1}}-p_{T-}U_n})R\cdot e^{-r(T-\tau_{n-1})},$$

and then we can get the terminal condition on the interval $[\tau_{n-2}, \tau_{n-1})$ from (2.6), which is:

$$\begin{aligned} \hat{C}(p,\tau_{(n-1)-}) &= \hat{C}(p,\tau_{n-1})e^{-p_{\tau_{(n-1)-}}U_{n-1}} + R\cdot e^{-r(T-\tau_{n-1})}(1-e^{-p_{\tau_{(n-1)-}}U_{n-1}}) \\ &= e^{-r(T-\tau_{n-1})}\left(e^{D(\tau_{n-1},T-)-\sum\limits_{j=n-1}^{n} p_{\tau_{j-}}U_j} + R(1-e^{D(\tau_{n-1},T-)-\sum\limits_{j=n-1}^{n} p_{\tau_{j-}}U_j}) \right) \end{aligned}$$

where $D(\tau_{n-1},T-) = A(\tau_{n-1},T-) - B(\tau_{n-1},T-)p(\tau_{(n-1)})$.

From (2.6) with $j = n-1$, we have:

$$\begin{cases} \dfrac{\partial \hat{C}}{\partial t} + \dfrac{1}{2}s^2\dfrac{\partial^2 \hat{C}}{\partial p^2} + a\dfrac{\partial \hat{C}}{\partial p} - (r+p_t)\hat{C} - R\cdot e^{-r(T-t)}p_t = 0 \\ \qquad\qquad (\tau_{n-2} \le t < \tau_{n-1}, p > 0), \\ \hat{C}(p,\tau_{(n-1)-}) = e^{-r(T-\tau_{n-1})}\left(e^{D(\tau_{n-1},T-)-\sum\limits_{j=n-1}^{n} p_{\tau_{j-}}U_j} + R(1-e^{D(\tau_{n-1},T-)-\sum\limits_{j=n-1}^{n} p_{\tau_{j-}}U_j}) \right) \\ \qquad\qquad (p > 0). \end{cases}$$

As before, letting $\hat{C} = ue^{-r(T-t)}$, we have

$$\begin{cases} \dfrac{\partial u}{\partial t} + \frac{1}{2}s^2\dfrac{\partial^2 u}{\partial p^2} + a\dfrac{\partial u}{\partial p} - p_t(u-R) = 0 & (\tau_{n-2} \le t < \tau_{n-1}, p > 0), \\ \\ u(p, \tau_{(n-1)-}) = e^{D(\tau_{n-1}, T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}} U_j}(1-R) + R & (p > 0). \end{cases} \tag{2.16}$$

Again, using the change of unknown function

$$\hat{u} = u - R,$$

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} + \frac{1}{2}s^2\dfrac{\partial^2 \hat{u}}{\partial p^2} + a\dfrac{\partial \hat{u}}{\partial p} - p_t\hat{u} = 0 & (\tau_{n-2} \le t < \tau_{n-1}, p > 0), \\ \hat{u}(p, \tau_{(n-1)-}) = u - R = e^{D(\tau_{n-1}, T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}} U_j}(1-R) & (p > 0). \end{cases}$$

Using the change of unknown function $W = W(p,t)$ so that

$$\hat{u} = We^{D(\tau_{n-1}, T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}} U_j}(1-R),$$

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}s^2\dfrac{\partial^2 W}{\partial p^2} + a\dfrac{\partial W}{\partial p} - p_tW = 0 & (\tau_{n-2} \le t < \tau_{n-1}, p > 0), \\ W(p, \tau_{(n-1)-}) = 1 & (p > 0). \end{cases}$$

Again we set $W(p,t) = e^{A(t,\tau_{(n-1)-}) - B(t,\tau_{(n-1)-})p}$.

So we can write $\hat{C}(p,t)$ in terms of W as

$$\hat{C}(p,t) = (We^{D(\tau_{n-1}, T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}} U_j})e^{-r(T-t)} + (1 - We^{D(\tau_{n-1}, T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}} U_j})R \cdot e^{-r(T-t)}.$$

As before we find $A(t, \tau_{(n-1)_-})$ and $B(t, \tau_{(n-1)_-})$ as follows:

$$A(t, \tau_{(n-1)-}) = -\int_t^{\tau_{(n-1)-}} \left(b(s)B(s, \tau_{(n-1)-}) - \frac{1}{2}d(s)B^2(s, \tau_{(n-1)-}) \right) ds$$

and

$$B(t, \tau_{(n-1)-}) = \begin{cases} \dfrac{1 - e^{-c(\tau_{n-1} - t)}}{c}, & dp = (b(t) - c \cdot p)dt + \sqrt{d(t)} \cdot dW_1, c \ne 0 \\ \tau_{n-1} - t, & dp = (b(t) - c \cdot p)dt + \sqrt{d(t)} \cdot dW_1, c = 0 \\ -\sqrt{\frac{2}{c}} \cdot \dfrac{e^{\sqrt{\frac{c}{2}}(\tau_{n-1} - t)} - e^{-\sqrt{\frac{c}{2}}(\tau_{n-1} - t)}}{e^{\sqrt{\frac{c}{2}}(\tau_{n-1} - t)} + e^{-\sqrt{\frac{c}{2}}(\tau_{n-1} - t)}}, & \\ & dp = b(t)dt + \sqrt{d(t) + K \cdot p} \cdot dW_1. \end{cases}$$

So for $\tau_{n-2} \leq t < \tau_{n-1}$, we have

$$\hat{C}(p,t) = \left(e^{D(\tau_{(n-1)-},T-)+A(t,\tau_{(n-1)-})-B(t,\tau_{(n-1)-})p_t - \sum_{j=n-1}^{n} p_{\tau_{j-}} U_j} \right) e^{-r(T-t)} + \left(1 - e^{D(\tau_{(n-1)-},T-)+A(t,\tau_{(n-1)-})-B(t,\tau_{(n-1)-})p_t - \sum_{j=n-1}^{n} p_{\tau_{j-}} U_j} \right) R \cdot e^{-r(T-t)}.$$

By extending this backwards, we have the following result.

Theorem 1. Under Assumptions 1 through 5, the price of the defaultable corporate coupon-bond for any $0 \leq t < T$, with $\tau_{j-1} \leq t < \tau_j$, is given by

$$\hat{G}(p,t) = \hat{C}(p,t) + \sum_{\tau_j \geq t} \hat{C}(p,t;\tau_j) \tag{2.17}$$

$$\hat{C}(p,t) = \left(e^{A(t,T-)-B(t,T-)p_t - \sum_{k=j}^{n} p_{\tau_{k-}} U_k} \right) e^{-r(T-t)} + \left(1 - e^{A(t,T-)-B(t,T-)p_t - \sum_{k=j}^{n} p_{\tau_{k-}} U_k} \right) R \cdot e^{-r(T-t)} \tag{2.18}$$

where

$$A(t,T-) = -\int_t^{\tau_{j-}} \left(b(s)B(s,\tau_{j-}) - \frac{1}{2}d(s)B^2(s,\tau_{j-}) \right) ds - \sum_{k=j+1}^{n} \int_{\tau_{k-1}}^{\tau_{k-}} \left(b(s)B(s,\tau_{k-}) - \frac{1}{2}d(s)B^2(s,\tau_{k-}) \right) ds \tag{2.19}$$

and

$$B(t,T-)=\begin{cases}\dfrac{n-(j-1)-e^{-c(\tau_j-t)}-\sum\limits_{k=j+1}^{n}e^{-c(\tau_k-\tau_{k-1})}}{c}, \\ \qquad \text{if } dp=(b(t)-c\cdot p)dt+\sqrt{d(t)}\cdot dW_1+p_{\tau_j-}U_jI_{\{\tau_j\in(t,t+dt]\}}, c\neq 0 \\ T-t, \\ \qquad \text{if } dp=(b(t)-c\cdot p)dt+\sqrt{d(t)}\cdot dW_1+p_{\tau_j-}U_jI_{\{\tau_j\in(t,t+dt]\}}, c=0 \\ -\sqrt{\frac{2}{c}}\cdot\dfrac{e^{\sqrt{\frac{c}{2}}(\tau_j-t)}+\sum\limits_{k=j+1}^{n}e^{\sqrt{\frac{c}{2}}(\tau_k-\tau_{k-1})}-e^{-\sqrt{\frac{c}{2}}(\tau_j-t)}-\sum\limits_{k=j+1}^{n}e^{-\sqrt{\frac{c}{2}}(\tau_k-\tau_{k-1})}}{e^{\sqrt{\frac{c}{2}}(\tau_j-t)}+\sum\limits_{k=j+1}^{n}e^{\sqrt{\frac{c}{2}}(\tau_k-\tau_{k-1})}+e^{-\sqrt{\frac{c}{2}}(\tau_j-t)}+\sum\limits_{k=j+1}^{n}e^{-\sqrt{\frac{c}{2}}(\tau_k-\tau_{k-1})}}, \\ \qquad \text{if } dp=b(t)dt+\sqrt{d(t)+K\cdot p}\cdot dW_1+p_{\tau_j-}U_jI_{\{\tau_j\in(t,t+dt]\}}\end{cases} \tag{2.20}$$

with, on each time interval, $\tau_{j-1}\leq t<\tau_j$

$$B(t,\tau_{j-})=\begin{cases}\dfrac{1-e^{-c(\tau_j-t)}}{c}, & dp=(b(t)-c\cdot p)dt+\sqrt{d(t)}\cdot dW_1, c\neq 0 \\ \tau_j-t, & dp=(b(t)-c\cdot p)dt+\sqrt{d(t)}\cdot dW_1, c=0 \\ -\sqrt{\frac{2}{c}}\cdot\dfrac{e^{\sqrt{\frac{c}{2}}(\tau_j-t)}-e^{-\sqrt{\frac{c}{2}}(\tau_j-t)}}{e^{\sqrt{\frac{c}{2}}(\tau_j-t)}+e^{-\sqrt{\frac{c}{2}}(\tau_j-t)}}, & dp=b(t)dt+\sqrt{d(t)+K\cdot p}\cdot dW_1.\end{cases} \tag{2.21}$$

In this section, we assumed that the solution to (2.11) is given in the exponential form. This is because the intensity model is considered to be an extension of the hazard rate model, which has a solution in the exponential form. It should be also noted that the Black-Scholes equation admits the exponential solution form via exponential transformation. While the Black-Scholes equation has constant drift and constant volatility, we assumed that the drift and the volatility of our PDE are dependent on the intensity and time.

2.2 Corporate Coupon-Bond with Constant Interest Rate - Expected and Unexpected Default

Under this section, we consider default event as both exogenous and endogenous event. While exogenous cause occurs outside the company's control (so we will use default intensity as in the previous section), in case of endogenous cause, the company decides to file bankruptcy when its total asset hits the predetermined barrier.

2.2.1 Formulation

Assumption 1: The firm assets $V = V(t) = V_t$ consists of *m* shares of traded stock, whose price at time *t* is $S = S(t) = S_t$, and *n* coupon-bond certificates, whose price at time *t* is $C = C(t) = C_t$:

$$V_t = mS_t + nC_t. \tag{2.22}$$

The firm assets value also follows the geometric Brownian motion (drift a_V, volatility s_V: constants) on $[t, t + dt]$ if coupon is not due on the interval,

$$dV = a_V V_t dt + s_V V_t dW_2, \tag{2.23}$$

and on predetermined coupon payment dates $t = \tau_j$, where *j* refers to *j*-th interest payment, $j = 1, \ldots, n$, the jump of V_t is given by

$$\Delta V_{\tau_j} = V_{\tau_j} - V_{\tau_j -} = ncC(T) = nc,$$

where c is the coupon rate of the bonds and *T* is the maturity of the bond. (Here we assume that the bonds are redeemed at their face value; therefore, $C(T) = 1$.)

Assumption 2: Unexpected default intensity is given by

$$p_t = p_0 + \int_0^t a(p,t)\ dt + \int_0^t s(p,t)\ dW_1 + \sum_{j=1}^{N_t} p_{\tau_j -} U_j,$$

where N_t is the number of interest payments up to time *t*. This is right- continuous, adapted process with finite and predetermined discontinuities. We assume that $E(dW_1 \cdot dW_2) = 0$, that is, unexpected default is not correlated with the asset value of the company.

Assumption 3: Expected default occurs when

$$V \leq V_b(t); \qquad V_b(t) = V_B \text{ or } V_B e^{-r(T-t)},$$

where V_B is constant and T is the maturity of the bonds.

Assumption 4: The defaultable corporate coupon-bond price is given by the function $G = G(V, p, t)$, which constitutes of $C = C(V, p, t)$, the value at time t of the principal portion only, and c_i, the i-th coupon with $c_i C(V, p, t; \tau_i)$ to be the value at time t of i-th coupon due on τ_i. Therefore, we have

$$G(V, p, t) = C(V, p, t) + \sum_{\tau_i \geq t} c_i C(V, p, t; \tau_i).$$

Assumption 5: Expected and unexpected default recovery is $R_d = R \cdot e^{-r(T-t)}$; $0 \leq R \leq 1$; constant.

Problem: Under these assumptions, we will find the corporate bond price of defaultable corporate coupon-bond with both expected and unexpected default, which is given as a function of V, p and t, that is $G = G(V, p, t)$.

2.2.2 Derivation of the Model

We will form a portfolio by buying one bond certificate under consideration and selling Λ_1 shares of traded stock and Λ_2 certificates of corporate coupon bond with unexpected default only, whose pricing was considered in the previous section and here assumed to be traded. That is,

$$\Pi = C - \Lambda_1 S - \Lambda_2 \hat{C}. \tag{2.24}$$

So the price change of the portfolio over a small increment of time dt is given by

$$d\Pi = dC - \Lambda_1 dS - \Lambda_2 d\hat{C}. \tag{2.25}$$

From (2.22), $S = \frac{V - nC}{m}$. Substituting this in (2.24) and (2.25),

$$\Pi = (1 + \frac{\Lambda_1 n}{m})C - \frac{\Lambda_1}{m} V - \Lambda_2 \hat{C} \tag{2.26}$$

and

$$d\Pi = (1 + \frac{\Lambda_1 n}{m})dC - \frac{\Lambda_1}{m}dV - \Lambda_2 d\hat{C}.$$

If there is no unexpected default in time interval $[t, t+dt]$ with probability $1 - p_t dt$, noting that we assumed that unexpected default is not correlated with the total asset value of the company, by Ito formula, the value change in the portfolio is given by

$$\begin{aligned}
d\Pi &= \left(1 + \frac{\Lambda_1 n}{m}\right) dC - \frac{\Lambda_1}{m} dV - \Lambda_2 d\hat{C} \\
&= \left(1 + \frac{\Lambda_1 n}{m}\right) \Bigg[\Bigg(\Bigg(\frac{\partial C}{\partial t} + \frac{1}{2}[s_p^2 \frac{\partial^2 C}{\partial p^2} + s_V^2 V^2 \frac{\partial^2 C}{\partial V^2}]\Bigg) dt\Bigg) + \frac{\partial C}{\partial V} dV + \frac{\partial C}{\partial p} dp \\
&\quad + \{C(V, p_{\tau_j}, t) - C(V, p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}} \\
&\quad + \{C(V - nc, p_t, t) - C(V, p_t, t)\} I_{\{\tau_j \in (t, t+dt]\}}\Bigg] - \frac{\Lambda_1}{m} dV \\
&\quad - \Lambda_2 \Big((\frac{\partial \hat{C}}{dt} + \frac{1}{2} s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2}) dt + \frac{\partial \hat{C}}{dp} dp + \{\hat{C}(p_{\tau_j}, t) - \hat{C}(p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}}\Big).
\end{aligned} \tag{2.27}$$

We will choose Λ_1 and Λ_2 so that we can get rid of uncertainty caused by dV and dp terms, i.e.,

$$(1 + \frac{\Lambda_1 n}{m})\frac{\partial C}{\partial V} - \frac{\Lambda_1}{m} = 0, \ (1 + \frac{\Lambda_1 n}{m})\frac{\partial C}{\partial p} - \Lambda_2 \frac{\partial \hat{C}}{\partial p} = 0.$$

Solving for Λ_1 and Λ_2,

$$\begin{aligned}
\Lambda_1 &= m \frac{\partial C}{\partial V} \left(1 - n \frac{\partial C}{\partial V}\right)^{-1}, \\
\Lambda_2 &= \frac{\partial C}{\partial p} \left(\frac{\partial \hat{C}}{\partial p}\right)^{-1} \left(1 - n \frac{\partial C}{\partial V}\right)^{-1}, \quad \text{and} \\
&\left(1 + \frac{\Lambda_1 n}{m}\right) = \left(1 - n \frac{\partial C}{\partial V}\right)^{-1}.
\end{aligned}$$

Substituting these in (2.27),

$$\begin{aligned}
d\Pi &= \left(1 - n \frac{\partial C}{\partial V}\right)^{-1} \Bigg[\Bigg(\frac{\partial C}{\partial t} + \frac{1}{2}[s_p^2 \frac{\partial^2 C}{\partial p^2} + s_V^2 V^2 \frac{\partial^2 C}{\partial V^2}]\Bigg) dt \\
&\quad + \{C(V, p_{\tau_j}, t) - C(V, p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}} + \{C(V - nc, p_t, t) - C(V, p_t, t)\} I_{\{\tau_j \in (t, t+dt]\}}\Bigg] \\
&\quad - \frac{\partial C}{\partial p} \left(\frac{\partial \hat{C}}{\partial p}\right)^{-1} \left(1 - n \frac{\partial C}{\partial V}\right)^{-1} \Big([\frac{\partial \hat{C}}{dt} + \frac{1}{2} s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2}] dt + \{\hat{C}(p_{\tau_j}, t) - \hat{C}(p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}}\Big) \\
&.
\end{aligned} \tag{2.28}$$

In case of default, with probability $p_t dt$, the price change in C and $\hat{C}$ are given by

$$dC = R_d - C \quad \text{and} \quad d\hat{C} = \hat{R} - \hat{C},$$

where $\hat{R}$ refers to the recovery rate of bond price with unexpected default only. Then, the price change in the portfolio is

$$\begin{aligned} d\Pi &= \left(1 + \frac{\Lambda_1 n}{m}\right)(R_d - C) - \frac{\Lambda_1}{m}(dV) - \Lambda_2(\hat{R} - \hat{C}) \\ &= \left(1 - n\frac{\partial C}{\partial V}\right)^{-1}\left[(R_d - C) - \frac{\partial C}{\partial V}dV - \frac{\partial C}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}(\hat{R} - \hat{C})\right]. \end{aligned} \tag{2.29}$$

By the arbitrage principle, the expectation of $d\Pi$ must be equal to $r\Pi dt$. That is we have the following:

$$(2.28) \times (1 - p_t dt) + (2.29) \times p_t dt = r\Pi dt \quad (= r \times (2.26) \times dt).$$

Therefore,

$$\begin{aligned} &\left(\frac{\partial C}{\partial t} + \frac{1}{2}\left[s_p^2\frac{\partial^2 C}{\partial p^2} + s_V^2 V^2\frac{\partial^2 C}{\partial V^2}\right] + rV\frac{\partial C}{\partial V} - rC - \Big\{\{C(V, p_{\tau_j}, t) - C(V, p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}}\right. \\ &\qquad \left. + \{C(V - nc, p_t, t) - C(V, p_t, t)\}I_{\{\tau_j \in (t, t+dt]\}}\Big\}p_t + (R_d - C)p_t\right)dt \\ &+ \Big\{\{C(V, p_{\tau_j}, t) - C(V, p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}} + \{C(V - nc, p_t, t) - C(V, p_t, t)\}I_{\{\tau_j \in (t, t+dt]\}}\Big\} \\ &\qquad = \frac{\partial C}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}\left[\Big\{\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}s_p^2\frac{\partial^2 \hat{C}}{\partial p^2} + (\hat{R} - \hat{C})p_t + r\hat{C}\right. \\ &\quad \left. - \{\{\hat{C}(p_{\tau_j}, t) - \hat{C}(p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}}\}p_t\Big\}dt + \{\hat{C}(p_{\tau_j}, t) - \hat{C}(p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}}\right]. \end{aligned} \tag{2.30}$$

From (2.2), the terms inside the bracket on the right hand side of the equation is equal to $-a_p(p,t)\frac{\partial \hat{C}}{\partial p}dt$, then, (2.30) becomes

$$\begin{aligned} &\Big\{\frac{\partial C}{\partial t} + \frac{1}{2}\left[s_p^2\frac{\partial^2 C}{\partial p^2} + s_V^2 V^2\frac{\partial^2 C}{\partial V^2}\right] + rV\frac{\partial C}{\partial V} + a_p\frac{\partial C}{\partial p} - rC + p(R_d - C)\Big\}dt \\ &\qquad - (1 - p_t dt)\Big\{\{C(V, p_{\tau_j}, t) - C(V, p_{\tau_j -}, t)\}I_{\{\tau_j \in (t, t+dt]\}} \\ &\qquad\qquad + \{C(V - nc, p_t, t) - C(V, p_t, t)\}I_{\{\tau_j \in (t, t+dt]\}}\Big\} = 0. \end{aligned} \tag{2.31}$$

From the financial point of view, it is reasonable to consider that as every coupon date approaches, the bondholders start to take into consideration the possibility of default due to the jump (decrease) in total asset. Therefore, the bond price converges to what the investors would expect of what it would be in the future, i.e. its expectation. So, let us assume the following:

Assumption 6: The defaultable corporate coupon bond price at time $t = \tau_{j-}$ is the expectation of the price at time $t = \tau_j$. □

To find the expectation, we need the following lemma:

Lemma 2 The endogenous survival probability at $t = \tau_j$, given the survival at time $t = \tau_{j-}$ seen from time $t = \tau_{j-1}$, denoted by $Q(\tau_{j_-}, \tau_j; \tau_{j-1})$ (or $Q(\tau_{j_-}, \tau_j)$, for short) is given by

$$
\begin{aligned}
Q(\tau_{j_-}, \tau_j) = \Bigg[& \Phi\Big(\frac{\tilde{b}}{\sqrt{\tau_j - \tau_{j-1}}} - \sqrt{\tau_j - \tau_{j-1}}(\frac{a_V}{s_V} - \frac{s_V}{2})\Big) \\
& - \exp\Big\{2\tilde{m}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\Big\}\Phi\Big(\frac{-\tilde{b}}{\sqrt{\tau_j - \tau_{j-1}}} - \frac{2\tilde{m}}{\sqrt{\tau_j - \tau_{j-1}}} - \sqrt{\tau_j - \tau_{j-1}}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\Big)\Bigg] \Bigg/ \\
& \exp\Big\{2\tilde{m}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\Big\}\Phi\Big(\frac{-\tilde{b}}{\sqrt{\tau_j - \tau_{j-1}}} - \frac{2\tilde{m}}{\sqrt{\tau_j - \tau_{j-1}}} - \sqrt{\tau_j - \tau_{j-1}}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\Big), \quad (2.32)
\end{aligned}
$$

where $\tilde{b} = \frac{1}{s_V} \ln \frac{V_b + nc}{V_{\tau_{j-1}}}$ and $\tilde{m} = \frac{1}{s_V} \ln \frac{V_b}{V_{\tau_{j-1}}}$.

Proof.

The endogenous survival probability at $t = \tau_j$, given the survival at time $t = \tau_{j-}$ seen from time $t = \tau_{j-1}$ is:

$$
\begin{aligned}
& Q(\tau_{j_-}, \tau_j) \\
& = \frac{\mathbb{P}\Big(\text{Survival at } \tau_j \text{ seen from} \tau_{j-1} \cap \text{Survival at } \tau_{j-} \text{ seen from} \tau_{j-1}\Big)}{\mathbb{P}\Big(\text{Survival at } \tau_{j-} \text{ seen from } \tau_{j-1}\Big)} \\
& = \frac{\mathbb{P}\Big(\min_{\tau_{j-1} \le s \le \tau_{j-}} V(s) > V_b, V(\tau_{j-}) > V_b + nc\Big)}{\mathbb{P}\Big(\min_{\tau_{j-1} \le s \le \tau_{j-}} V(s) > V_b\Big)}. \quad (2.33)
\end{aligned}
$$

Therefore we need to find the two probabilities in Equation (2.33). To find them, we will follow the method shown by Shreve.

Let $B(t)$ be a Brownian motion without drift and define

$$M(t) = \min_{0 \le s \le t} B(t).$$

Then by the reflection principle, we have:

$$\begin{aligned} \mathbb{P}\big(M(t) < m, B(t) > b\big) &= \mathbb{P}\big(B(t) < 2m - b\big) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{2m-b} \exp\Big\{ -\frac{x^2}{2t} \Big\}\, dx, \quad m < 0, m < b. \end{aligned}$$

Then the joint density $f(m,b)$ of this probability is given by:

$$\begin{aligned} f(m,b) &= \frac{\partial^2}{\partial m \partial b} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{2m-b} \exp\Big\{ -\frac{x^2}{2t} \Big\}\, dx \\ &= -\frac{\partial}{\partial m} \Big[\frac{1}{\sqrt{2\pi t}} \exp\Big\{ -\frac{(2m-b)^2}{2t} \Big\} \Big] \\ &= \frac{2(2m-b)}{t\sqrt{2\pi t}} \exp\Big\{ -\frac{(2m-b)^2}{2t} \Big\} \end{aligned}$$

so that

$$\begin{aligned} &\mathbb{P}\big(M(t) < m, B(t) > b\big) \\ &= \int_{x=b}^{\infty} \int_{y=-\infty}^{m} \frac{2(2y-x)}{t\sqrt{2\pi t}} \exp\Big\{ -\frac{(2y-x)^2}{2t} \Big\}\, dy\, dx, \quad m < 0, m < b. \end{aligned}$$

Now let

$$\tilde{B}(t) = \theta t + B(t) \quad \text{and} \quad \tilde{M}(t) = \min_{0 \le s \le t} \tilde{B}(s),$$

where $B(t), 0 \le s \le t$ is a Brownian motion without drift on $(\Omega, \mathscr{F}, \mathbb{P})$. Define

$$\begin{aligned} Z(s) &= \exp\Big\{ -\theta B(s) - \frac{1}{2}\theta^2 s \Big\} \\ &= \exp\Big\{ -\theta[\tilde{B}(s) - \theta s] - \frac{1}{2}\theta^2 s \Big\} \\ &= \exp\Big\{ -\theta \tilde{B}(s) + \frac{1}{2}\theta^2 s \Big\}, \\ \tilde{\mathbb{P}}(A) &= \int_A Z(s)\, d\mathbb{P} \end{aligned}$$

and set $\tilde{M}(t) = \min_{0 \le s \le t} \tilde{B}(t)$. Then, under $\tilde{\mathbb{P}}$, $\tilde{B}(t)$ is a Brownian motion without drift, so

$$\begin{aligned} &\tilde{\mathbb{P}}\big(\tilde{M}(t) < \tilde{m}, \tilde{B}(t) > \tilde{b}\big) \\ &\qquad = \int_{x=\tilde{b}}^{\infty} \int_{y=-\infty}^{\tilde{m}} \frac{2(2y-x)}{t\sqrt{2\pi t}} \exp\Big\{ -\frac{(2y-x)^2}{2t} \Big\}\, dy\, dx, \quad \tilde{m} < 0, \tilde{m} < \tilde{b}. \end{aligned}$$

Then under $\mathbb{P}$-measure, we have

$$\begin{aligned}&\mathbb{P}\big(\tilde{M}(t) < \tilde{m}, \tilde{B}(t) > \tilde{b}\big)\\ &= \int_{x=\tilde{b}}^{\infty}\int_{y=-\infty}^{\tilde{m}} \frac{2(2y-x)}{t\sqrt{2\pi t}} \exp\Big\{-\frac{(2y-x)^2}{2t}\Big\} \exp\{\theta x - \frac{1}{2}\theta^2 t\}\, dy\, dx, \quad \tilde{m} < 0, \tilde{m} < \tilde{b}.\end{aligned} \tag{2.34}$$

Now let us define $V(t)$ as in Assumption 1, that is,

$$dV(t) = a_V V(t) + s_V(t) dB(t) + ncI_{\{\tau_j \in [t, t+dt]\}}. \tag{2.35}$$

For $t \in (0, \tau_1)$, The solution for (2.35) is given by

$$\begin{aligned} V(t) &= V_0 \exp\Big\{s_V B(t) + (a_V - \frac{1}{2}s_V^2)t\Big\} \\ &= V_0 \exp\Big\{s_V \big[B(t) + \underbrace{(\frac{a_V}{s_V} - \frac{s_V}{2}}_{\theta})t\big]\Big\} \\ &= V_0 \exp\Big\{s_V \tilde{B}(t)\Big\} \end{aligned}$$

where

$$\theta = \frac{a_V}{s_V} - \frac{s_V}{2} \quad \text{and} \quad \tilde{B}(t) = \theta t + B(t).$$

So, by setting $\tilde{M}(t) = \min_{0 \le s \le t} \tilde{B}(t)$ as before, we have

$$\min_{0 \le s \le t} V(s) = V_0 \exp\{s_V \tilde{M}(t)\},$$

and

$$V(t) > V_b + nc \quad \Rightarrow \quad \tilde{B}(t) > \frac{1}{s_V} \ln \frac{V_b + nc}{V_0} \tag{2.36}$$

$$\min_{0 \le s \le t} V(s) < V_b \quad \Rightarrow \quad \tilde{M}(t) < \frac{1}{s_V} \ln \frac{V_b}{V_0} \tag{2.37}$$

and by letting $\frac{1}{s_V} \ln \frac{V_b + nc}{V_0} = \tilde{b}$ and $\frac{1}{s_V} \ln \frac{V_b}{V_0} = \tilde{m}$,

$$V_b < V_0 < V_b + nc \quad \Rightarrow \quad \tilde{m} < 0 < \tilde{b}. \tag{2.38}$$

So, by (2.36) through (2.38), we have

$$\mathbb{P}\big(\min_{0\le s\le t} V(s) < V_b, V(t) > V_b + nc\big) = \mathbb{P}\{\tilde{M}(t) < \tilde{m}, \tilde{B}(t) > \tilde{b}\}.$$

By integrating (2.34), we have

$$\begin{aligned}
&\mathbb{P}\{\tilde{M}(t) < \tilde{m}, \tilde{B}(t) > \tilde{b}\} \\
&= \int_{x=\tilde{b}}^{\infty}\int_{y=-\infty}^{\tilde{m}} \frac{2(2y-x)}{t\sqrt{2\pi t}} \exp\Big\{-\frac{(2y-x)^2}{2t}\Big\} \exp\{\theta\tilde{b} - \frac{1}{2}\theta^2 t\}\, dy\, dx \\
&= -\int_{x=\tilde{b}}^{\infty} \Big[\frac{1}{\sqrt{2\pi t}} \exp\Big\{-\frac{(2y-x)^2}{2t} + \theta x - \frac{1}{2}\theta^2 t\Big\}\Big]_{y=-\infty}^{\tilde{m}} dx \\
&= -\frac{1}{\sqrt{2\pi t}} \int_{x=\tilde{b}}^{\infty} \Big(\exp\Big\{-\frac{(2\tilde{m}-x)^2}{2t} + \theta x - \frac{1}{2}\theta^2 t\Big\}\Big)\, dx \\
&= \frac{1}{\sqrt{2\pi t}} e^{2\tilde{m}\theta} \int_{-\infty}^{x=-\tilde{b}} \exp\Big\{-\frac{\big(x-(2\tilde{m}+t\theta)\big)^2}{2t}\Big\}\, dx \\
&\quad \Big(\text{by letting} \quad z = \frac{x-(2\tilde{m}+t\theta)}{\sqrt{t}}\Big) \\
&= \frac{1}{\sqrt{2\pi}} e^{2\tilde{m}\theta} \int_{z=-\infty}^{\frac{-\tilde{b}-(2\tilde{m}+t\theta)}{\sqrt{t}}} e^{-\frac{z^2}{2}}\, dz \\
&= e^{2\tilde{m}\theta}\Phi\Big(\frac{-\tilde{b}-2\tilde{m}-t\theta}{\sqrt{t}}\Big) \\
&\quad \Big(\text{Substituting} \quad \theta = \frac{a_V}{s_V} - \frac{s_V}{2} \quad \text{back}\Big) \\
&= \exp\Big\{2\tilde{m}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\Big\}\Phi\Big(\frac{-\tilde{b}}{\sqrt{t}} - \frac{2\tilde{m}}{\sqrt{t}} - \sqrt{t}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\Big)
\end{aligned}$$

where $\tilde{b} = \frac{1}{s_V}\ln\frac{V_b+nc}{V_0}$ and $\tilde{m} = \frac{1}{s_V}\ln\frac{V_b}{V_0}$.

Since $V(t)$ is lognormal on the interval without any jump, the probability $V(t) > V_b + nc$, seen from time $t = 0$ is:

$$\mathbb{P}\Big(V(t) > V_b + nc\Big) = \Phi\left(\frac{\ln\frac{V_0}{V_b+nc} + \big(a_V - \frac{s_V^2}{2}\big)t}{s_V\sqrt{t}}\right) = \Phi\left(-\frac{\tilde{b}}{\sqrt{t}} - \sqrt{t}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\right). \quad (2.39)$$

Therefore,

$$
\begin{aligned}
&\mathbb{P}\big(\min_{0\le s\le \tau_{1-}} V(s) > V_b, V(\tau_{1-}) > V_b + nc\big)\\
&= \mathbb{P}\big(V(\tau_{1-}) > V_b + nc\big) - \mathbb{P}\big(\min_{0\le s\le \tau_{1-}} V(s) < V_b, V(\tau_{1-}) > V_b + nc\big)\\
&= \Phi\bigg(- \frac{\tilde{b}}{\sqrt{\tau_1}} - \sqrt{\tau_1}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg)\\
&\qquad - \exp\bigg\{2\tilde{m}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg\}\Phi\bigg(\frac{-\tilde{b}}{\sqrt{\tau_1}} - \frac{2\tilde{m}}{\sqrt{\tau_1}} - \sqrt{\tau_1}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg)
\end{aligned}
$$

where $\tilde{b} = \frac{1}{s_V}\ln\frac{V_b+nc}{V_0}$ and $\tilde{m} = \frac{1}{s_V}\ln\frac{V_b}{V_0}$.

Now, taking $\tilde{m} = \tilde{b}$ in (2.34) and replacing $V_b + nc$ by V_b in (2.39), we have

$$
\begin{aligned}
&\mathbb{P}\{\min_{0\le s\le \tau_{1-}} V(s) > V_b, V(\tau_{1-}) > V_b\}\\
&= \mathbb{P}\{V(\tau_{1-}) > V_b\} - \mathbb{P}\{\min_{0\le s\le \tau_{1-}} V(s) < V_b, V(\tau_{1-}) > V_b\}\\
&= \Phi\bigg(- \frac{\tilde{m}}{\sqrt{\tau_1}} - \sqrt{t}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg) - \exp\bigg\{2\tilde{m}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg\}\Phi\bigg(- \frac{3\tilde{m}}{\sqrt{t}} - \sqrt{t}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg)
\end{aligned}
$$

where $\tilde{m} = \frac{1}{s_V}\ln\frac{V_b}{V_0}$.

Therefore,

$$
\begin{aligned}
Q(\tau_{1-}, \tau_1) = &\bigg[\Phi\bigg(\frac{\tilde{b}}{\sqrt{\tau_1}} - \sqrt{\tau_1 1}\big(\frac{a_V}{s_V} - \frac{s_V}{2}\big)\bigg)\\
&\quad - \exp\bigg\{2\tilde{m}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg\}\Phi\bigg(\frac{-\tilde{b}}{\sqrt{\tau_1}} - \frac{2\tilde{m}}{\sqrt{\tau_1}} - \sqrt{\tau_1}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg)\bigg] \bigg/\\
&\qquad \exp\bigg\{2\tilde{m}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg\}\Phi\bigg(\frac{-\tilde{b}}{\sqrt{\tau_1}} - \frac{2\tilde{m}}{\sqrt{\tau_1}} - \sqrt{\tau_1}\Big(\frac{a_V}{s_V} - \frac{s_V}{2}\Big)\bigg),
\end{aligned}
$$

where $\tilde{b} = \frac{1}{s_V}\ln\frac{V_b+nc}{V_0}$ and $\tilde{m} = \frac{1}{s_V}\ln\frac{V_b}{V_0}$.

And for $j = 2\ldots n$, we have (2.32). □

If we let the endogenous survival probability at time $t = \tau_j$, given the survival at time $t = \tau_{j-}$, is $Q(\tau_{j-}, \tau_j)$, and the exogenous survival probability at time $t = \tau_j$, given the survival at time $t = \tau_{j-}$ is given by $P(\tau_{j-}, \tau_j)$, then Assumption 6 is expressed as

$$
\begin{aligned}
C(\tau_{j-}) &= E[C(\tau_j)]\\
&= C(\tau_j)\cdot Q(\tau_{j-}, \tau_j))P(\tau_{j-}, \tau_j) + R\Big[1 - Q(\tau_{j-}, \tau_j)P(\tau_{j-}, \tau_j)\Big] \qquad (2.40)
\end{aligned}
$$

and especially

$$\begin{aligned}C(T-) &= E[C(T)]\\ &= 1\cdot Q(T-,T)P(T-,T)+R\Big[1-Q(T-,T)P(T-,T)\Big].\end{aligned}$$

For convenience, hereinafter we refer the survival probability (both endogenous and exogenous combined) at time $t=\tau_j$ given the survival at $t=\tau_{j-}$ as $S(\tau_{j-},\tau_j)$, that is,

$$S(\tau_{j-},\tau_j)=Q(\tau_{j-},\tau_j)P(\tau_{j-},\tau_j) \tag{2.41}$$

As in the previous section, if we consider time intervals $[\tau_{j-1},\tau_j)$, (2.31) reduces to

$$\begin{cases}\dfrac{\partial C}{\partial t}+\dfrac{1}{2}\big[s_p^2\dfrac{\partial^2 C}{\partial p^2}+s_V^2V^2\dfrac{\partial^2 C}{\partial V^2}\big]+a_p\dfrac{\partial C}{\partial p}+rV\dfrac{\partial C}{\partial V}+(r+p)C & \\ \quad +e^{-r(T-t)}Rp=0 & (\tau_{j-1}\le t<\tau_j, V>V_b, p>0)\\ C(V_b(t),p,t)=e^{-r(T-t)} & (\tau_{j-1}\le t<\tau_j, p>0)\\ C(V,p,\tau_{j-})=C(V,p,\tau_j)\cdot S(\tau_{j-},\tau_j)+R\Big[1-S(\tau_{j-},\tau_j)\Big] & (V>V_b,p>0)\\ \lim\limits_{V\to\infty}C(V,p,t)=e^{-r(T-t)} & (\tau_{j-1}\le t<\tau_j, p>0)\\ \lim\limits_{p\to\infty}C(V,p,t)=R & (\tau_{j-1}\le t<\tau_j, V>V_b).\end{cases}$$

And especially for t in $[\tau_{n-1},T)$, we have

$$\begin{cases}\dfrac{\partial C}{\partial t}+\dfrac{1}{2}\big[s_p^2\dfrac{\partial^2 C}{\partial p^2}+s_V^2V^2\dfrac{\partial^2 C}{\partial V^2}\big]+a_p\dfrac{\partial C}{\partial p}+rV\dfrac{\partial C}{\partial V}+(r+p)C & \\ \quad +e^{-r(T-t)}Rp=0 & (\tau_{n-1}\le t<T, p>0, V>V_b, p>0)\\ C(V_b(t),p,t)=e^{-r(T-t)} & (\tau_{n-1}\le t<T, p>0)\\ C(V,p,T)=1 & (V>V_b,p>0)\\ C(V,p,T-)=1\cdot S(T-,T)+R\Big[1-S(T-,T)\Big] & (V>V_b,p>0)\\ \lim\limits_{V\to\infty}C(V,p,t)=e^{-r(T-t)} & (\tau_{n-1}\le t<T, p>0)\\ \lim\limits_{p\to\infty}C(V,p,t)=R & (\tau_{n-1}\le t<T, V>V_b).\end{cases} \tag{2.42}$$

As before, we will try to solve (2.42) for $t\in[\tau_{n-1},T)$, then solve for other t backwards. Using the change of unknown function

$$C = ue^{-r(T-t)},$$

(2.42) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\big[s_p^2 \dfrac{\partial^2 u}{\partial p^2} + s_V^2 V^2 \dfrac{\partial^2 u}{\partial V^2}\big] + a_p \dfrac{\partial u}{\partial p} + rV\dfrac{\partial u}{\partial V} & \\ \qquad -p(u-R) = 0 & (\tau_{n-1} \le t < T, p > 0, V > V_b, p > 0) \\ u(V_b(t), p, t) = R & (\tau_{n-1} \le t < T, p > 0) \\ u(V, p, T) = 1 & (V > V_b, p > 0) \\ u(V, p, T-) = 1 \cdot S(T-, T) + R\Big[1 - S(T-, T)\Big] & \\ \qquad = R + S(T-, T)\big[1 - R\big] & (V > V_b, p > 0) \\ \lim\limits_{V \to \infty} u(V, p, t) = Ce^{r(T-t)} & (\tau_{n-1} \le t < T, p > 0) \\ \lim\limits_{p \to \infty} u(V, p, t) = Re^{r(T-t)} & (\tau_{n-1} \le t < T, V > V_b). \end{cases}$$

Using the change of unknown function again,

$$\hat{u} = u - R,$$

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} + \dfrac{1}{2}\big[s_p^2 \dfrac{\partial^2 \hat{u}}{\partial p^2} + s_V^2 V^2 \dfrac{\partial^2 \hat{u}}{\partial V^2}\big] & \\ \qquad + a_p \dfrac{\partial u}{\partial p} + rV\dfrac{\hat{\partial} u}{\partial V} - p\hat{u} = 0 & (\tau_{n-1} \le t < T, p > 0, V > V_b, p > 0) \\ \hat{u}(V_b(t), p, t) = 0 & (\tau_{n-1} \le t < T, p > 0) \\ \hat{u}(V, p, T) = 1 - R & (V > V_b, p > 0) \\ \hat{u}(V, p, T-) = (1-R)S(T-, T) & (V > V_b, p > 0) \\ \lim\limits_{V \to \infty} \hat{u}(V, p, t) = 0 & (\tau_{n-1} \le t < T, p > 0) \\ \lim\limits_{p \to \infty} \hat{u}(V, p, t) = R\big(e^{r(T-t)-1}\big) & (\tau_{n-1} \le t < T, V > V_b). \end{cases} \tag{2.43}$$

Using the change of unknown function again and letting

$$\hat{u} = W(V, p, t)S(1-R), \tag{2.44}$$

$C(V,p,t)$ can be expressed in terms $W(V,p,t)$ as

$$\begin{aligned} C(V,p,t) &= ue^{-r(T-t)} = (\hat{u}+R)\cdot e^{-r(T-t)} \\ &= [WS(1-R)+R]\cdot e^{-r(T-t)} \\ &= [WS-WSR+R]\cdot e^{-r(T-t)} \\ &= WS\cdot e^{-r(T-t)} + (1-WS)R\cdot e^{-r(T-t)}. \end{aligned}$$

We can interpret that the price at time t is an expectation of the value of the bond at time t. So we can regard WS as survival probability , and $(1-WS)$ as default probability at time t. Using (2.44), (2.43) we have the following pricing model:

Theorem 3. Under Assumptions 1 though 6, the price of the defaultable zero-coupon bond with expected and unexpected default at time $t \in [\tau_{n-1}, T)$ is given by the following:

$$C(V,p,t) = WS\cdot e^{-r(T-t)} + (1-WS)R\cdot e^{-r(T-t)}$$

where S is given by (2.41) and $W(V,p,t)$ satisfies the following PDE and determining conditions:

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{1}{2}\big[s_p^2\frac{\partial^2 W}{\partial p^2} + s_V^2 V^2 \frac{\partial^2 W}{\partial V^2}\big] & \\ \qquad\qquad + a_p \frac{W}{\partial p} + rV\frac{\partial \hat{W}}{\partial V} - pW = 0 & (\tau_{n-1} \le t < T, p>0, V>V_b, p>0) \\ W(V_b(t),p,t) = 0 & (\tau_{n-1} \le t < T, p>0) \\ W(V,p,T-) = 1 & (V>V_b, p>0) \\ \lim\limits_{V\to\infty} W(V,p,t) = 0 & (\tau_{n-1} \le t < T, p>0) \\ \lim\limits_{p\to\infty} W(V,p,t) = \frac{R\left(e^{r(T-t)-1}\right)}{S(1-R)} & (\tau_{n-1} \le t < T, V>V_b). \end{cases} \tag{2.45}$$

2.2.3 Particular Solution

In addition to Assumptions 1 though 6, we try to find a particular solution $W(V,p,t)$ in the separative form

$$W(V,p,t) = f(V,t)\cdot g(p,t).$$

Substituting this in (2.45), the PDE in (2.45) becomes

$$\left[\frac{\partial f}{\partial t}+\frac{1}{2}s_v^2V^2\frac{\partial^2 f}{\partial V^2}+rV\frac{\partial f}{\partial V}\right]g+\left[\frac{\partial g}{\partial t}+\frac{1}{2}s_v^2\frac{\partial^2 g}{\partial p^2}+a_p\frac{\partial g}{\partial p}-pg\right]f=0$$

To get the solution for (2.45), we further assume the following:

$$\begin{aligned}\frac{\partial f}{\partial t}+\frac{1}{2}s_v^2V^2\frac{\partial^2 f}{\partial V^2}+rV\frac{\partial f}{\partial V}=0,\\ \frac{\partial g}{\partial t}+\frac{1}{2}s_v^2\frac{\partial^2 g}{\partial p^2}+a_p\frac{\partial g}{\partial p}-pg=0.\end{aligned}$$

Then, we need to solve the following two problems.

$$\begin{cases}\dfrac{\partial f}{\partial t}+\frac{1}{2}s_v^2V^2\dfrac{\partial^2 f}{\partial V^2}+rV\dfrac{\partial f}{\partial V}=0 & (\tau_{n-1}\le t<T, V>V_b)\\ f(V_b(t),t)=0 & (\tau_{n-1}\le t<T)\\ f(V,T-)=1 & (V>V_b)\end{cases} \tag{2.46}$$

and

$$\begin{cases}\dfrac{\partial g}{\partial t}+\frac{1}{2}s_p^2\dfrac{\partial^2 g}{\partial p^2}+a_p\dfrac{\partial g}{\partial p}-pg=0 & (\tau_{n-1}\le t<T, p>0)\\ g(p,T-)=1 & (p>0).\end{cases} \tag{2.47}$$

First, we will solve (2.46). To do so, let

$$\begin{aligned}\frac{V}{V_b}=e^y \quad\Rightarrow\quad y=\ln\frac{V}{V_b}, \quad \text{and}\\ \zeta \quad = \quad \frac{((T_-)-t)s_V^2}{2}\end{aligned}$$

and

$$f(V,t)=V_b v(y,t).$$

Then, (2.46) becomes

$$\begin{cases}\dfrac{\partial v}{\partial t}-\dfrac{\partial^2 v}{\partial y^2}+\Bigg(\underbrace{1-\dfrac{2}{s_v^2}r}_{k}\Bigg)\dfrac{\partial v}{\partial y}=0 & (0\le\zeta<\dfrac{(T_-)s_V^2}{2}, 0<y<\infty)\\ v(0,\zeta)=0 & (0\le\zeta<\dfrac{(T_-)s_V^2}{2})\\ v(y,0)=\dfrac{1}{V_b} & (0<y<\infty).\end{cases} \tag{2.48}$$

Assuming that the solution $v(y,\zeta)$ is in the following type

$$v = \omega \cdot e^{\alpha y + \beta \zeta},$$

where α and β are to be chosen later, and letting $k = 1 - \frac{2}{s_v^2} r$, the first equation in (2.48) becomes

$$\omega_\zeta = \omega_{yy} + (\alpha^2 - k\alpha - \beta)\omega + (2\alpha - k)\omega_y$$

where the subscripts refer to the partial derivatives with respect to the subscript. We chose α and β so that the coefficients of ω and ω_y are equal to 0, that is

$$\begin{aligned}
\alpha &= \frac{k}{2} = \frac{1}{2} - \frac{r}{s_V^2} \\
\beta &= \alpha^2 - k\alpha = -\frac{k^2}{4}.
\end{aligned}$$

Substituting these in (2.48), it becomes a heat equation with the initial condition:

$$\begin{cases}
\omega_\zeta = \omega_{yy} & (0 \le \zeta < \frac{(T_-)s_V^2}{2}, 0 < y < \infty) \\
\omega(y,0) = \frac{1}{V_b} e^{-\frac{k}{2}y} & (0 < y < \infty).
\end{cases}$$

Define

$$\bar{h}(y) = \begin{cases}
h(y) = \omega(y,0) = \frac{1}{V_b} e^{-\frac{k}{2}y}, & (y > 0) \\
0, & \text{otherwise.}
\end{cases}$$

Let $\omega_1(y,\zeta)$ be a solution of the initial value problem

$$\begin{cases}
\omega_\zeta = \omega_{yy} & (0 \le \zeta, -\infty < y < \infty) \\
\omega(y,0) = \hat{h}(y) & (-\infty < y < \infty).
\end{cases}$$

and $\omega_2(y,\zeta) = \omega_1(-y,\zeta)$ (and so $\omega_2(0,\zeta) = \omega_1(0,\zeta)$). Then, by the solution formula and the image solution method

$$\omega_1(y,\zeta) = \frac{1}{2\sqrt{\pi\zeta}} \int_{-\infty}^{\infty} \hat{h}(\xi) \exp\left(- \frac{(y-\xi)^2}{4\zeta} \right) d\xi = \frac{1}{2\sqrt{\pi\zeta}} \int_0^{\infty} h(\xi) \exp\left(- \frac{(y-\xi)^2}{4\zeta} \right) d\xi,$$

$$\omega_2(y,\zeta) = \frac{1}{2\sqrt{\pi\zeta}} \int_0^{\infty} h(\xi) \exp\left(- \frac{(y+\xi)^2}{4\zeta} \right) d\xi,$$

and

$$\omega(y,\zeta) = \omega_1(y,\zeta) - \omega_2(y,\zeta).$$

Computing $\omega_1(y,\zeta)$, we have

$$\begin{aligned}
\omega_1(y,\zeta) &= \frac{1}{2\sqrt{\pi\zeta}}\int_0^\infty h(\xi)\exp\left(-\frac{(y-\xi)^2}{4\zeta}\right)d\xi \\
&\quad \Big(\text{letting} \quad z = \frac{\xi - y}{\sqrt{2\zeta}}\Big) \\
&= \frac{1}{V_b\sqrt{2\pi}}\int_{-\frac{y}{\sqrt{2\zeta}}}^\infty e^{-\frac{k}{2}(z\sqrt{2\zeta}+y)}\cdot e^{-\frac{1}{2}z^2}\,dz = \frac{1}{V_b\sqrt{2\pi}}e^{-\frac{k}{2}y+\frac{\zeta}{4}k^2}\int_{-\frac{y}{\sqrt{2\zeta}}}^\infty e^{-\frac{1}{2}(z+\frac{\sqrt{2\zeta}k}{2})^2}\,dz \\
&\quad \Big(\text{letting} \quad \eta = z + \frac{\sqrt{2\zeta}k}{2}\Big) \\
&= \frac{1}{V_b\sqrt{2\pi}}e^{-\frac{k}{2}y+\frac{\zeta}{4}k^2}\int_{-\frac{y}{\sqrt{2\zeta}}+\frac{\sqrt{2\zeta}k}{2}}^\infty e^{-\frac{\eta^2}{2}}\,d\eta \\
&= \frac{1}{V_b\sqrt{2\pi}}e^{-\frac{k}{2}y+\frac{\zeta}{4}k^2}\Phi(d_1)
\end{aligned}$$

where

$$d_1 = \frac{y}{\sqrt{2\zeta}} - \frac{\sqrt{2\zeta}k}{2} = \frac{\ln\frac{V}{V_b}}{\sqrt{(T_-)-t)s_V}} + \sqrt{(T_-)-t}\Big(\frac{s_V}{2} - \frac{r}{s_V}\Big). \tag{2.49}$$

Similarly,

$$\begin{aligned}
\omega_2(y,\zeta) &= \frac{1}{2\sqrt{\pi\zeta}}\int_0^\infty h(\xi)\exp\left(-\frac{(y+\xi)^2}{4\zeta}\right)d\xi \\
&= \frac{1}{V_b\sqrt{2\pi}}e^{\frac{k}{2}y+\frac{\zeta}{4}k^2}\Phi(d_2)
\end{aligned}$$

where

$$d_2 = -\frac{y}{\sqrt{2\zeta}} + \frac{\sqrt{2\zeta}k}{2} = \frac{\ln\frac{V_b}{V}}{\sqrt{(T_-)-t)s_V}} + \sqrt{(T_-)-t}\Big(\frac{s_V}{2} - \frac{r}{s_V}\Big). \tag{2.50}$$

Therefore,

$$\begin{aligned}
f(V,t) &= V_b\cdot v(y,t) \\
&= V_b\cdot e^{\alpha y+\beta\zeta}\big(\omega_1(y,\zeta) - \omega_2(y,\zeta)\big) \\
&= \Phi(d_1) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(d_2)
\end{aligned}$$

where d_1 and d_2 are as defined above.

Notice that the IVP of $g(p,t)$ is the same as (2.11).

Hence,

$$W(V,p,t) = e^{A(t,T-)-B(t,T-)p}\left[\Phi(d_1) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(d_2)\right]$$

where d_1 and d_2 are defined by (2.49) and (2.50), and $A(t,T_-)$ and $B(t,T_-)$ are defined by (2.15), and (2.16). So $C(V,p,t)$ is given by

$$\begin{aligned}
C(V,p,t) &= W(V,p,t)S(T-,T)e^{-r(T-t)} + \big(1 - W(V,p,t)S(T-,T))\big)R\cdot e^{-r(T-t)} \\
&= e^{A(t,T-)-B(t,T-)p}\left[\Phi(d_1) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(d_2)\right]S(T-,T)\cdot e^{-r(T-t)} \\
&\quad + \left\{1 - e^{A(t,T-)-B(t,T-)p}\left[\Phi(d_1) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(d_2)\right]S(T-,T)\right\}R\cdot e^{-r(T-t)}
\end{aligned}$$

and when $t = \tau_{n-1}$, we have

$$\begin{aligned}
C(V,p,\tau_{n-1}) &= e^{A(\tau_{n-1},T-)-B(\tau_{n-1},T-)p}\left[\Phi(\tilde{d}_{1,n}) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right]S(T-,T)\cdot e^{-r(T-\tau_{n-1})} \\
&+ \left\{1 - e^{A(\tau_{n-1},T-)-B(\tau_{n-1},T_-)p}\left[\Phi(\tilde{d}_{1,n}) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right]S(T-,T)\right\}R\cdot e^{-r(T-\tau_{n-1})}
\end{aligned}$$

where, for $j = 1\cdots n$,

$$d_{1,j} = \frac{\ln\frac{V_{\tau_{j-}}}{V_b}}{\sqrt{T/n}s_V} + \sqrt{T/n}\Big(\frac{s_V}{2} - \frac{r}{s_V}\Big) \tag{2.51}$$

and

$$d_{2,j} = \frac{\ln\frac{V_b}{V_{\tau_{j-}}}}{\sqrt{T/n}s_V} + \sqrt{T/n}\Big(\frac{s_V}{2} - \frac{r}{s_V}\Big). \tag{2.52}$$

We will solve for $t < \tau_{n-1}$, by solving the PDEs backwards on each interval $[\tau_{j-1},\tau_j), j = 1\cdots n-1$. On $[\tau_{n-2},\tau_{n-1})$, using (2.40) for the terminal condition, the IVP becomes

$$
\begin{cases}
\dfrac{\partial C}{\partial t} + \dfrac{1}{2}\big[s_p^2\dfrac{\partial^2 C}{\partial p^2} + s_V^2 V^2\dfrac{\partial^2 C}{\partial V^2}\big] + a_p\dfrac{\partial C}{\partial p} + rV\dfrac{\partial C}{\partial V} + (r+p)C + e^{-r(T-t)}Rp = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad (\tau_{n-2} \le t < \tau_{n-1}, p > 0, V > V_b, p > 0) \\
C(V, p, \tau_{(n-1)-}) \\
\quad = e^{r(T-\tau_{n-1})}\Bigg[e^{A(\tau_{n-1},T-)-B(\tau_{n-1},T-)p}\left(\Phi(\tilde{d}_{1,n}) - \left(\dfrac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right)S(T-,T)S(\tau_{(n-1)-},\tau_{n-1}) \\
\qquad + R\Bigg\{1 - e^{A(\tau_{n-1},T-)-B(\tau_{n-1},T-)p}\left(\Phi(\tilde{d}_{1,n}) - \left(\dfrac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right)S(T-,T)S(\tau_{(n-1)-},\tau_{n-1})\Bigg\}\Bigg] \\
\qquad\qquad\qquad\qquad\qquad\qquad (V > V_b, p > 0).
\end{cases}
\tag{2.53}
$$

As before, letting $C = ue^{-r(T-t)}$, (2.53) becomes

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \dfrac{1}{2}\big[s_p^2\dfrac{\partial^2 u}{\partial p^2} + s_V^2 V^2\dfrac{\partial^2 u}{\partial V^2}\big] + a_p\dfrac{\partial u}{\partial p} + rV\dfrac{\partial u}{\partial V} - p(u-R) = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad (\tau_{n-2} \le t < \tau_{n-1}, p > 0, V > V_b, p > 0) \\
u(V, p, \tau_{(n-1)-}) \\
\quad = e^{A(\tau_{n-1},T_-)-B(\tau_{n-1},T_-)p}\left(\Phi(\tilde{d}_{1,n}) - \left(\dfrac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right)S(T_-,T)S(\tau_{(n-1)_-},\tau_{n-1})(1-R) + R \\
\qquad\qquad\qquad\qquad\qquad\qquad (V > V_b, p > 0).
\end{cases}
$$

Using the change of unknown function again,

$$
\hat{u} = u - R,
$$

$$
\begin{cases}
\dfrac{\partial \hat{u}}{\partial t} + \dfrac{1}{2}\big[s_p^2\dfrac{\partial^2 \hat{u}}{\partial p^2} + s_V^2 V^2\dfrac{\partial^2 \hat{u}}{\partial V^2}\big] + a_p\dfrac{\partial u}{\partial p} + rV\dfrac{\hat{\partial u}}{\partial V} - p\hat{u} = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad (\tau_{n-2} \le t < \tau_{n-1}, p > 0, V > V_b, p > 0) \\
\hat{u}(V, p, \tau_{(n-1)_-}) \\
\quad = e^{A(\tau_{n-1},T_-)-B(\tau_{n-1},T_-)p}\left(\Phi(\tilde{d}_{1,n}) - \big(\frac{V}{V_b}\big)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right)S(T_-,T)S(\tau_{(n-1)_-},\tau_{n-1})(1-R) \\
\qquad\qquad\qquad\qquad\qquad\qquad (V > V_b, p > 0).
\end{cases}
\tag{2.54}
$$

Using the change of unknown function again and letting

$$
\begin{aligned}
\hat{u} = W(V,p,t)e^{A(\tau_{n-1},T-)-B(\tau_{n-1},T-)p}&\left(\Phi(\tilde{d}_{1,n}) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right)\cdot \\
&S(T-,T)S(\tau_{(n-1)-},\tau_{n-1})(1-R)
\end{aligned}
$$

Then (2.53) reduces to the same IVP as (2.44), so the solution is given by

$$W(V,p,t) = e^{A(t,\tau_{(n-1)-})-B(t,\tau_{(n-1)-})p}\left[\Phi(d_1) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(d_2)\right] \tag{2.55}$$

where

$$d_1 = \frac{\ln\frac{V}{V_b}}{\sqrt{(\tau_{(n-1)-})-t)}s_V} + \sqrt{(\tau_{(n-1)-})-t}\Big(\frac{s_V}{2} - \frac{r}{s_V}\Big)$$

and

$$d_2 = \frac{\ln\frac{V_b}{V}}{\sqrt{(\tau_{(n-1)-})-t)}s_V} + \sqrt{(\tau_{(n-1)-})-t}\Big(\frac{s_V}{2} - \frac{r}{s_V}\Big).$$

For $t \in (\tau_{n-2}, \tau_{n-1})$,

$$\begin{aligned}
&C(V,p,t)\\
&= W(V,p,t)e^{A(\tau_{n-1},T_-)-B(\tau_{n-1},T_-)p}\left(\Phi(\tilde{d}_{1,n}) - \Big(\frac{V}{V_b}\Big)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right)\\
&\qquad \cdot S(T_-,T)S(\tau_{(n-1)_-},\tau_{n-1})e^{-r(T-t)}\\
&\quad + \Bigg[W(V,p,t)e^{A(\tau_{n-1},T_-)-B(\tau_{n-1},T_-)p}\left(\Phi(\tilde{d}_{1,n}) - \Big(\frac{V}{V_b}\Big)^{1-\frac{2r}{s_V^2}}\Phi(\tilde{d}_{2,n})\right)\\
&\qquad \cdot S(T_-,T)S(\tau_{(n-1)_-},\tau_{n-1})e^{-r(T-t)}\Bigg]Re^{-r(T-t)},
\end{aligned}$$

where $W(V,p,t)$, $\tilde{d}_{1,n}$, $\tilde{d}_{2,n}$ are given by (2.55),(2.51) and (2.52) respectively.

Repeating this backwards, for any $t \in (\tau_{j-1}, \tau_j), j = 1, \cdots, n$ we obtain the following:

Theorem 4. Under the assumptions 1 through 6, and additional assumptions made in this subsection, the price of corporate coupon-bond with unexpected and expected default is given by

$$G(V,p,t) = C(V,p,t) + \sum_{\tau_i \geq t} c_i C(V,p,t;\tau_i)$$

where

$$
\begin{aligned}
&C(V,p,t)\\
&= e^{A(t,\tau_{j-})-B(t,\tau_{j-})p+\sum_{i=j}^{n-1} A(\tau_i,\tau_{(i+1)-})-B(\tau_i,\tau_{(i+1)-})p} \cdot \left[\Phi(d_1) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}} \Phi(d_2)\right] S(\tau_{j-},\tau_j)\\
&\quad \prod_{i=j+1}^{n} \left[\left(\Phi(\tilde{d}_{1,n}) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}} \Phi(\tilde{d}_{2,n})\right) S(\tau_{(i-1)-},\tau_i)\right] e^{-r(T-t)}\\
&\quad \Bigg(1 - e^{A(t,\tau_{j-})-B(t,\tau_{j-})p+\sum_{i=j}^{n-1} A(\tau_i,\tau_{(i+1)-})-B(\tau_i,\tau_{(i+1)-})p}\\
&\qquad \cdot \left[\Phi(d_1) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}} \Phi(d_2)\right] S(\tau_{j-},\tau_j)\\
&\qquad \prod_{i=j+1}^{n} \left[\left(\Phi(\tilde{d}_{1,n}) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}} \Phi(\tilde{d}_{2,n})\right) S(\tau_{(i-1)-},\tau_i)\right]\Bigg) e^{-r(T-t)}. \qquad (2.56)
\end{aligned}
$$

Chapter 3

Credit Derivatives Pricing with Constant Interest Rate

3.1 Credit Default Swaption with Constant Interest Rate - Unexpected Default

In the existing literature (see Schonbucher (2003a), and Hull et al. (2003)), the value of the credit default swaption was given assuming that the price of underlying asset, i.e., the credit default swap follows the geometric Brownian motion. However, when the information regarding the forward credit default swap is sparse, it is not convenient to form a pricing model based on the price of forward credit default swap. In Hull's paper, the pricing formula for forward credit default swap is given; however, the formula does not suggest that the forward credit default swap follows the geometric Brownian motion. This discrepancy was observed since the arbitrage principle was used to price the forward credit default swap, and then a PDE approach was taken to evaluate the price of credit default swaption. This approach will also be problematic when the reference entity has no existing credit default swap, which, if there exists, can be used to infer the drift and volatility of forward credit default swap.

In this paper, we will take a different approach. First, we apply the PDE approach to evaluate the forward credit default swap, and then use the arbitrage principle to find the price of credit default swaption. This approach makes sense since the underlying asset of the credit default swaption does not exist until the date the option expires. Also, by taking this approach, the expected fee for the forward credit default swap takes the jumps in default intensity into consideration.

3.1.1 Formulation

The following assumptions are valid for Chapter 3 only.

Assumption 1: Let $t = 0$ and $t = T_0$ be the time when the credit default swaption (hereinafter, the "swaption") starts and expires respectively. $t = T_0$ is also when the forward credit default swap takes effect upon the exercise of the swaption. Let $t = T_N$ be the expiration of the credit default swap and, for simplicity, let us assume that $T_j, j = 1, \cdots, N$ falls on the interest payment date of the coupon bonds issued by the reference entity of the credit default swap and the credit default swap terminates on the day the coupon bonds are to be redeemed, that is $\tau_n = T_N$ where τ_n is as defined in subsection 2.1.1. Let z^* be the exercise price of the swaption. This is actually the fee the swaption holder pays on the notional amount of the credit default swap once the swaption is exercised. Let $z(t), t \geq T_0$ be the annual fee of forward credit default swap. Define

$$\mathcal{Z}(t) := z(t) \cdot E[\sum_{i:T_i>t}^{N} b(T_i)\Gamma_i I_{\{T_i \leq \tau\}} + b(\tau)\Gamma^* I_{\{T_0 \leq \tau \leq T_N\}}]$$

where $b(T_i) = \exp\{-\int_t^{T_i} r\ ds\}$, τ is the time of default, and Γ_i and Γ^* are the lengths of time interval since the last fee payment till T_i and the default date, respectively. Therefore, $\mathcal{Z}(t)$ is the present value at time t of the forward credit default swap. Observe that by the arbitrage principle, the present value of the total fee leg is the same as the present value of the protection leg. That is:

$$\mathcal{Z}(t) = E[b(\tau)(1 - R) \cdot I_{\{\tau \leq T_N\}}]. \tag{3.1}$$

Assumption 2: Default is an exogenous event. Unexpected default probability on any interval $[t, t + dt]$ is given by,

$$dp = a_p(p,t)dt + s_p(p,t)dW_1 + p_{\tau_j -}U_j I_{\{\tau_j \in (t,t+dt]\}}, \tag{3.2}$$

where $I_{\{\tau_j \in (t,t+dt]\}}$ is an indicator function taking 1 when $\tau_j \in (t, t + dt]$ and 0 otherwise, W_1 is a standard Brownian motion, and U_j is defined the same as in Assumption 1 in Chapter 1 .

Assumption 3: Default recovery is given in the form of face value exogenous recovery ($R \cdot e^{-r(T-t)}$ where R is constant with $0 \leq R \leq 1$ and T is the maturity of the bond) or in the form of market price exogenous recovery ($R\times$ bond price at default time).

Assumption 4: The price of defaultable corporate zero-coupon bond price is given by the function $\hat{C} = \hat{C}(p,t)$, whose solution is given by (2.18), and the price of the swaption is given by $\hat{X} = \hat{X}(p,t)$.

Problem: Under this setting and above assumptions, we shall find the price of the credit default swaption $\hat{X}(p,t)$.

3.1.2 Derivation of the model

We construct a portfolio by hedging $\hat{X}(p,t)$ with the reference entity's zero-coupon bonds with exogenous default. So the value of the portfolio is:

$$\Pi = \hat{X} - \Lambda\hat{C}.$$

The change of value of this portfolio over a small time increment $[t, t+dt]$ is given by

$$d\Pi = d\hat{X} - \Lambda d\hat{C}.$$

If there is no default over $[t, t+dt]$ (with probability $1 - p_t dt$), then we have

$$\begin{aligned} d\Pi = &\frac{\partial \hat{X}}{\partial t}dt + \frac{\partial \hat{X}}{\partial p}dp + \frac{1}{2}s_p^2\frac{\partial^2 \hat{X}}{\partial p^2}dt \\ &-\Lambda\left[\frac{\partial \hat{C}}{\partial t}dt + \frac{\partial \hat{C}}{\partial p}dp + \frac{1}{2}s_p^2\frac{\partial^2 \hat{C}}{\partial p^2}dt + \{\hat{C}(p_{T_j}, t) - \hat{C}(p_{T_j -}, t)\}I_{\{T_j \in (t, t+dt]\}}\right]. \end{aligned}$$

To get rid of the uncertainty caused by dp term, we choose $\Lambda = \frac{\partial \hat{X}}{\partial p}\left(\frac{\partial C}{\partial p}\right)^{-1}$. Then, we have

$$\begin{aligned} d\Pi = &\left(\frac{\partial \hat{X}}{\partial t} + \frac{1}{2}s_p^2\frac{\partial^2 \hat{X}}{\partial p^2}\right)dt \\ &-\frac{\partial \hat{X}}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}\left[\left(\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}s_p^2\frac{\partial^2 \hat{C}}{\partial p^2}\right)dt \right.\\ &\left. + \{\hat{C}(p_{T_j}, t) - \hat{C}(p_{T_j -}, t)\}I_{\{T_j \in (t, t+dt]\}}\right]. \end{aligned} \tag{3.3}$$

If there is default with probability $p_t dt$ before the inception of the forward credit default swap, the swaption contract becomes void; therefore, we have:

$$d\Pi = -\hat{X} - \frac{\partial \hat{X}}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}(R - \hat{C}). \tag{3.4}$$

Now by the arbitrage principle, we must have $d\Pi = r\Pi dt$.

Taking the expectation of $d\Pi$, by the Ito Lemma, and setting this equal to $r\Pi dt$, i.e., setting $(3.3) \times (1 - p_t dt) + (3.4) \times p_t dt = r\Pi dt$, we have

$$\left[\frac{\partial \hat{X}}{\partial t} + \frac{1}{2}s_p^2\frac{\partial^2 \hat{X}}{\partial p^2} - r\hat{X} - \hat{X}p_t\right]dt$$
$$- \frac{\partial \hat{X}}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}\Bigg[\bigg(\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}s_p^2\frac{\partial^2 \hat{C}}{\partial p^2} - (R - \hat{C})p_t$$
$$+ \{\hat{C}(p_{T_j}, t) - \hat{C}(p_{T_j -}, t)\}I_{\{T_j \in (t, t+dt]\}}p_t dt - r\hat{C}\bigg)dt$$
$$+ \{\hat{C}(p_{T_j}, t) - \hat{C}(p_{T_j -}, t)\}I_{\{T_j \in (t, t+dt]\}}\Bigg] = 0.$$

By (2.2),

$$(\frac{\partial \hat{C}}{\partial p}\bigg)^{-1}\Bigg[\bigg(\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}s_p^2\frac{\partial^2 \hat{C}}{\partial p^2} - (R - \hat{C})p_t$$
$$+ \{\hat{C}(p_{T_j}, t) - \hat{C}(p_{T_j -}, t)\}I_{\{T_j \in (t, t+dt]\}}p_t dt - r\hat{C}\bigg)dt$$
$$+ \{\hat{C}(p_{T_j}, t) - \hat{C}(p_{T_j -}, t)\}I_{\{T_j \in (t, t+dt]\}}\Bigg] = -a_p dt.$$

Because the duration of the swaption is relatively short (typically three months to six months), and we assume that the expiration date of the swaption falls on one of the coupon payment date of the bonds issued by the reference entity, we can assume that there is no jump in default intensity during the life time of option. Therefore,

$$\frac{\partial \hat{X}}{\partial t} + \frac{1}{2}s_p^2\frac{\partial^2 \hat{X}}{\partial p^2} + a_p\frac{\partial \hat{X}}{\partial p} - (p_t + r)\hat{X} = 0.$$

Note that the present value of forward credit default swap is given by (3.1),

$$\begin{aligned} \hat{X}(T_0) &= [Z(T_0) - Z^*]^+ \\ &= \Big[E[b(\tau)(1 - R) \cdot I_{\{\tau \leq T_N\}}] - z^* \cdot E[b(\tau) \cdot I_{\{\tau \leq T_N\}}]\Big]^+. \end{aligned} \tag{3.5}$$

From the financial point of view, we can expect the value of the swaption right before the jump (i.e., at $t = T_0$) to be the expectation of the value at $t = T_0$. So we can assume that the terminal condition as follows:

$$\hat{X}(T_{0-}) = P(T_{0-}, T_0)\Big[E[b(\tau)(1 - R) \cdot I_{\{\tau \leq T_N\}}] - z^* \cdot E[b(\tau) \cdot I_{\{\tau \leq T_N\}}]\Big]^+ \tag{3.6}$$

where $P(T_{0-}, T_0)$ is given by (2.4).

Letting $\hat{X}(p,t) = W\hat{X}(p,T_{0-})$, we have

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}s_p^2\dfrac{\partial^2 W}{\partial p^2} + a_p\dfrac{\partial W}{\partial p} - (p_t + r)W = 0, & (0 \le t < T_0, p > 0) \\ W(p, T_{0-}) = 1, & (p > 0). \end{cases}$$

As in subsection 2.1.2, we will restrict $a_p(p,t)$ and $s^2(p,t)$ to the following cases:

Assumption 5: $a_p(p,t)$ and $s^2(p,t)$ are linear in p, i.e.,

$$\begin{aligned} a_p(p,t) &= b(t) - c(t)p, \\ s^2(p,t) &= d(t) + e(t)p. \end{aligned} \qquad \square$$

We again will try to find the solution in the form of

$$W(p,t) = e^{A(t,T_0) - B(t,T_0)p}.$$

Then, we have

$$A' + \frac{1}{2}dtB^2 - b(t)B - r - p\Big(B' - \frac{1}{2}e(t)B^2 - c(t)B + 1\Big) = 0.$$

Since this should hold for any value of p, we have the system of equations.

$$\begin{cases} A' + \dfrac{1}{2}dtB^2 - b(t)B - r = 0 \\ B' - \dfrac{1}{2}e(t)B^2 - c(t)B + 1. \end{cases} \tag{3.7}$$

Noting that $A(T_{0-}, T_{0-}) = B(T_{0-}, T_{0-}) = 0$ since $W(p, T_{0-}) = 1$ and the solution for (3.7) is given by (2.16) and (2.18) (by replacing $T-$ by T_{0-}), we have the following conclusion.

Theorem 5. Under the assumptions in Subsection 3.1.1, the price of the credit default swaption with unexpected default is possibly given by

$$\hat{X}(t,p) = \hat{X}(T_0,p) \cdot e^{A(t,T_0) - B(t,T_0)p}, \tag{3.8}$$

where

$$A(t,T_0) = -\int_t^{T_0} \Big(b(s)B(s,T_0) - \frac{1}{2}d(s)B^2(s,T_0) + r\Big)\, ds, \tag{3.9}$$

$$B(t,T_0) = \begin{cases} \dfrac{1-e^{-c(T_0-t)}}{c}, & dp = \big(b(t) - c\cdot p)dt + \sqrt{dt}\cdot dW_1, c \neq 0\big) \\ T_0 - t, & dp = \big(b(t) - c\cdot p)dt + \sqrt{dt}\cdot dW_1, c = 0\big) \\ -\sqrt{\frac{2}{c}}\cdot\dfrac{\exp(\sqrt{\frac{c}{2}}(T_0-t) - \exp(-\sqrt{\frac{c}{2}}(T_0-t)}{\exp(\sqrt{\frac{c}{2}}(T_0-t) + \exp(-\sqrt{\frac{c}{2}}(T_0-t)}, & \\ & dp = b(t)dt + \sqrt{d(t) + K\cdot p}\cdot dW_1 \end{cases} \tag{3.10}$$

and $\hat{X}(T_0, p)$ is given by (3.5).

3.2 Credit Derivatives Pricing with Constant Interest Rate - Expected and Unexpected Default

3.2.1 Formulation

Assumption 1: We assume the same setting for the credit default swaption as in Section 3.1: its inception and expiration, the onset and fee structure of the underlying forward credit default swap.

Assumption 2: Default event is both exogenous and endogenous. Unexpected default probability on any interval $[t, t+dt]$ is given by

$$dp = a_p(p,t)dt + s_p(p,t)dW_1 + p_{\tau_j -} U_j I_{\{\tau_j \in (t,t+dt]\}}, \tag{3.11}$$

where $I_{\{\tau_j \in (t,t+dt]\}}$ is an indicator function taking 1 when $\tau_j \in (t, t+dt]$ and 0 otherwise. Expected default occurs when the firm assets $V = V(t) = V_t$ falls below the barrier, say $V_b(t) = V_B$. As in Section 2.2, the firm asset is the sum of its coupon bonds and stocks as in (2.22), and follows the geometric Brownian motion given by (2.21).

Assumption 3: Expected and unexpected default recovery is given as the form of face value exogenous recovery ($R\cdot e^{-r(T-t)}$, $0 \le R \le 1$: constant, T: maturity of the bond).

Assumption 4: The price of defaultable corporate zero-coupon bond with both expected and unexpected default is given by the function $C = C(V,p,t)$ and the price of the swaption is given by $X = X(V,p,t)$.

Problem: Under these setting and assumptions, we will find the credit default swaption $X = X(V, p, t)$.

3.2.2 Derivation of the model

We construct a portfolio by hedging $X = X(V, p, t)$ with Λ_1 zero-coupon bonds of the reference entity with exogenous and endogenous default, and Λ_2 stocks of the reference entity. So the value of the portfolio is:

$$\Pi = X - \Lambda_1 S - \Lambda_2 C,$$

and the change of value of this portfolio over a small time increment $[t, t + dt]$ is given by

$$d\Pi = dX - d\Lambda_1 S - d\Lambda_2 C.$$

Since $S = \dfrac{V - nC}{m}$, we have

$$\begin{aligned} \Pi &= X - \Lambda_1 \left(\frac{V - nC}{m} \right) - \Lambda_2 C \\ &= X - \frac{\Lambda_1}{m} V - \left(\Lambda_1 \frac{n}{m} - \Lambda_2 \right) C. \end{aligned}$$

By the same argument as before, since the duration of the swaption is relatively short, we assume that there is no jump in default intensity and the value of total assets during the term of the swaption, except the swaption expiration falls on a coupon payment date. Therefore, if there is no default (with probability $1 - p_t dt$),

$$\begin{aligned} d\Pi &= dX - \frac{\Lambda_1}{m} dV - \left(\Lambda_1 \frac{n}{m} - \Lambda_2 \right) dC \\ &= \frac{\partial X}{\partial V} dV + \left\{ \frac{\partial X}{\partial t} + \frac{1}{2} \big(s_V^2 V^2 \frac{\partial^2 X}{\partial V^2} + s_p^2 \frac{\partial^2 X}{\partial p^2} \big) \right\} dt + \frac{\partial X}{\partial p} dp - \frac{\Lambda_1}{m} dV \\ &\quad - \big(\Lambda_1 \frac{n}{m} - \Lambda_2 \big) \left[\frac{\partial C}{\partial V} dV + \left\{ \frac{\partial C}{\partial t} + \frac{1}{2} \big(s_V^2 V^2 \frac{\partial^2 C}{\partial V^2} + s_p^2 \frac{\partial^2 C}{\partial p^2} \big) \right\} dt + \frac{\partial C}{\partial p} dp \right] \end{aligned} \tag{3.12}$$

Let us choose Λ_1 and Λ_2 so that we can get rid of uncertainty caused by dp and dV terms. That is,

$$\begin{aligned} &\frac{\partial X}{\partial V} - \frac{\Lambda_1}{m} - \big(\Lambda_1 \frac{n}{m} - \Lambda_2 \big) \frac{\partial C}{\partial V} = 0, \quad \text{and} \\ &\frac{\partial X}{\partial p} - \big(\Lambda_1 \frac{n}{m} - \Lambda_2 \big) \frac{\partial C}{\partial p} = 0. \end{aligned}$$

Solving for Λ_1 and Λ_2,

$$\Lambda_1 = m\left[\frac{\partial X}{\partial V} - \frac{\partial X}{\partial p}\cdot\frac{\partial C}{\partial V}\left(\frac{\partial C}{\partial p}\right)^{-1}\right], \quad \text{and}$$
$$\Lambda_2 = n\left[\frac{\partial X}{\partial V} - \frac{\partial X}{\partial p}\cdot\frac{\partial C}{\partial V}\left(\frac{\partial C}{\partial p}\right)^{-1}\right] - \frac{\partial X}{\partial p}\left(\frac{\partial C}{\partial p}\right)^{-1},$$

and

$$\frac{\Lambda_1}{m} = \frac{\partial X}{\partial V} - \frac{\partial X}{\partial p}\cdot\frac{\partial C}{\partial V}\left(\frac{\partial C}{\partial p}\right)^{-1}, \quad \text{and}$$
$$\Lambda_1\frac{n}{m} - \Lambda_2 = \frac{\partial X}{\partial p}\left(\frac{\partial C}{\partial p}\right)^{-1}.$$

Putting these back in (3.11), we have

$$d\Pi = \left[\frac{\partial X}{\partial t} + \frac{1}{2}\big(s_V^2V^2\frac{\partial^2 X}{\partial V^2} + s_p^2\frac{\partial^2 X}{\partial p^2}\big)\right]dt - \frac{\partial X}{\partial p}\left(\frac{\partial C}{\partial p}\right)^{-1}\left[\frac{\partial C}{\partial t} + \frac{1}{2}\big(s_V^2V^2\frac{\partial^2 C}{\partial V^2} + s_p^2\frac{\partial^2 C}{\partial p^2}\big)\right]dt.$$

When there is a default (with probability $p_t dt$), the change of this portfolio's value is given by

$$\begin{aligned} d\Pi &= -X - \frac{\Lambda_1}{m}dV - \big(\Lambda_1\frac{n}{m} - \Lambda_2\big)(R-C) \\ &= -X - \left[\frac{\partial X}{\partial V} - \frac{\partial X}{\partial p}\cdot\frac{\partial C}{\partial V}\left(\frac{\partial C}{\partial p}\right)^{-1}\right]dV - \frac{\partial X}{\partial p}\left(\frac{\partial C}{\partial p}\right)^{-1}(R-C). \end{aligned}$$

By the arbitrage principle, the expectation of $d\Pi$ is equal to $r\Pi dt$. Ignoring the higher order of infinitesimal terms of dt, we have

$$\begin{aligned} &\frac{\partial X}{\partial t} + \frac{1}{2}\left(s_V^2V^2\frac{\partial^2 X}{\partial V^2} + s_p^2\frac{\partial^2 X}{\partial p^2}\right) - Xp_t - rX + rV\frac{\partial X}{\partial V} \\ &- \frac{\partial X}{\partial p}\left(\frac{\partial C}{\partial p}\right)^{-1}\left[\frac{\partial C}{\partial t} + \frac{1}{2}\left(s_V^2V^2\frac{\partial^2 C}{\partial V^2} + s_p^2\frac{\partial^2 C}{\partial p^2}\right) + rV\frac{\partial C}{\partial V} + (R-C)p_t - rC\right] = 0. \quad (3.13) \end{aligned}$$

But by (2.2), the expression in the bracket in the second line in the equation is equal to $-a_p\left(\frac{\partial C}{\partial p}\right)^{-1}$. So (3.14) reduces to

$$\frac{\partial X}{\partial t} + \frac{1}{2}\left(s_V^2V^2\frac{\partial^2 X}{\partial V^2} + s_p^2\frac{\partial^2 X}{\partial p^2}\right) + a_p\frac{\partial X}{\partial p} + rV\frac{\partial X}{\partial V} - (p_t + r)X = 0.$$

Again, the terminal condition is given by (3.6); here the survival probability is given by $Q\cdot P$, where Q is the survival measure based on the intensity given by (2.3) , P is the survival measure given by (2.4), and the jump in the survival probability $S(T_{0-}, T_0)$ at time $t = T_0$ is given by (2.41). Therefore we have the following model.

Theorem 6. Under Assumptions 1 though 4, the price of the credit default swaption with expected and unexpected default probability is given by the following:

$$\begin{cases} \dfrac{\partial X}{\partial t} + \frac{1}{2}\left(s_V^2 V^2 \dfrac{\partial^2 X}{\partial V^2} + s_p^2 \dfrac{\partial^2 X}{\partial p^2} \right) + a_p \dfrac{\partial X}{\partial p} + rV \dfrac{\partial X}{\partial V} - (p_t + r)X = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \le t < T_0, p > 0, V > V_b) \\ X(V, p, T_{0-}) = S(T_{0-}, T_0) \Big[E[b(\tau)(1-R) \cdot I_{\{\tau \le T_N\}}] - z^* \cdot E[b(\tau) \cdot I_{\{\tau \le T_N\}}] \Big]^+, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (p > 0, V > V_b) \\ \lim\limits_{V \to \infty} X(V, p, t) = 0, \qquad\qquad\qquad (0 \le t < T_0, p > 0) \\ \lim\limits_{p \to \infty} X(V, p, t) = (1-R) \cdot e^{-r(T_N - t)}, \qquad (0 \le t < T_0, V > V_b) \end{cases}$$

where $S(T_{0-}, T_0)$ at time $t = T_0$ is given by (2.41).

3.2.3 Particular Solution

In this subsection, we will find a particular solution to a limited case.

Letting $X(V, p, t) = WX(V, p, T_0)$, we have

$$\begin{cases} \dfrac{\partial W}{\partial t} + \frac{1}{2}\left(s_V^2 V^2 \dfrac{\partial^2 W}{\partial V^2} + s_p^2 \dfrac{\partial^2 W}{\partial p^2} \right) + a_p \dfrac{\partial W}{\partial p} + rV \dfrac{\partial W}{\partial V} - (p_t + r)W = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \le t < T_0, p > 0, V > V_b) \\ W(V, p, T_{0-}) = 1, \qquad\qquad\qquad (p > 0, V > V_b) \\ \lim\limits_{V \to \infty} W(V, p, t) = 0, \qquad\qquad\qquad (0 \le t < T_0, p > 0) \\ \lim\limits_{p \to \infty} W(V, p, t) = X(V, p, T_0), \qquad (0 \le t < T_0, V > V_b). \end{cases} \tag{3.14}$$

We assume that

$$W(V, p, t) = f(V, t) \cdot g(p, t).$$

Substituting this into (3.14), we obtain

$$\left[\frac{\partial f}{\partial t} + \frac{1}{2} s_V^2 V^2 \frac{\partial^2 f}{\partial V^2} + rV \frac{\partial f}{\partial V} \right] g + \left[\frac{\partial g}{\partial t} + \frac{1}{2} s_p^2 \frac{\partial^2 g}{\partial p^2} + a_p \frac{\partial g}{\partial p} - (p_t + r) g \right] f = 0.$$

To find a solution, we further assume that

$$\frac{\partial f}{\partial t} + \frac{1}{2}s_v^2 V^2 \frac{\partial^2 f}{\partial V^2} + rV\frac{\partial f}{\partial V} = 0,$$
$$\frac{\partial g}{\partial t} + \frac{1}{2}s_v^2 \frac{\partial^2 g}{\partial p^2} + a_p\frac{\partial g}{\partial p} - pg = 0.$$

Then we have the following two systems:

$$\begin{cases} \dfrac{\partial f}{\partial t} + \frac{1}{2}s_V^2 V^2 \dfrac{\partial^2 f}{\partial V^2} + rV\dfrac{\partial f}{\partial V} = 0, & (0 \le t < T_0, V > V_b) \\ f(V_b, t) = 0, & (0 \le t < T_0) \\ f(V, T_0) = 1, & (V > V_b) \end{cases} \tag{3.15}$$

and

$$\begin{cases} \dfrac{\partial g}{\partial t} + \frac{1}{2}s_p^2 \dfrac{\partial^2 g}{\partial p^2} + a_p\dfrac{\partial g}{\partial p} - (p_t + r)g = 0, & (0 \le t < T_0, p > 0) \\ g(p, T_{0-}) = 1, & (p > 0). \end{cases} \tag{3.16}$$

Noting that (3.15) is the same as (2.46) and (3.16) is the same as (2.47), we will have the following result.

Theorem 7. Under the setting and assumptions in Subsection 3.2.1 and further assumptions made in this subsection, the price of the credit default swaption with expected and unexpected default is given by

$$X(V,t,p) = X(V,T_0,p) \cdot e^{A(t,T_{0-}) - B(t,T_{0-})p}\left[\Phi(d_1) - \left(\frac{V}{V_b}\right)^{1-\frac{2r}{s_V^2}}\Phi(d_2)\right] \tag{3.17}$$

where $X(V,T_0,p)$ is defined in (3.5), $A(t,T_{0-})$ and $B(t,T_{0-})$ are given by (3.9) and (3.10) respectively, and d_1 and d_2 are defines as follows:

$$d_1 = \frac{\ln\frac{V}{V_b}}{\sqrt{(T_0-t)}s_V} + \sqrt{(T_0-t)}\Big(\frac{s_V}{2} - \frac{r}{s_V}\Big)$$
$$d_2 = \frac{\ln\frac{V_b}{V}}{\sqrt{(T_0-t)}s_V} + \sqrt{(T_0-t)}\Big(\frac{s_V}{2} - \frac{r}{s_V}\Big). \tag{3.18}$$

Chapter 4

Defaultable Corporate Coupon-Bond Pricing with Stochastic Interest Rate

4.1 Corporate Coupon Bond with Stochastic Interest Rate - Unexpected Default

4.1.1 Formulation

Under this section, we will assume the following.

Assumption 1: The risk free short term interest rate $r(t) = r_t$ follows Vasicek model (1977):

$$dr_t = \theta(\nu(t) - r(t))dt + s_r(t)dW_1(t).$$

where θ is a constant, $\nu(t)$ and $s_r(t)$ are deterministic functions of t, and $W_1(t)$ is a standard Brownian motion.

Under this assumption, the price $Z(t)$ of default-free zero-coupon bond satisfies the following PDE:

$$\begin{cases} \dfrac{\partial Z}{\partial t} + \frac{1}{2}s_r^2\dfrac{\partial^2 Z}{\partial r^2} + \theta(\nu(t) - r(t))\dfrac{\partial Z}{\partial r} - rZ = 0, \\ Z(r,T) = 1. \end{cases} \tag{4.1}$$

and

$$\begin{aligned} Z(r,t;T) &= e^{\bar{A}(t,T) - \bar{B}(t,T)r} \\ \bar{B}(t,T) &= \frac{1 - e^{-\theta(T-t)}}{\theta} \\ \bar{A}(t,T) &= -\int_t^T \left\{\theta\nu(s)\bar{B}(s,T) - \frac{1}{2}s_r^2\bar{B}^2(s,T)\right\} ds. \end{aligned} \tag{4.2}$$

Assumption 2: Unexpected default probability in $[t, t+dt]$ is $p_t dt$, the default intensity p_t satisfies

$$dp = a_p(r,p,t)dt + s_p(r,p,t)dW_2 + p_{\tau_j\_}U_j I_{\{\tau_j \in (t,t+dt]\}}$$

with

$$
\begin{aligned}
a_p(r,p,t) &= \alpha(t) + \beta(t)r + \gamma(t)p, \\
s_p^2 &= \delta(t) + \epsilon(t)r + \eta(t)p,
\end{aligned}
$$

where $W_2(t)$ is a standard Brownian motion, and Unexpected default recover is given by $R_d = R{\cdot}Z$, where $R, 0 \leq R \leq 1$, is a constant, and Z is the price at the default time of default-free zero-coupon bond.

Assumption 3: Covariance $Cov(dW_1, dW_2) = \rho$.

Assumption 4: The defaultable corporate coupon-bond price is given by the function $\hat{G} = \hat{G}(r,p,t)$, which constitutes of $\hat{C} = \hat{C}(r,p,t)$, the value at time t of the principal portion only, and c_i, the i-th coupon with $c_i\hat{C}(r,p,t;\tau_i)$ to be the value at time t of i-th coupon due on τ_i. Therefore, we have

$$
\hat{G}(r,p,t) = \hat{C}(r,p,t) + \sum_{\tau_i \geq t} c_i\hat{C}(r,p,t;\tau_i).
$$

Problem: Under these assumptions, we shall find the price of defaultable corporate coupon-bond $\hat{G} = \hat{G}(r,p,t)$.

4.1.2 Derivation of the model

As in Chapter 2, to hedge the risk of p_t we construct a portfolio by hedging one bond with another bond with different maturity. We will include default-free zero-coupon bond here to hedge the risk caused by $r(t)$. Let us denote the priced of a bond with maturity T_i and default recovery R_i by $C_i(r,p,t:T_i), i = 1,2$. Then, the portfolio is:

$$
\Pi = \hat{C}_1 - \Lambda_1 Z - \Lambda_2 \hat{C}_2.
$$

The change of values in the portfolio over a small time increment $[t, t+dt]$, if there is no default with probability $1 - p_t dt$, is

$$\begin{aligned} d\Pi &= \left(\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_1}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_1}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_1}{\partial p^2}\big]\right) dt + \frac{\partial \hat{C}_1}{\partial r} dr + \frac{\partial \hat{C}_1}{\partial p} dp \\ &\quad + \{\hat{C}_1(r, p_{\tau_j}, t) - \hat{C}_1(r, p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}} - \Lambda_1 \left[\left(\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2}\right) dt + \frac{\partial Z}{\partial r} dr\right] \\ &\quad - \Lambda_2 \Bigg[\left(\frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_2}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_2}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_2}{\partial p^2}\big]\right) dt + \frac{\partial \hat{C}_2}{\partial r} dr + \frac{\partial \hat{C}_2}{\partial p} dp \\ &\quad + \{\hat{C}_2(r, p_{\tau_j}, t) - \hat{C}_2(r, p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}}\Bigg] \\ &= \left(\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_1}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_1}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_1}{\partial p^2}\big]\right) dt - \Lambda_1 \left(\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2}\right) dt \\ &\quad - \Lambda_2 \left(\frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_2}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_2}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_2}{\partial p^2}\big]\right) dt \\ &\quad + \left(\frac{\partial \hat{C}_1}{\partial r} - \Lambda_1 \frac{\partial Z}{\partial r} - \Lambda_2 \frac{\partial \hat{C}_2}{\partial r}\right) dr + \left(\frac{\partial \hat{C}_1}{\partial p} - \Lambda_2 \frac{\partial \hat{C}_2}{\partial p}\right) dp \\ &\quad + \{\hat{C}_1(r, p_{\tau_j}, t) - \hat{C}_1(r, p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}} \\ &\quad + \{\hat{C}_2(r, p_{\tau_j}, t) - \hat{C}_2(r, p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}}. \end{aligned} \tag{4.3}$$

We will choose Λ_1 and Λ_2 so that we can get rid of uncertainty caused by dr and dp terms. That is,

$$\Lambda_1 = \left[\frac{\partial \hat{C}_1}{\partial r} - \frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_1}{\partial p}\right)^{-1} \frac{\partial \hat{C}_2}{\partial r}\right]\left(\frac{\partial \hat{Z}}{\partial r}\right)^{-1} \text{and}$$
$$\Lambda_2 = \frac{\partial \hat{C}_1}{\partial p}\left(\frac{\partial \hat{C}_1}{\partial p}\right)^{-1}.$$

Then (4.3) becomes

$$\begin{aligned} d\Pi &= \left(\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_1}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_1}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_1}{\partial p^2}\big]\right) dt - \Lambda_1 \left(\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2}\right) dt \\ &\quad - \Lambda_2 \left(\frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_2}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_2}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_2}{\partial p^2}\big]\right) dt \\ &\quad + \{\hat{C}_1(r, p_{\tau_j}, t) - \hat{C}_1(r, p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}} \\ &\quad + \{\hat{C}_2(r, p_{\tau_j}, t) - \hat{C}_2(r, p_{\tau_j -}, t)\} I_{\{\tau_j \in (t, t+dt]\}}. \end{aligned} \tag{4.4}$$

If there is a default, with probability $p_t dt$, then the price change in the portfolio is

$$d\Pi = (R_1 - \hat{C}_1) - \Lambda_1 dZ - \Lambda_2 (R_2 - \hat{C}_2). \tag{4.5}$$

By the Arbitrage Principle, we should have $E[d\Pi] = r_t \Pi dt$, i.e., $(4.4) \times (1 - p_t dt) + (4.5) \times p_t dt = r_t \Pi dt$. By the Ito Lemma, and considering the time period with no jump (i.e., $\tau_j \notin [t, t+dt]$), we have

$$\begin{aligned}
&\frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_1}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_1}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_1}{\partial p^2}\big] - rC_1 + p_t(R_1 - C_1) \\
&-\Lambda_2 \Big\{ \frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_2}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_2}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_2}{\partial p^2}\big] - rC_1 + p_t(R_2 - C_2) \\
&-\Lambda_1 \Big(\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2} - rZ \Big) = 0.
\end{aligned}$$

Note that by (4.1)

$$\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2} - rZ = -\theta(\nu(t) - r(t)) \frac{\partial Z}{\partial r}. \tag{4.6}$$

Substituting (4.6) and simplifying, we obtain

$$\begin{aligned}
&\Big\{ \frac{\partial \hat{C}_1}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_1}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_1}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_1}{\partial p^2}\big] \\
&\qquad + \theta(\nu(t) - r(t)) \frac{\partial \hat{C}_1}{\partial r} - rC_1 + p_t(R_1 - C_1) \Big\} \Big(\frac{\partial \hat{C}_1}{\partial p} \Big)^{-1} \\
&= \Big\{ \frac{\partial \hat{C}_2}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}_2}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}_2}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}_2}{\partial p^2}\big] \\
&\qquad + \theta(\nu(t) - r(t)) \frac{\partial \hat{C}_2}{\partial r} - rC_2 + p_t(R_2 - C_2) \Big\} \Big(\frac{\partial \hat{C}_2}{\partial p} \Big)^{-1}.
\end{aligned}$$

The left hand side of the equation is a function of T_1 but not T_2 and the right hand side is a function of T_2 but not T_1 so that both sides must be a function independent of T_1 and T_2, say $-\kappa(r, p, t)$. So we have

$$\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}\big[s_r^2 \frac{\partial^2 \hat{C}}{\partial r^2} + 2\rho s_r s_p \frac{\partial^2 \hat{C}}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2}\big] + \theta(\nu(t) - r(t)) \frac{\partial \hat{C}}{\partial r} - r\hat{C} + p_t(R_d - \hat{C}) = -\kappa(r, p, t) \frac{\partial \hat{C}}{\partial p}.$$

Here $\kappa(r, p, t)$ is a risk neutral drift of p_t. We can write $\kappa(r, p, t)$ in the form of $\kappa(r, p, t) = a_p(r, p, t) - s_p(r, p, t) \cdot \lambda(r, p, t)$ or $\lambda(r, p, t) = \frac{a_p(r,p,t) - \kappa(r,p,t)}{s_p(r,p,t)}$. The function $\lambda(r, p, t)$ is called a market price of risk p_t and measures an extra compensation per unit of risk for taking on the risk incurred by p_t. In a model where we can dynamically hedge the portfolio, we can eliminate such risk totally, therefore in the computation below, we assume $\lambda(r, p, t) = 0$. That is, $\kappa(r, p, t) = a_p(r, p, t)$.

As before, in addition to the previous assumptions, let us also assume the following.

Assumption 5: The defaultable coupon bond price at time $t = \tau_{j-}$ is the expectation of the price at time $t = \tau_j$. $\square$

That is,

$$\hat{C}(r, p, \tau_{j-}) = E(\hat{C}(t, p, \tau_j)), \tag{4.7}$$

where expectation is taken with respect to the survival probability $P(\cdot, \cdot)$. Especially, if $t = T- = \tau_{n-}$, we have

$$\hat{C}(r, p, T-) = 1 \cdot P(T-, T) + R \cdot (1 - P(T-, T))$$

and $\hat{C}(r, p, T) = 1$ if there is not default until $t = T$.

Then, using $R_d = R \cdot Z$, on the time interval $[\tau_{n-1}, T)$ we have a PDE with the terminal condition.

$$\begin{cases} \dfrac{\partial \hat{C}}{\partial t} + \frac{1}{2}\big[s_r^2 \dfrac{\partial^2 \hat{C}}{\partial r^2} + 2\rho s_r s_p \dfrac{\partial^2 \hat{C}}{\partial r \partial p} + s_p^2 \dfrac{\partial^2 \hat{C}}{\partial p^2}\big] + \theta(\nu(t) - r(t))\dfrac{\partial \hat{C}}{\partial r} + a_p(r, p, t)\dfrac{\partial \hat{C}}{\partial p} \\ \qquad -(r + p_t)\hat{C} + pRZ = 0, \qquad (\tau_{n-1} \le t < T, r > 0, p > 0) \\ \hat{C}(r, p, T-) = 1 \cdot P(T-, T) + R \cdot (1 - P(T-, T)) \\ \qquad = e^{-p_{T-}U_n} + R \cdot (1 - e^{-p_{T-}U_n}) \qquad (r > 0, p > 0). \end{cases} \tag{4.8}$$

In order to find an explicit solution, assume that the drift and volatility of $p(t)$ is not correlated to the short term rate $r(t)$. That is,

$$\begin{aligned} a_p &= \alpha(t) + \gamma(t)p, \\ s_p^2 &= \delta(t) + \eta(t)p. \end{aligned} \tag{4.9}$$

Using the change of unknown function and variable as

$$\hat{C}(r, p, t) = Z(r, t) \cdot u(p, t),$$

and using (4.1), (4.8) reduces to

$$\begin{cases} \dfrac{\partial u}{\partial t} + \frac{1}{2}s_p^2 \dfrac{\partial^2 u}{\partial p^2} + a_p \dfrac{\partial u}{\partial p} - p(u - R) = 0, \qquad (\tau_{n-1} \le t < T, p > 0) \\ u(p, T-) = e^{-p_{T-}U_n} + R \cdot (1 - e^{-p_{T-}U_n}), \quad (r > 0, p > 0). \end{cases} \tag{4.10}$$

Noting that this system is the same as (2.8), we have the solution for $t \in [\tau_{n-1}, T)$,

$$\hat{C}(r,p,t) = Z(r,t)\big[e^{A(t,T-)-B(t,T-)p_t} + (1 - e^{A(t,T-)-B(t,T-)p_t} \cdot R\big]$$

where

$$A(t,T-) = -\int_t^{T-} \left(\alpha(s)B(s,T-) - \frac{1}{2}\Lambda(s)B^2(s,T-)\right)\,ds$$

and

$$B(t,T-) = \begin{cases} \frac{1-e^{-\gamma(T-t)}}{\gamma}, \\ \qquad\qquad dp = (\alpha(t) - \gamma\cdot p)dt + \sqrt{\delta(t)}\cdot dW_2, \gamma \neq 0 \\ \tau_n - t, \qquad dp = (\alpha(t) - \gamma\cdot p)dt + \sqrt{\delta(t)}\cdot dW_2, \gamma = 0 \\ -\sqrt{\frac{2}{\gamma}}\cdot\frac{exp(\sqrt{\frac{\gamma}{2}}(T-t)) - exp(-\sqrt{\frac{\gamma}{2}}(T-t))}{exp(\sqrt{\frac{\gamma}{2}}(T-t)) + exp(-\sqrt{\frac{\gamma}{2}}(T-t))}, \\ \qquad\qquad dp = \alpha(t)dt + \sqrt{\delta(t) + k\cdot p}\cdot dW_2. \end{cases}$$

Now, using (4.7), we can obtain the terminal condition for the period $[\tau_{n-2}, \tau_{n-1})$, and we have the following PDE with the terminal condition:

$$\begin{cases} \dfrac{\partial\hat{C}}{\partial t} + \frac{1}{2}\big[s_r^2\dfrac{\partial^2\hat{C}}{\partial r^2} + 2\rho s_r s_p\dfrac{\partial^2\hat{C}}{\partial r\partial p} + s_p^2\dfrac{\partial^2\hat{C}}{\partial p^2}\big] + \theta(\nu(t) - r(t))\dfrac{\partial\hat{C}}{\partial r} \\ \qquad + a_p(r,p,t)\dfrac{\partial\hat{C}}{\partial p} - (r + p_t)\hat{C} + pRZ = 0, \qquad (\tau_{n-2} \le t < \tau_{n-1}, r > 0, p > 0) \\ \hat{C}(r,p,\tau_{(n-1)-}) = Z(r,\tau_{n-1})\cdot E[C(r,p,\tau_{n-1})] \\ \qquad = Z(r,\tau_{n-1})\cdot\Big(e^{D(\tau_{n-1},T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}}U_j} + R\cdot(1 - e^{D(\tau_{n-1},T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}}U_j})\Big) \\ \qquad = Z(r,\tau_{n-1})\cdot\Big(e^{D(\tau_{n-1},T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}}U_j}(1-R) + R\Big) \qquad (r > 0, p > 0) \end{cases}$$

where $D(\tau_{n-1}, T-) = A(\tau_{n-1}, T-) - B(\tau_{n-1}, T-)p(\tau_{n-1})$. Again, letting $\hat{C} = Z(r,t)\cdot u(p,t)$, and using (4.1), we have the following PDE with the terminal condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \frac{1}{2}s_p^2\dfrac{\partial^2 u}{\partial p^2} + a_p\dfrac{\partial u}{\partial p} - p(u - R) = 0, \qquad (\tau_{n-2} \le t < \tau_{n-1}, p > 0) \\ u(p,\tau_{(n-1)-}) = e^{D(\tau_{n-1},T-) - \sum\limits_{j=n-1}^{n} p_{\tau_{j-}}U_j}(1-R) + R, \quad (r > 0, p > 0). \end{cases}$$

Noting that this is the same system as (2.16), for $t \in [\tau_{n-2}, \tau_{n-1})$, we have the following solution.

$$\hat{C}(r,p,t) = Z(r,t)\big[e^{A(t,\tau_{(n-1)-})-B(t,\tau_{(n-1)-})p_t} + (1 - e^{A(t,\tau_{(n-1)-})-B(t,\tau_{(n-1)-})p_t} \cdot R\big]$$

where

$$A(t,\tau_{(n-1)-}) = -\int_t^{\tau_{(n-1)-}} \left(\alpha(s)B(s,\tau_{(n-1)-}) - \frac{1}{2}\Lambda(s)B^2(s,\tau_{(n-1)-})\right) ds$$

and

$$B(t,\tau_{(n-1)-}) = \begin{cases} \frac{1-e^{-\gamma(\tau_{(n-1)-}-t)}}{\gamma}, & dp = (\alpha(t)-\gamma\cdot p)dt + \sqrt{\delta(t)}\cdot dW_2, \gamma \neq 0 \\ \tau_{n-1}-t, & dp = (\alpha(t)-\gamma\cdot p)dt + \sqrt{\delta(t)}\cdot dW_2, \gamma = 0 \\ -\sqrt{\frac{2}{\gamma}}\cdot\frac{exp(\sqrt{\frac{\gamma}{2}}(\tau_{n-1}-t))-exp(-\sqrt{\frac{\gamma}{2}}(\tau_{n-1}-t))}{exp(\sqrt{\frac{\gamma}{2}}(\tau_{n-1}-t))+exp(-\sqrt{\frac{\gamma}{2}}(\tau_{n-1}-t))}, & \\ & dp = \alpha(t)dt + \sqrt{\delta(t)+K\cdot p}\cdot dW_2. \end{cases}$$

By extending this backwards, we have the following result.

Theorem 8. Under Assumptions 1 through 5, the price of the corporate coupon-bond with unexpected default and stochastic short term rate, for any $0 \le t < T$ with $\tau_{j-1} \le t < \tau_j$ is given by

$$\hat{G}(r,p,t) = \hat{C}(r,p,t) + \sum_{\tau_i \ge t} c_i \hat{C}(r,p,t;\tau_i)$$

where

$$\begin{aligned} \hat{C}(r,p,t) \quad = \quad & Z(r,t)\cdot\left[e^{A(t,T-)-B(t,T-)p_t-\sum\limits_{k=j}^{n} p_{\tau_{k-}}U_k} + \right. \\ & \left.\left(1-e^{A(t,T-)-B(t,T-)p_t-\sum\limits_{k=j}^{n} p_{\tau_{k-}}U_k}\right)\cdot R\right] \end{aligned} \tag{4.11}$$

where, $Z(r,t)$ is given by (4.2), $c_i\hat{C}(r,p,t;\tau_i)$ corresponds to $i-$th coupon, whose price is given by the product of coupon rate c_i and zero-coupon of the same company maturing at time τ_i, and

$$\begin{aligned} A(t,T-) = & -\int_t^{\tau_{j-}} \left(\alpha(s)B(s,\tau_{j-}) - \frac{1}{2}\Lambda(s)B^2(s,\tau_{j-})\right) ds \\ & - \sum_{k=j+1}^{n}\int_{\tau_{k-1}}^{\tau_{k-}} \left(\alpha(s)B(s,\tau_{k-}) - \frac{1}{2}\Lambda(s)B^2(s,\tau_{k-})\right) ds \end{aligned} \tag{4.12}$$

and

$$B(t,T-)=\begin{cases}\dfrac{n-(j-1)-e^{-\gamma(\tau_j-t)}-\sum\limits_{k=j+1}^{n}e^{-\gamma(\tau_k-\tau_{k-1})}}{\gamma}, \\ \qquad \text{if } dp=(\alpha(t)-\gamma\cdot p)dt+\sqrt{\delta(t)}\cdot dW_2+p_{\tau_j-}U_jI_{\{\tau_j\in(t,t+dt]\}},\gamma\neq 0 \\ T-t, \\ \qquad \text{if } dp=(\alpha(t)-\gamma\cdot p)dt+\sqrt{\delta(t)}\cdot dW_2+p_{\tau_j-}U_jI_{\{\tau_j\in(t,t+dt]\}},\gamma=0 \\ -\sqrt{\dfrac{2}{\gamma}}\cdot\dfrac{e^{\sqrt{\frac{\gamma}{2}}(\tau_j-t)}+\sum\limits_{k=j+1}^{n}e^{\sqrt{\frac{\gamma}{2}}(\tau_k-\tau_{k-1})}-e^{-\sqrt{\frac{\gamma}{2}}(\tau_j-t)}-\sum\limits_{k=j+1}^{n}e^{-\sqrt{\frac{\gamma}{2}}(\tau_k-\tau_{k-1})}}{e^{\sqrt{\frac{\gamma}{2}}(\tau_j-t)}+\sum\limits_{k=j+1}^{n}e^{\sqrt{\frac{\gamma}{2}}(\tau_k-\tau_{k-1})}+e^{-\sqrt{\frac{\gamma}{2}}(\tau_j-t)}+\sum\limits_{k=j+1}^{n}e^{-\sqrt{\frac{\gamma}{2}}(\tau_k-\tau_{k-1})}}, \\ \qquad \text{if } dp=\alpha(t)dt+\sqrt{\delta(t)+K\cdot p}\cdot dW_2+p_{\tau_j-}U_jI_{\{\tau_j\in(t,t+dt]\}}\end{cases} \tag{4.13}$$

with, on each time interval, $\tau_{j-1}\leq t<\tau_j$

$$B(t,\tau_{j-})=\begin{cases}\dfrac{1-e^{-\gamma(\tau_j-t)}}{\gamma}, & dp=(\alpha(t)-\gamma\cdot p)dt+\sqrt{\delta(t)}\cdot dW_2,\gamma\neq 0 \\ \tau_j-t, & dp=(\alpha(t)-\gamma\cdot p)dt+\sqrt{\delta(t)}\cdot dW_2,\gamma=0 \\ -\sqrt{\dfrac{2}{\gamma}}\cdot\dfrac{e^{\sqrt{\frac{\gamma}{2}}(\tau_j-t)}-e^{-\sqrt{\frac{\gamma}{2}}(\tau_j-t)}}{e^{\sqrt{\frac{\gamma}{2}}(\tau_j-t)}+e^{-\sqrt{\frac{\gamma}{2}}(\tau_j-t)}}, & dp=\alpha(t)dt+\sqrt{\delta(t)+K\cdot p}\cdot dW_2.\end{cases} \tag{4.14}$$

4.2 Corporate Coupon Bond with Stochastic Interest Rate - Expected and Unexpected Default

4.2.1 Formulation

Under this section, we will assume the following.

Assumption 1: As in the previous section, risk free short term interest rate $r(t) = r_t$ follows Vasicek model (1977):

$$dr_t = \theta(\nu(t) - r(t))dt + s_r dW_1(t).$$

where θ is a constant, $\nu(t)$ and s_r are deterministic functions of t.

Again, as in the previous section, the price $Z(t)$ satisfies the PDE given by (4.1) and its solution is given by (4.2).

Assumption 2: Unexpected default probability in $[t, t+dt]$ is $p_t dt$, the default intensity p_t satisfies

$$dp = a_p(r,p,t)dt + s_p(r,p,t)dW_2 + p_{\tau_j} U_j I_{\{\tau_j \in (t,t+dt]\}}$$

with

$$\begin{aligned} a_p(r,p,t) &= \alpha(t) + \beta(t)r + \gamma(t)p, \\ s_p^2 &= \delta(t) + \epsilon(t)r + \eta(t)p, \end{aligned}$$

and unexpected default recovery is given by $R_d = R \cdot Z$, where $R : 0 \le r \le 1$, constant, and Z is the price at the default time of default free zero coupon bond.

Assumption 3: The firm assets $V(t) = V_t$ consists of m shares of traded stock, whose price at time t is $S(t) = S_t$, and n coupon-bond certificates, whose price at time t is $C(t) = C_t$:

$$V(t) = mS(t) + nC(t).$$

The firm assets value follows a geometric Brownian motion:

$$dV = a_V V_t dt + s_V V_t dW_3,$$

a_V and S_V are constant, and on predetermined coupon payment dates $t = \tau_j$, where j refers to j-th interest payment, $j = 1, \cdots, n$, the jump of V_t is given by

$$\Delta V_{\tau_j} = V_{\tau_j} - V_{\tau_{j\_}} = ncC(T) = nc,$$

where c is the coupon rate of the bonds and T is the maturity of the bond. (Here, we assume that the bonds are redeemed at their face value; therefore, $C(T) = 1$.) Expected default occurs when

$$V \leq V_b(t); \quad V_b(t) = V_B \quad \text{or} \quad V_B \cdot Z$$

and the default recovery is also given by $R_d = R \cdot Z, R : 0 \leq r \leq 1$, where Z is as defined above.

Assumption 4:

$$dW_i \cdot dW_j = \rho_{ij} dt, \quad i = 1, 2, 3.$$

However we assume that unexpected default and expected default are not correlated, i.e., $\rho_{23} = 0$.

Assumption 5: The defaultable corporate coupon-bond price is given by the function $G = G(r, V, p, t)$, which constitutes of $C = C(r, V, p, t)$, the value at time t of the principal portion only, and c_i, the i-th coupon with $c_i C(r, V, p, t; \tau_i)$ to be the value at time t of i-th coupon due on τ_i. Therefore, we have

$$G(r, V, p, t) = C(r, V, p, t) + \sum_{\tau_i \geq t} c_i C(r, V, p, t; \tau_i).$$

Problem: Under these assumptions, we shall find the price of defaultable coupon-bond with both expected and unexpected default, which is given as a function of r, V, p and t, that is $G = G(r, V, p, t)$.

4.2.2 Derivation of the Model

We will form the portfolio by buying one bond certificate under concern and hedge the risk incurred by V, p and r by selling Λ_1 shares of the traded stock, Λ_2 coupon-bond certificates with unexpected default only, whose price is given by $\hat{C}(r, p, t)$ and Λ_3 default free zero coupon bond. We denote the default recovery for $\hat{C}(r, p, t)$ by $\hat{R}$:

$$\Pi = C - \Lambda_1 S - \Lambda_2 \hat{C} - \Lambda_3 Z.$$

By the Arbitrage Principle, the price change of the portfolio over a small increment of time dt is equal to $r\Pi dt$, so that, after taking Assumption 3 into consideration, we have the following:

$$\begin{aligned} d\Pi &= dC - \Lambda_1 dS - \Lambda_2 d\hat{C} - \Lambda_3 dZ \\ &= \Big(1 + \frac{\Lambda_1 n}{m}\Big) dC - \frac{\Lambda_1}{m} dV - \Lambda_2 d\hat{C} - \Lambda_3 dZ \\ &= r\Pi dt. \end{aligned}$$

If there is no default over $[t, t+dt]$ with probability $1 - p_t dt$, then the change in the value of the portfolios is given by

$$\begin{aligned} d\Pi \;=\; & \Big(1 + \frac{\Lambda_1 n}{m}\Big)\Big[\Big\{\frac{\partial C}{\partial t} + \frac{1}{2}\Big(s_r^2 \frac{\partial^2 C}{\partial r^2} + 2\rho_{13} s_r s_V V \frac{\partial^2 C}{\partial r \partial V} + s_V^2 V^2 \frac{\partial^2 C}{\partial V^2} \\ & + 2\rho_{12} s_r s_p \frac{\partial^2 C}{\partial r \partial p} + s_p^2 \frac{\partial^2 C}{\partial p^2}\Big)\Big\} dt + \frac{\partial C}{\partial r} dr + \frac{\partial C}{\partial V} dV + \frac{\partial C}{\partial p} dp \\ & + \big\{C(r, V_{\tau_j}, p_{\tau_j}, t) - C(r, V_{\tau_{j-}}, p_{\tau_{j-}}, t)\big\} I_{\{\tau_j \in (t, t+dt]\}}\Big] \\ & - \frac{\Lambda_1}{m} dV - \Lambda_2 \Big[\Big\{\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}\Big(s_r^2 \frac{\partial^2 \hat{C}}{\partial r^2} + 2\rho_{12} s_r s_p \frac{\partial^2 \hat{C}}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2}\Big)\Big\} dt \\ & + \frac{\partial \hat{C}}{\partial r} dr + \frac{\partial \hat{C}}{\partial p} dp + \big\{\hat{C}(r, p_{\tau_j}, t) - \hat{C}(r, p_{\tau_{j-}}, t)\big\} I_{\{\tau_j \in (t, t+dt]\}}\Big] \\ & - \Lambda_3 \Big[\Big\{\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2}\Big\} dt + \frac{\partial Z}{\partial r} dr\Big]. \end{aligned}$$

We will choose Λ_1, Λ_2 and Λ_3 so that we can get rid of uncertainty caused by dr, dp and dV terms, that is,

$$\begin{aligned} & \Lambda_1 = m \frac{\partial C}{\partial V}\Big(1 - n\frac{\partial C}{\partial V}\Big)^{-1}, \text{and} \quad 1 + \frac{\Lambda_1 n}{m} = \Big(1 - n\frac{\partial C}{\partial V}\Big)^{-1} \\ & \Lambda_2 = \frac{\partial C}{\partial p}\Big(\frac{\partial \hat{C}}{\partial p}\Big)^{-1}\Big(1 - n\frac{\partial C}{\partial V}\Big)^{-1}, \quad \text{and} \\ & \Lambda_3 = \Big(1 - n\frac{\partial C}{\partial V}\Big)^{-1}\Big[\frac{\partial C}{\partial r} - \frac{\partial C}{\partial p}\Big(\frac{\partial \hat{C}}{\partial p}\Big)^{-1}\frac{\partial \hat{C}}{\partial r}\Big]\Big(\frac{\partial Z}{\partial r}\Big)^{-1}. \end{aligned}$$

On the time interval $[t, t + dt)$, we have

$$
\begin{aligned}
d\Pi &= \Big(1 - \frac{\partial C}{\partial V}\Big)^{-1}\Big[\Big\{\frac{\partial C}{\partial t} + \frac{1}{2}\Big(s_r^2 \frac{\partial^2 C}{\partial r^2} + 2\rho_{13} s_r s_V V \frac{\partial^2 C}{\partial r \partial V} + s_V^2 V^2 \frac{\partial^2 C}{\partial V^2} \\
&\quad + 2\rho_{12} s_r s_p \frac{\partial^2 C}{\partial r \partial p} + s_p^2 \frac{\partial^2 C}{\partial p^2}\Big)\Big\}dt \\
&\quad + \big\{C(r, V_{\tau_j}, p_{\tau_j}, t) - C(r, V_{\tau_{j-}}, p_{\tau_{j-}}, t)\big\} I_{\{\tau_j \in (t, t+dt]\}}\Big] \\
&\quad - \Big(1 - \frac{\partial C}{\partial V}\Big)^{-1} \frac{\partial C}{\partial p}\Big(\frac{\partial \hat{C}}{\partial p}\Big)^{-1}\Big[\Big\{\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}\Big(s_r^2 \frac{\partial^2 \hat{C}}{\partial r^2} + 2\rho_{12} s_r s_p \frac{\partial^2 \hat{C}}{\partial r \partial p} + s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2}\Big)\Big\}dt \\
&\quad + \big\{\hat{C}(r, p_{\tau_j}, t) - \hat{C}(r, p_{\tau_{j-}}, t)\big\} I_{\{\tau_j \in (t, t+dt]\}}\Big] \\
&\quad - \Big(1 - \frac{\partial C}{\partial V}\Big)^{-1}\Big(\frac{\partial Z}{\partial r}\Big)\Big[\frac{\partial C}{\partial r} - \frac{\partial C}{\partial p}\Big(\frac{\partial C}{\partial p}\Big)^{-1}\frac{\partial \hat{C}}{\partial r}\Big]\Big\{\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2}\Big\}dt. \qquad (4.15)
\end{aligned}
$$

In case of default with probability $p_t dt$, since

$$
dC = R_d - C \quad \text{and} \quad d\hat{C} = \hat{R} - \hat{C},
$$

the change in the value is given by

$$
\begin{aligned}
d\Pi &= \Big(1 + \frac{\partial \Lambda_1 n}{m}\Big)(R_d - C) - \frac{\Lambda_1}{m} dV - \Lambda_2(\hat{R} - \hat{C}) - \Lambda_3 dZ \\
&= \Big(1 - \frac{\partial C}{\partial V}\Big)^{-1}\Big[(R_d - C) - \frac{\partial C}{\partial V} dV - \frac{\partial C}{\partial p}\Big(\frac{\partial \hat{C}}{\partial p}\Big)^{-1}(\hat{R} - \hat{C}) \\
&\quad - \Big(\frac{\partial Z}{\partial r}\Big)^{-1}\Big\{\frac{\partial C}{\partial r}\Big(\frac{\partial \hat{C}}{\partial p}\Big)^{-1}\frac{\partial \hat{C}}{\partial r}\Big\} dZ\Big]. \qquad (4.16)
\end{aligned}
$$

We take the expectation of $d\Pi$ and equate it with $r\Pi dt$, that is $(4.15) \times (1 - p_t dt) + (4.16) \times p_t dt$.

Then, multiplying by $\left(1-\frac{\partial C}{\partial V}\right)^{-1}$, we obtain

$$\begin{aligned}
&\Big\{\frac{\partial C}{\partial t}+\frac{1}{2}\Big(s_r^2\frac{\partial^2 C}{\partial r^2}+2\rho_{13}s_r s_V V\frac{\partial^2 C}{\partial r\partial V}+s_V^2V^2\frac{\partial^2 C}{\partial V^2}\\
&+2\rho_{12}s_r s_p\frac{\partial^2 C}{\partial r\partial p}+s_p^2\frac{\partial^2 C}{\partial p^2}\Big)\Big\}dt\\
&+(1-p_t dt)\big\{C(r,V_{\tau_j},p_{\tau_j},t)-C(r,V_{\tau_{j-}},p_{\tau_{j-}},t)\big\}I_{\{\tau_j\in(t,t+dt]\}}\\
&-\frac{\partial C}{\partial p}\Big(\frac{\partial\hat{C}}{\partial p}\Big)^{-1}\Big\{\frac{\partial\hat{C}}{\partial t}+\frac{1}{2}\Big(s_r^2\frac{\partial^2\hat{C}}{\partial r^2}+2\rho_{12}s_r s_p\frac{\partial^2\hat{C}}{\partial r\partial p}+s_p^2\frac{\partial^2\hat{C}}{\partial p^2}\Big)\Big\}dt\\
&-(1-p_t dt)\frac{\partial C}{\partial p}\Big(\frac{\partial\hat{C}}{\partial p}\Big)^{-1}\big\{\hat{C}(r,p_{\tau_j},t)-\hat{C}(r,p_{\tau_{j-}},t)\big\}I_{\{\tau_j\in(t,t+dt]\}}\\
&-\Big(\frac{\partial Z}{\partial r}\Big)^{-1}\Big[\frac{\partial C}{\partial r}-\frac{\partial C}{\partial p}\Big(\frac{\partial\hat{C}}{\partial p}\Big)^{-1}\frac{\partial\hat{C}}{\partial r}\Big]\Big\{\frac{\partial Z}{\partial t}+\frac{1}{2}s_r^2\frac{\partial^2 Z}{\partial r^2}\Big\}dt\\
&+\Big\{(R_d-C)-\frac{\partial C}{\partial p}\Big(\frac{\partial\hat{C}}{\partial p}\Big)^{-1}(\hat{R}-\hat{C})\Big\}p_t dt\\
=\;&r\Big[C-\frac{\partial C}{\partial V}-\frac{\partial C}{\partial p}\Big(\frac{\partial\hat{C}}{\partial p}\Big)^{-1}\hat{C}-\Big\{\frac{\partial C}{\partial r}V-\frac{\partial C}{\partial p}\Big(\frac{\partial\hat{C}}{\partial p}\Big)^{-1}\frac{\partial\hat{C}}{\partial r}\Big\}\Big(\frac{\partial Z}{\partial r}\Big)^{-1}Z\Big]p_t dt.
\end{aligned}\tag{4.17}$$

Now, on $[\tau_j,\tau_{j+1})$, (4.17) becomes

$$\begin{aligned}
&\frac{\partial C}{\partial t}+\frac{1}{2}\Big(s_r^2\frac{\partial^2 C}{\partial r^2}+2\rho_{13}s_r s_V V\frac{\partial^2 C}{\partial r\partial V}+s_V^2V^2\frac{\partial^2 C}{\partial V^2}+2\rho_{12}s_r s_p\frac{\partial^2 C}{\partial r\partial p}+s_p^2\frac{\partial^2 C}{\partial p^2}\Big)\\
&+rV\frac{\partial C}{\partial V}-rC+p(R_d-C)\\
&-\frac{\partial C}{\partial p}\Big(\frac{\partial\hat{C}}{\partial p}\Big)^{-1}\Big[\frac{\partial\hat{C}}{\partial t}+\frac{1}{2}\Big(s_r^2\frac{\partial^2\hat{C}}{\partial r^2}+2\rho_{12}s_r s_p\frac{\partial^2\hat{C}}{\partial r\partial p}+s_p^2\frac{\partial^2\hat{C}}{\partial p^2}\Big)-r\hat{C}+p(\hat{R}-\hat{C})\Big]\\
&-\Big(\frac{\partial Z}{\partial r}\Big)^{-1}\Big[\frac{\partial C}{\partial r}-\frac{\partial C}{\partial p}\Big(\frac{\partial\hat{C}}{\partial p}\Big)^{-1}\frac{\partial\hat{C}}{\partial r}\Big]\Big\{\frac{\partial Z}{\partial t}+\frac{1}{2}s_r^2\frac{\partial^2 Z}{\partial r^2}-rZ\Big\}=0.
\end{aligned}\tag{4.18}$$

By (4.6) and (4.1), (4.18) becomes

$$\begin{aligned}
&\frac{\partial C}{\partial t}+\frac{1}{2}\Big(s_r^2\frac{\partial^2 C}{\partial r^2}+2\rho_{13}s_r s_V V\frac{\partial^2 C}{\partial r\partial V}+s_V^2V^2\frac{\partial^2 C}{\partial V^2}+2\rho_{12}s_r s_p\frac{\partial^2 C}{\partial r\partial p}+s_p^2\frac{\partial^2 C}{\partial p^2}\Big)\\
&+rV\frac{\partial C}{\partial V}-rC+p(R_d-C)+a_p\frac{\partial C}{\partial p}+\theta(\nu(t)-r)\frac{\partial C}{\partial r}=0.
\end{aligned}\tag{4.19}$$

As in the previous sections, we assume the following:

Assumption 6: The defaultable corporate coupon bond price at time $t=\tau_{j-}$ is the expectation of the price at time $t=\tau_j$.

So if we let $S(\tau_{j-}, \tau j)$ be the survival probability of the bond at time $t = \tau_j$ given the survival at time $t = \tau_{j-}$, then the price of the corporate bond at $t = \tau_{n-} = T_{-}$, which is the terminal condition for the PDE (4.18), is given by

$$C(r, V, p, T-) = 1 \cdot S(T-, T) + R \cdot [1 - S(T-, T)]. \tag{4.20}$$

Therefore we have the following model:

Theorem 9. Under Assumptions 1 through 6, the price of the defaultable corporate coupon bond price C on $[\tau_{n-1}, T_{-})$ is modeled by

$$\begin{cases} \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\Big(s_r^2 \dfrac{\partial^2 C}{\partial r^2} + 2\rho_{13} s_r s_V V \dfrac{\partial^2 C}{\partial r \partial V} + s_V^2 V^2 \dfrac{\partial^2 C}{\partial V^2} + 2\rho_{12} s_r s_p \dfrac{\partial^2 C}{\partial r \partial p} + s_p^2 \dfrac{\partial^2 C}{\partial p^2}\Big) \\ \qquad + rV \dfrac{\partial C}{\partial V} - rC + p(R_d - C) + a_p \dfrac{\partial C}{\partial p} + \theta(\nu(t) - r)\dfrac{\partial C}{\partial r} = 0 \\ C(r, V, p, T-) = 1 \cdot S(T-, T) + R \cdot [1 - S(T-, T)]. \end{cases}$$

4.2.3 Particular Solution

To solve (4.19) with (4.20) as the terminal condition, we use the change of unknown function and variable as follows:

$$x = \frac{V}{Z} \quad \text{and} \quad u(x, p, t) = \frac{C(r, V, p, t)}{Z}.$$

Then, (4.19) and (4.20) become

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\Big\{[s_r^2 \bar{B}^2(t) + s_V^2 + 2\rho_{13} s_r s_V \bar{B}(t)]x^2 \dfrac{\partial^2 u}{\partial x^2} + s_p^2(p, r, t)\dfrac{\partial^2 u}{\partial p^2} + 2\rho_{12} s_r s_p \bar{B}(t) x \dfrac{\partial^2 u}{\partial p \partial x}\Big\} \\ \quad + [a_p(p, r, t) - 2\rho_{12} s_r s_p \bar{B}(t)]\dfrac{\partial u}{\partial p} - p(u - R) = 0, \qquad (x > V_B, p > 0, \tau_{n-1} \le t < T) \\ u(x, p, T-) = 1 \cdot S(T-, T) + R_d \cdot [1 - S(T-, T)], \qquad (x > V_B, p > 0) \\ u(V_B, p, t) = R_d, \qquad (p > 0, \tau_{n-1} \le t < T) \end{cases}$$

where $\bar{B}(t)$ was defined in (4.2).

We will consider the case where $\beta(t) = \epsilon(t) \equiv 0$ and $\rho_{13} = 0$. So letting

$$\bar{s}^2(t) = s_r^2 \bar{B}^2(t) + s_V^2 + 2\rho_{13} s_r s_V \bar{B}(t),$$

$$\bar{\rho}(p, t) = \rho_{12} s_r s_p \bar{B}(t), \text{and}$$

$$\bar{a}_p(p, t) = a_p(p, r, t) - 2\rho_{12} s_r s_p \bar{B}(t),$$

and also letting

$$\hat{u} = u - R,$$

we have

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} + \dfrac{1}{2}\left\{ \bar{s}^2(t)x^2 \dfrac{\partial^2 \hat{u}}{\partial x^2} + s_p^2(p,t)\dfrac{\partial^2 \hat{u}}{\partial p^2} + 2\bar{\rho}x\dfrac{\partial^2 \hat{u}}{\partial p \partial x} \right\} \\ \qquad\qquad + \bar{a}_p(p,t)\dfrac{\partial \hat{u}}{\partial p} - p\hat{u} = 0, & (x > V_B, p > 0, \tau_{n-1} \le t < T) \\ \hat{u}(x,p,T-) = (1-R)\cdot S(T-,T), & (x > V_B, p > 0) \\ \hat{u}(V_B,p,t) = 0, & (p > 0, \tau_{n-1} \le t < T). \end{cases}$$

Now using the change of unknown function

$$\hat{u} = W(1-R)S(T,T-),$$

we have

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\left\{ \bar{s}^2(t)x^2 \dfrac{\partial^2 W}{\partial x^2} + s_p^2(p,t)\dfrac{\partial^2 W}{\partial p^2} + 2\bar{\rho}x\dfrac{\partial^2 W}{\partial p \partial x} \right\} \\ \quad + \bar{a}_p(p,t)\dfrac{\partial W}{\partial p} - pW = 0, & (x > V_B, p > 0, \tau_{n-1} \le t < T) \\ W(x,p,T-) = 1, & (x > V_B, p > 0) \\ W(V_B,p,t) = 0, & (p > 0, \tau_{n-1} \le t < T). \end{cases}$$

Now using $\rho_{13} = 0$,

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\left\{ \bar{s}^2(t)x^2 \dfrac{\partial^2 W}{\partial x^2} + s_p^2(p,t)\dfrac{\partial^2 W}{\partial p^2} \right\} + a_p(p,t)\dfrac{\partial W}{\partial p} - pW = 0, \\ & (x > V_B, p > 0, \tau_{n-1} \le t < T) \\ W(x,p,T-) = 1, & (x > V_B, p > 0) \\ W(V_B,p,t) = 0, & (p > 0, \tau_{n-1} \le t < T). \end{cases} \tag{4.21}$$

If we further assume that

$$W(x,p,t) = f(x,t)\cdot g(p,t).$$

Then, using this, the PDE in (4.21) becomes

$$\left[\frac{\partial f}{\partial t} + \frac{1}{2}\bar{s}^2(t)x^2\frac{\partial^2 f}{\partial x^2}\right]g + \left[\frac{\partial g}{\partial t} + \frac{1}{2}s_p^2(p,t)\frac{\partial^2 g}{\partial p^2} + a_p(p,t)\frac{\partial g}{\partial p} - pg\right]f = 0.$$

As in before, assuming the coefficient terms of g and f are zero, we have the following two PDE problems:

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{1}{2}\bar{s}^2(t)x^2\dfrac{\partial^2 f}{\partial x^2} = 0, & (x > V_B, \tau_{n-1} \le t < T) \\ f(V_B, t) = 0, & (\tau_{n-1} \le t < T) \\ f(x, T-) = 1, & (x > V_B) \end{cases} \tag{4.22}$$

$$\begin{cases} \dfrac{\partial g}{\partial t} + \dfrac{1}{2}s_p^2(p,t)\dfrac{\partial^2 g}{\partial p^2} + a_p(p,t)\dfrac{\partial g}{\partial p} - pg = 0, & (p > 0, \tau_{n-1} \le t < T) \\ g(p, T-) = 1, & (p > 0). \end{cases} \tag{4.23}$$

Notice that (4.23) is the same as (2.47). Now, to solve (4.22), using the following time scale transformation,

$$s = \int_{\tau_{n-1}}^{t} \bar{s}^2(u)\,du \quad \text{and} \quad \bar{T} = \int_{\tau_{n-1}}^{T} \bar{s}^2(u)\,du,$$

and letting

$$\bar{f}(x, s) = f(x, t),$$

(4.22) becomes

$$\begin{cases} \dfrac{\partial \bar{f}}{\partial s} + \dfrac{1}{2}x^2\dfrac{\partial^2 \bar{f}}{\partial x^2} = 0, & (x > V_B, \tau_{n-1} \le s < \bar{T}) \\ f(V_B, s) = 0, & (0 \le s < \bar{T}) \\ f(x, \bar{T}) = 1, & (x > V_B). \end{cases} \tag{4.24}$$

To solve this, let

$$\begin{aligned} &\frac{x}{V_B} = e^y, \quad \text{and so} \quad y = \ln\frac{x}{V_B} \\ &\zeta = \frac{\bar{T} - s}{2}, \\ &\bar{f}(x, s) = V_B \cdot v(y, \zeta). \end{aligned}$$

Then, (4.24) becomes

$$\begin{cases} \dfrac{\partial v}{\partial \zeta} - \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial v}{\partial y} = 0, & (-\infty < y < \infty, 0 \le s < \frac{\bar{T}}{2}) \\ v(0, \zeta) = 0, & (0 \le s < \frac{\bar{T}}{2}) \\ v(y, 0) = \frac{1}{V_B}, & (-\infty < y < \infty). \end{cases} \tag{4.25}$$

Using the change of unknown function $v = we^{\alpha y + \beta\zeta}$, the PDE in (4.25) becomes

$$w_\zeta = w_{yy} + (\alpha^2 - \alpha_\beta)w + (2\alpha - 1)w_y.$$

We choose α and β so that the coefficients of w and w_y are zero. That is, $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{4}$. Then, we have the following heat equation:

$$\begin{cases} w_\zeta = w_{yy}, & (-\infty < y < \infty, 0 \le \zeta < \frac{\bar{T}}{2}) \\ w(y,0) = \frac{1}{V_B}e^{-\frac{1}{2}y}, & (-\infty < y < \infty). \end{cases}$$

Define

$$\bar{h}(y) = \begin{cases} h(y) = w(y,0) = \frac{1}{V_B}e^{-\frac{1}{2}y} & (y > 0) \\ 0, & \text{otherwise} \end{cases}$$

let $w_1(y,\zeta)$ be a solution of the IVP,

$$\begin{cases} w_\zeta = w_{yy}, & (-\infty < y < \infty, 0 \le \zeta) \\ w(y,0) = \hat{h}(y), & (-\infty < y < \infty) \end{cases} \tag{4.26}$$

and $w_2(y,\zeta) = w_1(-y,\zeta)$ (and so $w_2(0,\zeta) = w_1(0,\zeta)$). Then, by the solution formula and the image solution method

$$\begin{aligned} w_1(y,\zeta) &= \frac{1}{2\sqrt{\pi\zeta}}\int_{-\infty}^{\infty} \hat{h}(\xi)\exp\left(-\frac{(y-\xi)^2}{4\zeta}\right)d\xi = \frac{1}{2\sqrt{\pi\zeta}}\int_0^{\infty} \exp\left(-\frac{(y-\xi)^2}{4\zeta}\right)d\xi \\ &= \frac{1}{V_B}e^{-\frac{1}{2}y+\frac{\zeta}{4}}\cdot\Phi(d_1) \\ w_2(y,\zeta) &= \frac{1}{2\sqrt{\pi\zeta}}\int_0^{\infty} h(\xi)\exp\left(-\frac{(y+\xi)^2}{4\zeta}\right)d\xi \\ &= \frac{1}{V_B}e^{\frac{1}{2}y+\frac{\zeta}{4}}\cdot\Phi(d_2) \end{aligned}$$

where $d_1 = \dfrac{\ln\frac{x}{V_B}}{\sqrt{\bar{T}-s}} - \dfrac{\sqrt{\bar{T}-s}}{2}$ and $d_2 = \dfrac{\ln\frac{V_B}{x}}{\sqrt{\bar{T}-s}} - \dfrac{\sqrt{\bar{T}-s}}{2}$. Since $w = w_1 - w_2$, we have

$$\begin{aligned} \bar{f}(x,s) &= V_B v(y,\zeta) = V_B w(y,\zeta)e^{\frac{1}{2}y-\frac{1}{4}\zeta} = V_B\big(w_1(y,\zeta) - w_2(y,\zeta)\big)e^{\frac{1}{2}y-\frac{1}{4}\zeta} \\ &= \Phi(d_1) - \frac{x}{V_B}\Phi(d_2) \end{aligned}$$

where d_1 and d_2 are given above. And so,

$$f(x,t) = \Phi(d_1^*) - \frac{x}{V_B}\Phi(d_2^*)$$

where $d_1^* = \frac{\ln \frac{V}{V_B Z}}{\sqrt{\int_t^T \bar{s}^2(u)\,du}} - \frac{\sqrt{\int_t^T \bar{s}^2(u)\,du}}{2}$ and $d_2^* = \frac{\ln \frac{V_B Z}{V}}{\sqrt{\int_t^T \bar{s}^2(u)\,du}} - \frac{\sqrt{\int_t^T \bar{s}^2(u)\,du}}{2}$. So the price of the corporate coupon bond with expected and unexpected default for $t \in [\tau_{n-1}, T)$ is given by

$$
\begin{aligned}
C(r,V,p,t) &= Z \cdot u(x,p,t) = Z(\hat{u} + R) = Z\big(W(1-R)S(T,T-) + R\big) \\
&= Z\big(f \cdot g(1-R)S(T,T-) + R\big) \\
&= RZ(t,T) + e^{A(t,T)-B(t,T)p}(1-R)\big[Z \cdot \Phi(d_1^*) - \frac{V}{V_B}\Phi(d_2^*)\big]
\end{aligned}
$$

Now, using this, $C(r,V,p,\tau_{n-1})$ is given by

$$
\begin{aligned}
C(r,V,p,\tau_{n-1}) &= RZ(\tau_{n-1},T) \\
&\quad + e^{A(\tau_{n-1},T)-B(\tau_{n-1},T)p} \cdot (1-R)\Big[Z(\tau_{n-1},T)\Phi(d^*_{1,\tau_{n-1}}) - \frac{V}{V_B}\Phi(d^*_{2,\tau_{n-1}})\Big]
\end{aligned}
$$

where $A(t,\cdot)$ and $B(t,\cdot)$ are given by (2.19) through (2.21) and d_{1,τ_j} and d_{2,τ_j} are given by

$$
\begin{aligned}
d^*_{1,\tau_j} &= \frac{\ln \frac{V}{V_B Z(\tau_j,\tau_{j+1})}}{\sqrt{\int_{\tau_j}^{\tau_{j+1}} \bar{s}^2(u)\,du}} - \frac{\sqrt{\int_{\tau_j}^{\tau_{j+1}} \bar{s}^2(u)\,du}}{2}, \quad \text{and} \\
d^*_{2,\tau_j} &= \frac{\ln \frac{V_B Z(\tau_j,\tau_{j+1})}{V}}{\sqrt{\int_{\tau_j}^{\tau_{j+1}} \bar{s}^2(u)\,du}} - \frac{\sqrt{\int_{\tau_j}^{\tau_{j+1}} \bar{s}^2(u)\,du}}{2}
\end{aligned}
$$

Then, on the time interval $[\tau_{n-2}, \tau_{n-1})$, by the arbitrage principle, the terminal condition becomes

$$
\begin{aligned}
C(r,V,p,\tau_{(n-1)_-}) &= \big[RZ(\tau_{n-1},T) + e^{A(\tau_{n-1},T)-B(\tau_{n-1},T)p} \cdot (1-R)[Z(\tau_{n-1},T)\Phi(d^*_{1,\tau_{n-1}}) \\
&\quad -\frac{V}{V_B}\Phi(d^*_{2,\tau_{n-1}})]\big] \cdot S(\tau_{(n-1)_-}, \tau_{n-1}) \\
&\quad + RZ(\tau_{n-1},T)\big(1 - S(\tau_{(n-1)_-}, \tau_{n-1})\big) \\
&= S(\tau_{(n-1)_-}, \tau_{n-1})\Big\{(1-R)e^{A(\tau_{n-1},T)-B(\tau_{n-1},T)p}[Z(\tau_{n-1},T)\Phi(d^*_{1,\tau_{n-1}}) \\
&\quad -\frac{V}{V_B}\Phi(d^*_{2,\tau_{n-1}})]\Big\} + RZ(\tau_{n-1},T).
\end{aligned}
$$

Solving (4.19) with this terminal condition gives the solution for $t \in [\tau_{n-2}, \tau_{n-1})$ as follows:

$$
\begin{aligned}
C(r,V,p,t) &= RZ(t,T) + e^{A(t,\tau_{n-1})-B(t,\tau_{n-1})p} \cdot (1-R)[Z(t,\tau_{n-1})\Phi(d_1^*) - \frac{V(t)}{V_B}\Phi(d_2^*)] \\
&\quad \times S(\tau_{(n-1)_-}, \tau_{n-1}) \cdot \Big\{(1-R)e^{A(\tau_{n-1},T)-B(\tau_{n-1},T)p}[Z(\tau_{n-1},T)\Phi(d^*_{1,\tau_{n-1}}) \\
&\quad -\frac{V(\tau_{n-1})}{V_B}\Phi(d^*_{2,\tau_{n-1}})]\Big\}
\end{aligned}
$$

where $A(t,\cdot)$, $B(t,\cdot)$, d_{1,τ_j} and d_{2,τ_j} are the same as above; and d_1 and d_2 are given, for any $t \in [\tau_j, \tau_{j+1})$, as

$$
\begin{aligned}
d_1^* &= \frac{\ln \frac{V}{V_B Z(t,\tau_{j+1})}}{\sqrt{\int_t^{\tau_{j+1}} \bar{s}^2(u)\, du}} - \frac{\sqrt{\int_t^{\tau_{j+1}} \bar{s}^2(u)\, du}}{2}, \quad \text{and} \\
d_2^* &= \frac{\ln \frac{V_B Z(t,\tau_{j+1})}{V}}{\sqrt{\int_t^{\tau_{j+1}} \bar{s}^2(u)\, du}} - \frac{\sqrt{\int_t^{\tau_{j+1}} \bar{s}^2(u)\, du}}{2}.
\end{aligned}
$$

By iterating this process, we shall find the solution for any $t \in [0, T)$ as follows:

Theorem 10. Under Assumptions 1 through 5, for any $t \in [0, T)$ with $t \in [\tau_{j-1}, \tau_j), j = 1, \ldots, n-1$, the price of corporate coupon bond with expected and unexpected default is given by

$$
G(r, V, p, t) = C(r, V, p, t) + \sum_{\tau_i \geq t} c_i C(r, V, p, t; \tau_i)
$$

where

$$
\begin{aligned}
C(r, V, p, t) \;=\; & RZ(t, T) + e^{A(t,\tau_j) - B(t,\tau_j)p} \cdot (1 - R)[Z(t, \tau_j)\Phi(d_1^*) - \frac{V(t)}{V_B}\Phi(d_2^*)] \\
& \times \prod_{k=j}^{n-1} S(\tau_{k_-}, \tau_k) \cdot \Big\{ (1 - R) e^{A(\tau_k,\tau_{k+1}) - B(\tau_k,\tau_{k+1})p} [Z(\tau_k, \tau_{k+1})\Phi(d^*_{1,\tau_{n-1}}) \\
& - \frac{V(\tau_k)}{V_B}\Phi(d^*_{2,\tau_k})] \Big\} \qquad (4.27)
\end{aligned}
$$

and $C(r, V, p, t; \tau_i)$ is the price of zero-coupon bond at time t of the same company with maturity τ_i.

Chapter 5

Credit Derivatives Pricing with Stochastic Interest Rate

In this chapter, we shall find the price of credit default swaption using the bond pricing models formulated in the previous chapter.

5.1 Credit Default Swaption with Stochastic Interest Rate - Unexpected Default

5.1.1 Formulation

We make the following assumptions in this section.

Assumption 1: We assume that the time structure of the credit default swaption and the bonds underlying the swaption are the same as in subsection 3.1.1. That is:

Let $t = 0$ and $t = T_0$ be the time when the credit default swaption (hereinafter, the ”swaption”) starts and expires respectively. $t = T_0$ is also when the forward credit default swap takes effect upon the exercise of the swaption. Let $t = T_N$ be the expiration of the credit default swap and for simplicity, let us assume that $T_j, j = 1, \cdots, N$ falls on the interest payment date of the coupon bonds issued by the reference entity of the credit default swap and the credit default swap terminates on the day the coupon bonds are to be redeemed, that is $\tau_n = T_N$ where τ_n is as defined in subsection 2.1.1. Let z^* be the exercise price of the swaption. This is actually the fee the swaption holder pays on the notional amount of the credit default swap once the swaption is exercised. Let $z(t), t \geq T_0$ be the fee leg of forward credit default swap. Define

$$\mathcal{Z}(t) := z(t) \cdot E[\sum_{i,T_i>t}^{N} b(T_i)\Lambda_i I_{\{T_i \leq \tau\}} + b(\tau)\Lambda^* I_{\{T_0 \leq \tau \leq T_N\}}]$$

where $b(T_i) = \exp\{\int_t^{T_i} r\ ds\}$, τ is the time of default, and Λ_i and Λ^* are the lengths of time interval since the last fee payment till T_i and the default date, respectively. Therefore, $S(t)$ is the present value at time t of the forward credit default swap. Observe that by the arbitrage principle, the present value of the total fee leg is the same as the present value of the protection leg. That is:

$$\mathcal{Z}(t) = E[b(\tau)(1-R) \cdot I_{\{\tau \le T_N\}}]. \tag{5.1}$$

Assumption 2: The risk free short term interest rate $r(t) = r_t$ follows the Vasicek model:

$$dr = \theta(\nu(t) - r(t))dt + s_r(t)dW_1(t)$$

where θ, $\nu(t)$, $s_r(t)$ and $W(t)$ are defined as in section 4.1.

Assumption 3: Default is an exogenous event. Unexpected default probability on any interval $[t, t+dt]$ is given by,

$$dp = a_p(p,t)dt + s_p(p,t)dW_2 + p_{\tau_j -}U_j I_{\{\tau_j \in (t, t+dt]\}},$$

where $I_{\{\tau_j \in (t,t+dt]\}}$ is an indicator function. $dW_1 \cdot dW_2 = \rho$. Default recovery is given as the form of face value exogenous recovery ($R \cdot e^{-r(T-t)}$, $0 \le R \le 1$: constant, T: maturity of the bond) or as the form of market price exogenous recovery ($R\times$ bond price at default time).

The price of defaultable corporate zero-coupon bond with exogenous default is given by the function $\hat{C} = \hat{C}(r, p, t)$.

Problem: Under these setting and assumptions, we will find the price of the credit default swaption $\hat{X}(r, p, t)$.

5.1.2 Derivation of the model

We construct a portfolio consisting of (i) one credit default swaption $\hat{X}(r, p, t)$, (ii) Λ_1 units of reference entity's coupon-bonds with exogenous default $\hat{C}$ (to get rid of the risk arisen by p), and (iii) Λ_2 units of default-free zero-coupon bond Z (to get rid of the risk arisen by r). Then, value of the portfolio is:

$$\Pi = \hat{X} - \Lambda_1 \hat{C} - \Lambda_2 Z.$$

So the change of value of this portfolio over a small time increment $[t, t+dt]$ is given by

$$d\Pi = d\hat{X} - d\Lambda_1 \hat{C} - d\Lambda_2 Z.$$

If there is no default over a small time increment $[t, t+dt]$ (with probability $1 - p_t dt$), then by Ito Lemma, the change of value in the portfolio over this period is given by

$$\begin{aligned} d\Pi = & \frac{\partial \hat{X}}{\partial t}dt + \frac{\partial \hat{X}}{\partial p}dp + \frac{\partial \hat{X}}{\partial r}dr + \frac{1}{2}\left\{ s_p^2 \frac{\partial^2 \hat{X}}{\partial p^2} + s_r^2 \frac{\partial^2 \hat{X}}{\partial r^2} + 2\rho s_p s_r \frac{\partial^2 \hat{X}}{\partial r \partial p} \right\} dt \\ & - \Lambda_1 \left[\frac{\partial \hat{C}}{\partial t}dt + \frac{\partial \hat{C}}{\partial p}dp + \frac{\partial \hat{C}}{\partial r}dr + \frac{1}{2}\left\{ s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2} + s_r^2 \frac{\partial^2 \hat{C}}{\partial r^2} + 2\rho s_p s_r \frac{\partial^2 \hat{C}}{\partial r \partial p} \right\} dt \right. \\ & \left. - \{\hat{C}(r, p_{\tau_j}, t) - \hat{C}(r, p_{\tau_{j\_}}, t)\} I_{\{\tau_j \in (t, t+dt]\}} \right] \\ & - \Lambda_2 \left[\frac{\partial Z}{\partial t}dt + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2}dt + \frac{\partial Z}{\partial r}dr \right]. \end{aligned}$$

To get rid of the uncertainty caused by dp and dr terms, we choose Λ_1 and Λ_2 as follows:

$$\begin{aligned} \Lambda_1 &= \frac{\partial \hat{X}}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}, \\ \Lambda_2 &= \left(\frac{\partial Z}{\partial r}\right)^{-1}\left[\frac{\partial \hat{X}}{\partial r} - \frac{\partial \hat{X}}{\partial p} \cdot \frac{\partial \hat{C}}{\partial r}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}\right]. \end{aligned}$$

Then, we have

$$\begin{aligned} d\Pi = & \left[\frac{\partial \hat{X}}{\partial t} + \frac{1}{2}\left\{ s_p^2 \frac{\partial^2 \hat{X}}{\partial p^2} + s_r^2 \frac{\partial^2 \hat{X}}{\partial r^2} + 2\rho s_p s_r \frac{\partial^2 \hat{X}}{\partial r \partial p} \right\} \right] dt \\ & - \frac{\partial \hat{X}}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1} \left[\left\{ \frac{\partial \hat{C}}{\partial t} + \frac{1}{2}\left(s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2} + s_r^2 \frac{\partial^2 \hat{C}}{\partial r^2} + 2\rho s_p s_r \frac{\partial^2 \hat{C}}{\partial r \partial p} \right) \right\} dt \right. \\ & \left. - \{\hat{C}(r, p_{\tau_j}, t) - \hat{C}(r, p_{\tau_{j\_}}, t)\} I_{\{\tau_j \in (t, t+dt]\}} \right] \\ & - \left(\frac{\partial Z}{\partial r}\right)^{-1}\left[\frac{\partial \hat{X}}{\partial r} - \frac{\partial \hat{X}}{\partial p} \cdot \frac{\partial \hat{C}}{\partial r}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}\right]\left[\frac{\partial Z}{\partial t}dt + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2}dt\right]. \end{aligned} \tag{5.2}$$

If there is default with probability $p_t dt$ before the inception of the forward credit default swap, the swaption contract becomes void; therefore, we have:

$$\begin{aligned} d\Pi &= -\hat{X} - \Lambda_1(R - \hat{C}) - \Lambda_2 dZ \\ &= -\hat{X} - \frac{\partial \hat{X}}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}(R - \hat{C}) - \left(\frac{\partial Z}{\partial r}\right)^{-1}\left[\frac{\partial \hat{X}}{\partial r} - \frac{\partial \hat{X}}{\partial p} \cdot \frac{\partial \hat{C}}{\partial r}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}\right] dZ. \end{aligned} \tag{5.3}$$

Then, by the arbitrage principle, we must have $(5.2) \times (1 - p_t dt) + (5.3) \times p_t dt = r\Pi dt$. So we have

$$
\begin{aligned}
&\left[\frac{\partial \hat{X}}{\partial t} + \frac{1}{2}\left\{s_p^2 \frac{\partial^2 \hat{X}}{\partial p^2} + s_r^2 \frac{\partial^2 \hat{X}}{\partial r^2} + 2\rho s_p s_r \frac{\partial^2 \hat{X}}{\partial r \partial p}\right\} - (p_t + r)X\right] dt \\
&\quad - \frac{\partial \hat{X}}{\partial p}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}\left[\left\{\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}\left(s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2} + s_r^2 \frac{\partial^2 \hat{C}}{\partial r^2} + 2\rho s_p s_r \frac{\partial^2 \hat{C}}{\partial r \partial p}\right) + (R - C)p_t - r\hat{C}\right\} dt\right. \\
&\quad \left. - \{\hat{C}(r, p_{\tau_j}, t) - \hat{C}(r, p_{\tau_{j\_}}, t)\} I_{\{\tau_j \in (t, t+dt]\}}(1 + p_t dt)\right] \\
&\quad - \left(\frac{\partial Z}{\partial r}\right)^{-1}\left[\frac{\partial \hat{X}}{\partial r} - \frac{\partial \hat{X}}{\partial p} \cdot \frac{\partial \hat{C}}{\partial r}\left(\frac{\partial \hat{C}}{\partial p}\right)^{-1}\right]\left[\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2} - rZ\right] dt = 0. \qquad (5.4)
\end{aligned}
$$

Noting that, by (4.8) and (4.6), for the time interval such that $\tau_j \notin [t, t + dt]$,

$$
\frac{\partial \hat{C}}{\partial t} + \frac{1}{2}\left(s_p^2 \frac{\partial^2 \hat{C}}{\partial p^2} + s_r^2 \frac{\partial^2 \hat{C}}{\partial r^2} + 2\rho s_p s_r \frac{\partial^2 \hat{C}}{\partial r \partial p}\right) + (R - C)p_t - r\hat{C} = -\theta(\nu(t) - r)\frac{\partial \hat{C}}{\partial r} - a_p \frac{\partial \hat{C}}{\partial p}
$$

and

$$
\frac{\partial Z}{\partial t} + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2} - rZ = -\theta(\nu(t) - r)\frac{\partial \hat{Z}}{\partial r},
$$

(5.4) becomes

$$
\frac{\partial \hat{X}}{\partial t} + \theta(\nu(t) - r)\frac{\partial \hat{X}}{\partial r} + \frac{1}{2}\left\{s_p^2 \frac{\partial^2 \hat{X}}{\partial p^2} + s_r^2 \frac{\partial^2 \hat{X}}{\partial r^2} + 2\rho s_p s_r \frac{\partial^2 \hat{X}}{\partial r \partial p}\right\} - (p_t + r)X = 0.
$$

Since the present value of the forward credit default swap is given by (5.1), the value of the swaption at $t = T_0$ is given by

$$
\begin{aligned}
\hat{X}(T_0) &= [\mathcal{Z}(T_0) - \mathcal{Z}^*]^+ \\
&= \Big[E[b(\tau)(1 - R) \cdot I_{\{\tau \le T_N\}}] - z^* \cdot E\Big[\sum_{i, T_i > t}^{N} b(T_i)\Lambda_i I_{\{T_i \le \tau\}} + b(\tau)\Lambda^* I_{\{T_0 \le \tau \le T_N\}}\Big]\Big]^+.
\end{aligned}
$$

From the financial point of view, we can expect the value of the swaption right before the jump (i.e., at $t = T_{0-}$), to be the expectation of the value at $t = T_0$. So we can assume that the terminal condition as follows:

$$
\begin{aligned}
\hat{X}(T_{0-}) &= P(T_{0-}, T_0) \cdot E\Bigg[\Big[E[b(\tau)(1 - R) \cdot I_{\{\tau \le T_N\}}] \\
&\qquad - z^* \cdot E\Big[\sum_{i, T_i > t}^{N} b(T_i)\Lambda_i I_{\{T_i \le \tau\}} + b(\tau)\Lambda^* I_{\{T_0 \le \tau \le T_N\}}\Big]\Big]^+\Bigg] \qquad (5.5)
\end{aligned}
$$

where $P(\cdot, \cdot)$ is defined by (2.4). Using the change of unknown function

$$
\hat{X}(r, p, t) = \hat{X}(T_{0-})W(r, p, t),
$$

we have the following PDE problem:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \theta(\nu(t) - r)\dfrac{\partial W}{\partial r} + a_p\dfrac{\partial W}{\partial p} + \dfrac{1}{2}\left\{ s_p^2\dfrac{\partial^2 W}{\partial p^2} + s_r^2\dfrac{\partial^2 W}{\partial r^2} + 2\rho s_p s_r\dfrac{\partial^2 W}{\partial r \partial p}\right\} - (p_t + r)W = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \le t < T_0, r > 0, p > 0) \\ W(r, p, T_{0-}) = 1, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (r > 0, p > 0). \end{cases} \tag{5.6}$$

To solve this, first we assume that the drift and volatility of $p(t)$ are not correlated to the short term rate $r(t)$, that is,

$$\begin{aligned} a_p &= \alpha(t) + \gamma(t)p, \\ s_p^2 &= \delta(t) + \eta(t), \text{and} \\ \rho &= 0. \end{aligned} \tag{5.7}$$

Using the change of unknown function again, we consider the following case.

$$W(r, p, t) = u(p, t) \cdot Z(r, t),$$

where $Z(r, t)$ is given by (4.2), we have

$$u\Big(\frac{\partial Z}{\partial t} + \frac{1}{2}s_r^2 u\frac{\partial^2 Z}{\partial r^2} + \frac{\partial Z}{\partial r}\big(\theta(\nu(t) - r)\big) - rZ\Big) + Z\Big(\frac{\partial u}{\partial t} + \frac{1}{2}s_p^2\frac{\partial^2 u}{\partial p^2} + a_p\frac{\partial u}{\partial p} - p_t u\Big) = 0.$$

By (4.1), (5.6) reduces to

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}s_p^2\dfrac{\partial^2 u}{\partial p^2} + a_p\dfrac{\partial u}{\partial p} - p_t u = 0, & (0 \le t < T_0, r > 0, p > 0) \\ u(r, p, T_{0-}) = 1, & (r > 0, p > 0). \end{cases}$$

We will seek the solution in the form of $u = e^{A(t,T_0) - B(t,T_0)p}$ so that $\hat{X}(r, p, t) = \hat{X}(r, p, T_{0-})Z(t, T_0)e^{A(t,T_0) - B(t,T_0)p}$. Noting the similarity of the equation with (4.9), we have the following solution.

Theorem 11. Under Assumptions 1 though 4, the price of the credit default swaption is given by

$$\hat{X}(r, p, t) = \hat{X}(r, p, T_{0-})Z(t, T_0)e^{A(t,T_0) - B(t,T_0)p} \tag{5.8}$$

where $\hat{X}(r,p,T_{0-})$ and $Z(t,T_0)$ are given by (5.5) and (4.2) respectively, and $A(t,T_0)$ and $B(t,T_0)$ are given by

$$A(t,T_0) = -\int_t^{T_0} \big(b(s)B(s,T_0)) - \frac{1}{2}\Lambda(s)B^2(s,T_0)\big)\, ds$$

$$B(t,T_0) = \begin{cases} \dfrac{1-e^{-\gamma(T_0-t)}}{\gamma}, & dp = \big(\alpha(t) - \gamma \cdot p\big)dt + \sqrt{\delta(t)}dW_2, \gamma \neq 0 \\ T_0 - t, & dp = \big(\alpha(t) - \gamma \cdot p\big)dt + \sqrt{\delta(t)}dW_2, \gamma = 0 \\ \sqrt{\dfrac{2}{\gamma}} \cdot \frac{\exp\left(\sqrt{\frac{\gamma}{2}}(T_0-t)\right) - \exp\left(-\sqrt{\frac{\gamma}{2}}(T_0-t)\right)}{\exp\left(\sqrt{\frac{\gamma}{2}}(T_0-t)\right) + \exp\left(-\sqrt{\frac{\gamma}{2}}(T_0-t)\right)}, & \\ & dp = \alpha(t)dt + \sqrt{\delta(t) + K \cdot p} \cdot dW_2, K : \text{constant} \end{cases} \quad \square$$

5.2 Credit Default Swaption with Stochastic Interest Rate - Expected and Unexpected Default

5.2.1 Formulation

Assumption 1: We assume that the same setting for the credit default swaption as in Section 5.1: its inception and expiration, the onset and fee structure of the underlying forward credit default swap.

Assumption 2: The risk free short term interest rate $r(t) = r_t$ follows the Vasicek model:

$$dr = \theta(\nu(t) - r(t))dt + s_r(t)dW_1(t)$$

where θ, $\nu(t)$, and $s_r(t)$ are defined as in section 4.1.

Assumption 3: Default event is both exogenous and endogenous. Unexpected default probability on any interval $[t, t + dt]$ is given by,

$$dp = a_p(p,t)dt + s_p(p,t)dW_2 + p_{\tau_j-}U_j I_{\{\tau_j \in (t,t+dt]\}},$$

where $I_{\{\tau_j \in (t,t+dt]\}}$ is an indicator function. Expected default occurs when the firm assets $V = V(t) = V_t$ falls below the barrier, say $V_b(t)$. As in Section 4.2, the firm assets $V(t)$ is the sum of its coupon bonds (whose price is $C(r,p,V,t)$) and stocks, and follows the geometric Brownian motion, given by

$$dV = a_V V_t dt + s_V V_t dW_3.$$

Default recovery is given as the form of face value exogenous recovery ($R_d = R{\cdot}Z, R : 0 \leq R \leq 1$: constant, where Z is the price of risk free zero coupon bond, given by (4.2).

Assumption 4:

$$dW_i \cdot dW_j = \rho_{ij} dt, \quad i = 1, 2, 3.$$

However, we assume that the unexpected default and expected default are not correlated, i.e., $\rho_{23} = 0$.

Problem: Under these setting and assumptions, we will find the price of the credit default swaption $X = X(r, p, V, t)$.

5.2.2 Derivation of the Model

We construct a portfolio by hedging $X(r, p, V, t)$ with the reference entity's coupon-bonds with expected and unexpected default, the reference entity's stock, and zero-coupon default-free bond, to get rid of the risk arisen by p, V, and r. So the value of the portfolio is:

$$\begin{aligned} \Pi &= X - \Lambda_1 S - \Lambda_2 C - \Lambda_3 Z \\ &= X - \Lambda_1 \left(\frac{V - nC}{m} \right) - \Lambda_2 C - \Lambda_3 Z \\ &= X - \frac{\Lambda_1}{m} V - \left(\frac{n}{m} \Lambda_1 - \Lambda_2 \right) C - \Lambda_3 Z. \end{aligned}$$

So the change of value of this portfolio over a small time increment $[t, t + dt]$ is given by

$$d\Pi = dX - \frac{\Lambda_1}{m} dV - \left(\frac{n}{m} \Lambda_1 - \Lambda_2 \right) dC - \Lambda_3 dZ.$$

If there is no default over a small time increment $[t, t+dt]$ (with probability $1 - p_t dt$), then by Ito Lemma, the change of value in the portfolio over this period is given by

$$\begin{aligned}
d\Pi = & \frac{\partial X}{\partial t}dt + \frac{\partial X}{\partial p}dp + \frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial V}dV \\
& + \frac{1}{2}\left\{ s_p^2 \frac{\partial^2 X}{\partial p^2} + s_r^2 \frac{\partial^2 X}{\partial r^2} + s_V^2 V^2 \frac{\partial^2 X}{\partial V^2} + 2\rho_{12} s_r s_p \frac{\partial^2 X}{\partial r \partial p} + 2\rho_{13} s_r s_V V \frac{\partial^2 X}{\partial r \partial V} \right\} dt \\
& - \frac{\Lambda_1}{m} V - \Big(\frac{n}{m}\Lambda_1 - \Lambda_2 \Big) \Big[\frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial p}dp + \frac{\partial C}{\partial r}dr + \frac{\partial C}{\partial V}dV \\
& + \frac{1}{2}\left\{ s_p^2 \frac{\partial^2 C}{\partial p^2} + s_r^2 \frac{\partial^2 C}{\partial r^2} + s_r^2 \frac{\partial^2 C}{\partial r^2} + s_V^2 V^2 \frac{\partial^2 C}{\partial V^2} + 2\rho_{12} s_r s_p \frac{\partial^2 \hat{C}}{\partial r \partial p} + 2\rho_{13} s_r s_V V \frac{\partial^2 \hat{C}}{\partial r \partial V} \right\} dt \\
& + \{ \hat{C}(r, p_{\tau_j}, V_{\tau_j}, t) - \hat{C}(r, p_{\tau_{j_}}, V_{\tau_{j_}}, t) \} I_{\{\tau_j \in (t, t+dt]\}} \Big] \\
& - \Lambda_3 \Big[\frac{\partial Z}{\partial t}dt + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2} dt + \frac{\partial Z}{\partial r}dr \Big].
\end{aligned}$$

To get rid of the uncertainty caused by dp, dr and dV terms, we choose Λ_1, Λ_2 and Λ_3 as follows:

$$\begin{aligned}
\Lambda_1 &= m \Big[\frac{\partial \hat{X}}{\partial V} - \frac{\partial X}{\partial p} \Big(\frac{\partial C}{\partial p} \Big)^{-1} \frac{\partial C}{\partial V} \Big], \\
\Lambda_2 &= n \frac{\partial X}{\partial V} - \frac{\partial X}{\partial p} \Big(\frac{\partial C}{\partial p} \Big)^{-1} \Big[1 + n \frac{\partial C}{\partial V} \Big], \\
\Lambda_3 &= \Big(\frac{\partial Z}{\partial r} \Big)^{-1} \Big[\frac{\partial X}{\partial r} - \frac{\partial X}{\partial p} \Big(\frac{\partial C}{\partial p} \Big)^{-1} \frac{\partial C}{\partial r} \Big].
\end{aligned}$$

Then, we have

$$\begin{aligned}
d\Pi = & \Big[\frac{\partial X}{\partial t} + \frac{1}{2}\Big\{ s_p^2 \frac{\partial^2 X}{\partial p^2} + s_r^2 \frac{\partial^2 X}{\partial r^2} + s_V^2 V^2 \frac{\partial^2 X}{\partial V^2} + 2\rho_{12} s_r s_p \frac{\partial^2 X}{\partial r \partial p} + 2\rho_{13} s_r s_V V \frac{\partial^2 X}{\partial r \partial V} \Big\} \Big] dt \\
& - \frac{\partial X}{\partial p} \Big(\frac{\partial C}{\partial p} \Big)^{-1} \Big[\Big\{ \frac{\partial C}{\partial t} + \frac{1}{2} \Big(s_p^2 \frac{\partial^2 C}{\partial p^2} + s_r^2 \frac{\partial^2 C}{\partial r^2} + s_V^2 V^2 \frac{\partial^2 C}{\partial V^2} + 2\rho_{12} s_r s_p \frac{\partial^2 C}{\partial r \partial p} \\
& + 2\rho_{13} s_r s_V V \frac{\partial^2 C}{\partial r \partial V} \Big) \Big\} dt + \{ C(r, p_{\tau_j}, V_{\tau_j}, t) - C(r, p_{\tau_{j_}}, V_{\tau_{j_}}, t) \} I_{\{\tau_j \in (t, t+dt]\}} \Big] \\
& - \Big(\frac{\partial Z}{\partial r} \Big)^{-1} \Big[\frac{\partial \hat{X}}{\partial r} - \frac{\partial X}{\partial p} \Big(\frac{\partial \hat{C}}{\partial p} \Big)^{-1} \frac{\partial C}{\partial r} \Big] \Big[\frac{\partial Z}{\partial t} dt + \frac{1}{2} s_r^2 \frac{\partial^2 Z}{\partial r^2} dt \Big]. \qquad (5.9)
\end{aligned}$$

If there is default with probability $p_t dt$ before the inception of the forward credit default swap, the swaption contract becomes void; therefore, we have:

$$\begin{aligned} d\Pi &= -X - \frac{\Lambda_1}{m}dV - \Big(\frac{n}{m}\Lambda_1 - \Lambda_2\Big)(R_d - C) - \Lambda_3 dZ \\ &= -X - \bigg[\frac{\partial X}{\partial V} - \frac{\partial X}{\partial p}\Big(\frac{\partial \hat{C}}{\partial p}\Big)^{-1}\frac{\partial C}{\partial V}\bigg]dV - \frac{\partial X}{\partial p}\Big(\frac{\partial \hat{C}}{\partial p}\Big)^{-1}(R_d - C) \\ &\quad -\Big(\frac{\partial Z}{\partial r}\Big)^{-1}\bigg[\frac{\partial \hat{X}}{\partial r} - \frac{\partial \hat{X}}{\partial p}\cdot\Big(\frac{\partial \hat{C}}{\partial p}\Big)^{-1}\frac{\partial \hat{C}}{\partial r}\bigg]dZ. \end{aligned} \tag{5.10}$$

Then, by the arbitrage principle, we must have $(5.9)\times(1-p_t dt)+(5.10)\times p_t dt = r\Pi dt$. We have

$$\begin{aligned} &\bigg[\frac{\partial X}{\partial t} + \frac{1}{2}\bigg\{s_p^2\frac{\partial^2 X}{\partial p^2} + s_r^2\frac{\partial^2 X}{\partial r^2} + s_V^2 V^2\frac{\partial^2 X}{\partial V^2} + 2\rho_{12}s_r s_p\frac{\partial^2 X}{\partial r\partial p} + 2\rho_{13}s_r s_V\frac{\partial^2 X}{\partial r\partial V}\bigg\} - (p_t + r)X\bigg]dt \\ &\quad -\frac{\partial X}{\partial p}\Big(\frac{\partial C}{\partial p}\Big)^{-1}\bigg[\bigg\{\frac{\partial C}{\partial t} - rV\frac{\partial C}{\partial V} + \frac{1}{2}\Big(s_p^2\frac{\partial^2 C}{\partial p^2} + s_r^2\frac{\partial^2 C}{\partial r^2} + s_V^2 V^2\frac{\partial^2 C}{\partial V^2} + 2\rho_{12}s_r s_p\frac{\partial^2 C}{\partial r\partial p} \\ &\quad +2\rho_{13}s_r s_V\frac{\partial^2 C}{\partial r\partial V}\Big) + (R_d - C)p_t - rC\bigg\}dt \\ &\quad +\{C(r, p_{\tau_j}, V_{\tau_j}, t) - C(r, p_{\tau_{j\_}}, V_{\tau_{j\_}}, t)\}I_{\{\tau_j\in(t,t+dt]\}}(1 + p_t dt)\bigg] \\ &\quad -\Big(\frac{\partial Z}{\partial r}\Big)^{-1}\bigg[\frac{\partial X}{\partial r} - \frac{\partial X}{\partial p}\Big(\frac{\partial C}{\partial p}\Big)^{-1}\frac{\partial C}{\partial r}\bigg]\bigg[\frac{\partial Z}{\partial t} + \frac{1}{2}s_r^2\frac{\partial^2 Z}{\partial r^2} - rZ\bigg]dt = 0. \end{aligned}$$

Noting that, by (4.18) and (4.6), for the time interval such that $\tau_j \notin [t, t+dt]$,

$$\begin{aligned} &\frac{\partial C}{\partial t} - rV\frac{\partial C}{\partial V} + \frac{1}{2}\Big(s_p^2\frac{\partial^2 C}{\partial p^2} + s_r^2\frac{\partial^2 C}{\partial r^2} + s_V^2 V^2\frac{\partial^2 C}{\partial V^2} + 2\rho_{12}s_r s_p\frac{\partial^2 C}{\partial r\partial p} \\ &\qquad +2\rho_{13}s_r s_V\frac{\partial^2 C}{\partial r\partial V}\Big) + (R_d - C)p_t - rC = -\theta(\nu(t) - r)\frac{\partial C}{\partial r} - a_p\frac{\partial C}{\partial p}, \quad \text{and} \\ &\frac{\partial Z}{\partial t} + \frac{1}{2}s_r^2\frac{\partial^2 Z}{\partial r^2} - rZ = -\theta(\nu(t) - r)\frac{\partial \hat{Z}}{\partial r}. \end{aligned} \tag{5.11}$$

(5.11) becomes

$$\begin{aligned} &\frac{\partial X}{\partial t} + a_p\frac{\partial X}{\partial p} + \theta(\nu(t) - r)\frac{\partial X}{\partial r} + rV\frac{\partial X}{\partial V} \\ &\quad +\frac{1}{2}\bigg\{s_p^2\frac{\partial^2 X}{\partial p^2} + s_r^2\frac{\partial^2 X}{\partial r^2} + s_V^2 V^2\frac{\partial^2 X}{\partial V^2} + 2\rho_{12}s_r s_p\frac{\partial^2 X}{\partial r\partial p} + 2\rho_{13}s_r s_V V\frac{\partial^2 X}{\partial r\partial V}\bigg\} \\ &\qquad -(p_t + r)X = 0. \end{aligned} \tag{5.12}$$

Since the present value of the forward credit default swap is given by (5.1), the value of the swaption at $t = T_0$ is given by

$$\begin{aligned} X(T_0) &= [\mathcal{Z}(T_0) - \mathcal{Z}^*]^+ \\ &= \big[E[b(\tau)(1-R)\cdot I_{\{\tau\le T_N\}}] \\ &\qquad -z^*\cdot E[\sum_{i,T_i>t}^{N} b(T_i)\Lambda_i I_{\{T_i\le\tau\}} + b(\tau)\Lambda^* I_{\{T_0\le\tau\le T_N\}}]\big]^+. \end{aligned}$$

As before, from the financial point of view, we can expect the value of the swaption right before the jump (i.e., at $t = T_{0-}$), to be the expectation of the value at $t = T_0$. So we can assume that the terminal condition as follows:

$$
\begin{aligned}
X(T_{0-}) \quad = \quad & S(T_{0-}, T_0) \cdot E\Bigg[\big[E[b(\tau)(1-R) \cdot I_{\{\tau \leq T_N\}}] \\
& -z^* \cdot E[\sum_{i, T_i > t}^{N} b(T_i)\Lambda_i I_{\{T_i \leq \tau\}} + b(\tau)\Lambda^* I_{\{T_0 \leq \tau \leq T_N\}}]\big]^+\Bigg]
\end{aligned}
$$

where $P(\cdot, \cdot)$ is defined by (2.4).

Therefore, we have the following pricing model:

Theorem 12. Under Assumptions 1 through 4, the price of credit default swaption with expected and unexpected default probability is modeled as follows:

$$
\begin{cases}
\dfrac{\partial X}{\partial t} + a_p \dfrac{\partial X}{\partial p} + \theta(\nu(t) - r)\dfrac{\partial X}{\partial r} + rV\dfrac{\partial X}{\partial V} \\
\quad + \dfrac{1}{2}\left\{ s_p^2 \dfrac{\partial^2 X}{\partial p^2} + s_r^2 \dfrac{\partial^2 X}{\partial r^2} + s_V^2 V^2 \dfrac{\partial^2 X}{\partial V^2} + 2\rho_{12} s_r s_p \dfrac{\partial^2 X}{\partial r \partial p} + 2\rho_{13} s_r s_V V \dfrac{\partial^2 X}{\partial r \partial V} \right\} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -(p_t + r)X = 0 \\
X(T_{0-}) = S(T_{0-}, T_0) \cdot E\Bigg[\big[E[b(\tau)(1-R) \cdot I_{\{\tau \leq T_N\}}] \\
\qquad\qquad -z^* \cdot E[\sum_{i, T_i > t}^{N} b(T_i)\Lambda_i I_{\{T_i \leq \tau\}} + b(\tau)\Lambda^* I_{\{T_0 \leq \tau \leq T_N\}}]\big]^+\Bigg].
\end{cases}
$$

5.2.3 Particular Solution

Under this subsection, we will find a particular solution with additional condition.
Using the change of unknown function

$$
\hat{X}(r, p, V, t) = X(T_{0-})W(r, p, V, t),
$$

(5.12) becomes the following PDE problem with the terminal condition.

$$\begin{cases} \dfrac{\partial W}{\partial t} + a_p \dfrac{\partial W}{\partial p} + \theta(\nu(t) - r)\dfrac{\partial W}{\partial r} + +rV\dfrac{\partial W}{\partial V} \\ \qquad +\dfrac{1}{2}\left\{ s_p^2 \dfrac{\partial^2 W}{\partial p^2} + s_r^2 \dfrac{\partial^2 X}{\partial r^2} + s_V^2 V^2 \dfrac{\partial^2 W}{\partial V^2} + 2\rho_{12} s_r s_p \dfrac{\partial^2 W}{\partial r \partial p} \right. \\ \qquad \left. +2\rho_{13} s_r s_V V \dfrac{\partial^2 W}{\partial r \partial V} \right\} - (p_t + r)W = 0, \qquad (0 \le t < T_0, p > 0, r > 0, V > V_B) \\ W(r, p, V, T_{0-}) = 1, \qquad (p > 0, r > 0, V > V_B) \\ W(r, p, V_B, t) = 0, \qquad (0 \le t < T_0, p > 0, r > 0). \end{cases} \tag{5.13}$$

To solve (5.13) as the terminal condition, we use the change of unknown function and variable as follows:

$$x = \frac{V}{Z} \quad \text{and} \quad u(x, p, t) = \frac{W(r, V, p, t)}{Z}.$$

Then, (5.14) reduces to

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\left\{ [s_r^2 \bar{B}^2(t) + s_V^2 + 2\rho_{13} s_r s_V \bar{B}(t)] x^2 \dfrac{\partial^2 u}{\partial x^2} + s_p^2(p, r, t)\dfrac{\partial^2 u}{\partial p^2} + 2\rho_{12} s_r s_p \bar{B}(t) x \dfrac{\partial^2 u}{\partial p \partial x} \right\} \\ \quad +[a_p - 2\rho_{12} s_r s_p \bar{B}(t)]\dfrac{\partial u}{\partial p} - pu = 0, \qquad (x > V_B, p > 0, 0 \le t < T_0) \\ u(x, p, T_{0-}) = 1, \qquad (x > V_B, p > 0) \\ u(V_B, p, t) = 0, \qquad (p > 0, 0 \le t < T_0). \end{cases}$$

We will consider the case where $\beta(t) = \epsilon(t) \equiv 0$ and $\rho_{13} = 0$. Letting

$$\bar{s}^2(t) = s_r^2 \bar{B}^2(t) + s_V^2 + 2\rho_{13} s_r s_V \bar{B}(t),$$
$$\bar{\rho}(p, t) = \rho_{12} s_r s_p \bar{B}(t), \text{and}$$
$$\bar{a}_p(p, t) = a_p(r, t) - 2\rho_{12} s_r s_p \bar{B}(t),$$

we have

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\left\{ \bar{s}^2(t) x^2 \dfrac{\partial^2 u}{\partial x^2} + s_p^2(p, t)\dfrac{\partial^2 u}{\partial p^2} + 2\bar{\rho} x \dfrac{\partial^2 \hat{u}}{\partial p \partial x} \right\} \\ \quad +\bar{a}_p(p, t)\dfrac{\partial u}{\partial p} - pu = 0, \qquad (x > V_B, p > 0, 0 \le t < T_0) \\ u(x, p, T_{0-}) = 1 \qquad (x > V_B, p > 0) \\ u(V_B, p, t) = 0, \qquad (p > 0, \tau_{n-1} \le t < T). \end{cases}$$

Noting that the above PDE is the same as (4.20) we have the price of the swaption as follows:

Theorem 13. Under the Assumptions 1 through 4, the price of the credit default swaption for the corporate coupon-bond with expected and unexpected default is given by

$$
\begin{aligned}
X(r,p,V,t) &= Z(t,T_0)\cdot S(T_{0-},T_0)\cdot E\Bigg[\big[E[b(\tau)(1-R)\cdot I_{\{\tau\le T_N\}}] \\
&\quad -z^*\cdot E[\sum_{i,T_i>t}^{N} b(T_i)\Lambda_i I_{\{T_i\le\tau\}} + b(\tau)\Lambda^* I_{\{T_0\le\tau\le T_N\}}]\big]^+\Bigg] \\
&\quad \left(\Phi(d_1^*) - \frac{x}{V_B}\Phi(d_2)\right)\cdot e^{A(t,T_0)-B(t,T_0)p}
\end{aligned}
\tag{5.14}
$$

where

$$
\begin{aligned}
d_1^* &= \frac{\ln\frac{V}{V_B Z(t,T_0)}}{\sqrt{\int_t^{T_0}\bar{s}^2(u)\,du}} - \frac{\sqrt{\int_t^{T_0}\bar{s}^2(u)\,du}}{2}, \quad \text{and} \\
d_2^* &= \frac{\ln\frac{V_B Z(t,T_0)}{V}}{\sqrt{\int_t^{T_0}\bar{s}^2(u)\,du}} - \frac{\sqrt{\int_t^{T_0}\bar{s}^2(u)\,du}}{2}.
\end{aligned}
\tag{5.15}
$$

Chapter 6

Data Analysis

In this chapter, we shall try to find the price of BB and BBB rated corporate coupon bond with unexpected default probability, and with the risk free interest rate r to be stochastic.

6.1 Data

Table 3 and Figure 2 shows the historical monthly data of 3-month treasury bill, and the historical yields of AAA, AA, A and BBB rated corporate-bonds from October 1991 through November/December 2000 (Source: Moody's Investors Service). We shall use the rate for 3-month treasury bill as the risk-free interest rate in percent.

For example, in the first row, the daily average of the yield-to-maturity of 3-month treasury bill in January 1991 was 6.41%, while the daily averages of the yield-to-maturity of AAA, AA, A, and BBB rated corporate coupon bonds in the same period were 9.04%, 9.37%, 9.61%, and 10.45% respectively.

Table 3: Historical Short-Term Rate and Yields of Corporate Bonds

Months	3M T-Bill	AAA	AA	A	BBB
Jan-91	6.41	9.04	9.37	9.61	10.45
Feb-91	6.12	8.83	9.16	9.38	10.07
Mar-91	6.09	8.93	9.21	9.5	10.09

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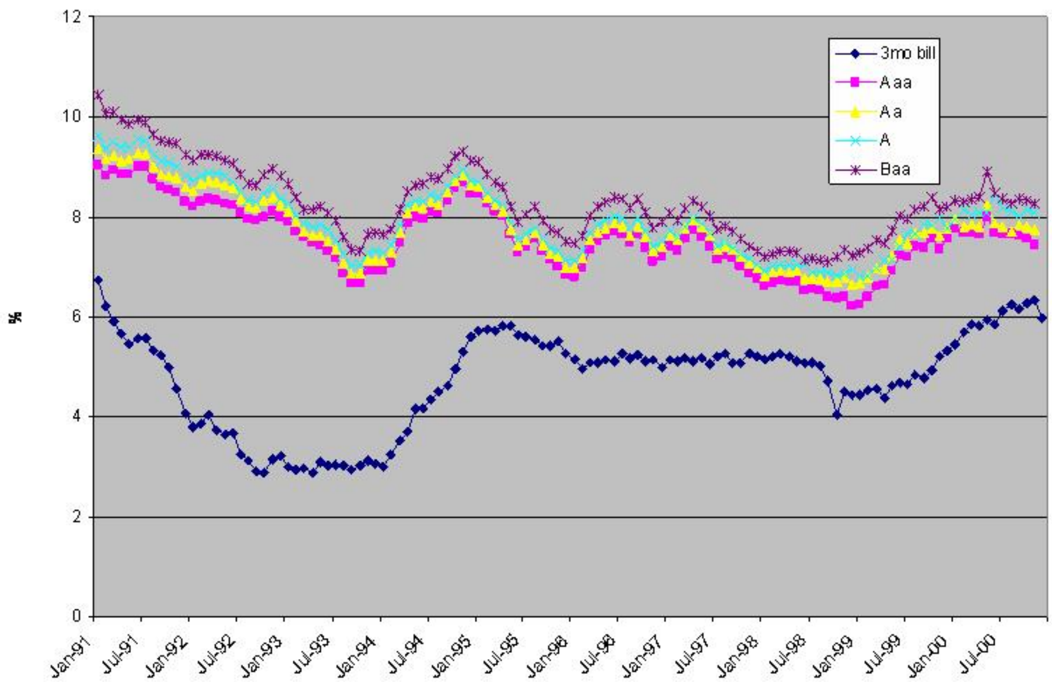

Figure 2.: Historical Risk-free Interest Rate and Corporate-Bond Yield

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Months	3M T-Bill	AAA	AA	A	BBB
Apr-91	5.83	8.86	9.12	9.39	9.94
May-91	5.63	8.86	9.15	9.41	9.86
Jun-91	5.75	9.01	9.28	9.55	9.96
Jul-91	5.75	9	9.25	9.51	9.89
Aug-91	5.50	8.75	8.99	9.26	9.65
Sep-91	5.37	8.61	8.86	9.11	9.51
Oct-91	5.14	8.55	8.83	9.08	9.49
Nov-91	4.69	8.48	8.78	9.01	9.45
Dec-91	4.18	8.31	8.61	8.82	9.26

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Months	3M T-Bill	AAA	AA	A	BBB
Jan-92	3.91	8.2	8.51	8.72	9.13
Feb-92	3.95	8.29	8.67	8.83	9.23
Mar-92	4.14	8.35	8.73	8.89	9.25
Apr-92	3.84	8.33	8.69	8.87	9.21
May-92	3.72	8.28	8.63	8.81	9.13
Jun-92	3.75	8.22	8.56	8.7	9.05
Jul-92	3.28	8.07	8.37	8.49	8.84
Aug-92	3.20	7.95	8.21	8.34	8.65
Sep-92	2.97	7.92	8.17	8.31	8.62
Oct-92	2.93	7.99	8.32	8.49	8.84
Nov-92	3.21	8.1	8.4	8.58	8.96
Dec-92	3.29	7.98	8.24	8.37	8.81
Jan-93	3.07	7.91	8.11	8.26	8.67
Feb-93	2.99	7.71	7.9	8.03	8.39
Mar-93	3.01	7.58	7.72	7.86	8.15
Apr-93	2.93	7.46	7.62	7.8	8.14
May-93	3.03	7.43	7.61	7.85	8.21
Jun-93	3.14	7.33	7.51	7.74	8.07
Jul-93	3.11	7.17	7.35	7.53	7.93
Aug-93	3.09	6.85	7.06	7.25	7.6
Sep-93	3.01	6.66	6.85	7.05	7.34
Oct-93	3.09	6.67	6.87	7.04	7.31
Nov-93	3.18	6.93	7.12	7.29	7.66
Dec-93	3.13	6.93	7.12	7.31	7.69
Jan-94	3.04	6.93	7.12	7.3	7.65
Feb-94	3.33	7.08	7.29	7.44	7.76

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Months	3M T-Bill	AAA	AA	A	BBB
Mar-94	3.59	7.48	7.69	7.82	8.13
Apr-94	3.78	7.88	8.08	8.22	8.52
May-94	4.27	7.99	8.19	8.32	8.62
Jun-94	4.25	7.97	8.17	8.31	8.65
Jul-94	4.46	8.11	8.31	8.44	8.8
Aug-94	4.61	8.07	8.25	8.38	8.74
Sep-94	4.75	8.34	8.49	8.61	8.98
Oct-94	5.10	8.57	8.71	8.82	9.2
Nov-94	5.45	8.68	8.83	8.94	9.32
Dec-94	5.76	8.46	8.62	8.73	9.11
Jan-95	5.90	8.46	8.6	8.7	9.08
Feb-95	5.94	8.26	8.39	8.48	8.85
Mar-95	5.91	8.12	8.24	8.33	8.7
Apr-95	5.84	8.03	8.12	8.23	8.6
May-95	5.85	7.65	7.74	7.86	8.2
Jun-95	5.64	7.3	7.43	7.53	7.9
Jul-95	5.59	7.41	7.54	7.65	8.04
Aug-95	5.57	7.57	7.69	7.79	8.19
Sep-95	5.42	7.32	7.45	7.56	7.93
Oct-95	5.44	7.12	7.27	7.39	7.75
Nov-95	5.52	7.02	7.18	7.32	7.68
Dec-95	5.29	6.82	6.99	7.13	7.49
Jan-96	5.15	6.81	6.99	7.12	7.47
Feb-96	4.96	6.99	7.16	7.31	7.63
Mar-96	5.10	7.35	7.52	7.68	8.03
Apr-96	5.09	7.5	7.68	7.83	8.19

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Months	3M T-Bill	AAA	AA	A	BBB
May-96	5.15	7.62	7.77	7.94	8.3
Jun-96	5.23	7.71	7.87	8.02	8.4
Jul-96	5.30	7.65	7.82	7.97	8.35
Aug-96	5.19	7.46	7.63	7.77	8.18
Sep-96	5.25	7.66	7.82	7.95	8.35
Oct-96	5.12	7.39	7.58	7.7	8.07
Nov-96	5.17	7.1	7.31	7.41	7.79
Dec-96	5.04	7.2	7.41	7.51	7.89
Jan-97	5.17	7.42	7.63	7.71	8.09
Feb-97	5.14	7.31	7.54	7.59	7.94
Mar-97	5.28	7.55	7.77	7.82	8.18
Apr-97	5.30	7.73	7.93	7.98	8.34
May-97	5.20	7.58	7.8	7.86	8.2
Jun-97	5.07	7.41	7.62	7.68	8.02
Jul-97	5.19	7.14	7.36	7.42	7.75
Aug-97	5.28	7.22	7.4	7.46	7.82
Sep-97	5.08	7.15	7.34	7.39	7.7
Oct-97	5.11	7	7.2	7.27	7.57
Nov-97	5.28	6.87	7.07	7.15	7.42
Dec-97	5.30	6.76	6.99	7.05	7.32
Jan-98	5.18	6.61	6.82	6.93	7.19
Feb-98	5.23	6.67	6.88	7.01	7.25
Mar-98	5.16	6.72	6.93	7.05	7.32
Apr-98	5.08	6.69	6.9	7.03	7.33
May-98	5.14	6.69	6.91	7.03	7.3
Jun-98	5.12	6.53	6.78	7.03	7.13

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Months	3M T-Bill	AAA	AA	A	BBB
Jul-98	5.09	6.55	6.78	6.88	7.15
Aug-98	5.04	6.52	6.77	6.89	7.14
Sep-98	4.74	6.41	6.69	6.89	7.09
Oct-98	4.07	6.37	6.69	6.82	7.18
Nov-98	4.53	6.41	6.79	6.85	7.34
Dec-98	4.50	6.22	6.65	6.95	7.23
Jan-99	4.45	6.24	6.68	6.8	7.29
Feb-99	4.56	6.4	6.79	6.84	7.39
Mar-99	4.57	6.62	6.98	6.97	7.53
Apr-99	4.41	6.64	6.96	7.14	7.48
May-99	4.63	6.93	7.23	7.13	7.72
Jun-99	4.72	7.23	7.52	7.4	8.02
Jul-99	4.69	7.19	7.48	7.69	7.95
Aug-99	4.87	7.4	7.68	7.65	8.15
Sep-99	4.82	7.39	7.68	7.84	8.2
Oct-99	5.02	7.55	7.79	7.84	8.38
Nov-99	5.23	7.36	7.62	7.99	8.15
Dec-99	5.36	7.55	7.78	7.79	8.19
Jan-00	5.50	7.78	7.96	7.96	8.33
Feb-00	5.73	7.68	7.82	8.15	8.29
Mar-00	5.86	7.68	7.83	8.06	8.37
Apr-00	5.82	7.64	7.82	8.07	8.4
May-00	5.99	7.99	8.24	8.07	8.9
Jun-00	5.86	7.67	7.87	8.49	8.48
Jul-00	6.14	7.65	7.81	8.18	8.35
Aug-00	6.28	7.65	7.7	8.11	8.26

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β_1	-0.084292805	Mean reversion rate	θ	1.01151
β_0	0.003526218	Reverting mean	ν	4.1833%
Standard error	0.00570581	Volatility	s_r	0.571%

Table 4: Parameters implied from the regression estimates

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Months	3M T-Bill	AAA	AA	A	BBB
Sep-00	6.18	7.62	7.83	8.02	8.35
Oct-00	6.29	7.55	7.81	8.13	8.34
Nov-00	6.36	7.45	7.75	8.09	8.28
Dec-00	5.94	7.21			

As we introduced in Chapter 4, we shall adopt Vasicek model for risk-free short term interest rate, i.e.,

$$dr_t = \theta(\nu(t) - r_t)dt + s_r dW_1(t). \tag{6.1}$$

where θ and s_r is a constant, and $\nu(t)$ is a deterministic function of t. We follow the method introduced by Chang (2006) to obtain the implied parameters. Using the data from Table 3, we run the following regression:

$$dr_t = \beta_0 + \beta_1 r_t + \epsilon. \tag{6.2}$$

Table 4 summarizes the implied parameters by this linear regression estimates. The mean reversion rate is calculated as a negative slope from $\beta_1 = -\theta dt$ and the reverting mean is calculated from $\beta_0 = \theta\nu dt$. The inverse of the mean reversion rate $(1/\lambda)$ can be interpreted as the number of periods elapsed between reversion, or speed of reversion. The reverting mean ν represents the level which the risk-free interest rate reverts to after wandering off. The volatility σ here is annualized standard deviation of the risk-free interest rate. In this case, we can expect that on the average $0.9886 (= 1/1.01151)$ year elapses between reversions, the level of the risk-free interest rate reverts to is 4.1833%, and the annualized standard deviation is 0.571%.

Therefore, we estimate the short-term rate by the following equation and Figure 3 below shows a simulation of this mean reverting model.

$$dr = 1.01151(0.04133 - r_t)dt + 0.00570581dW \tag{6.3}$$

where t is in years.

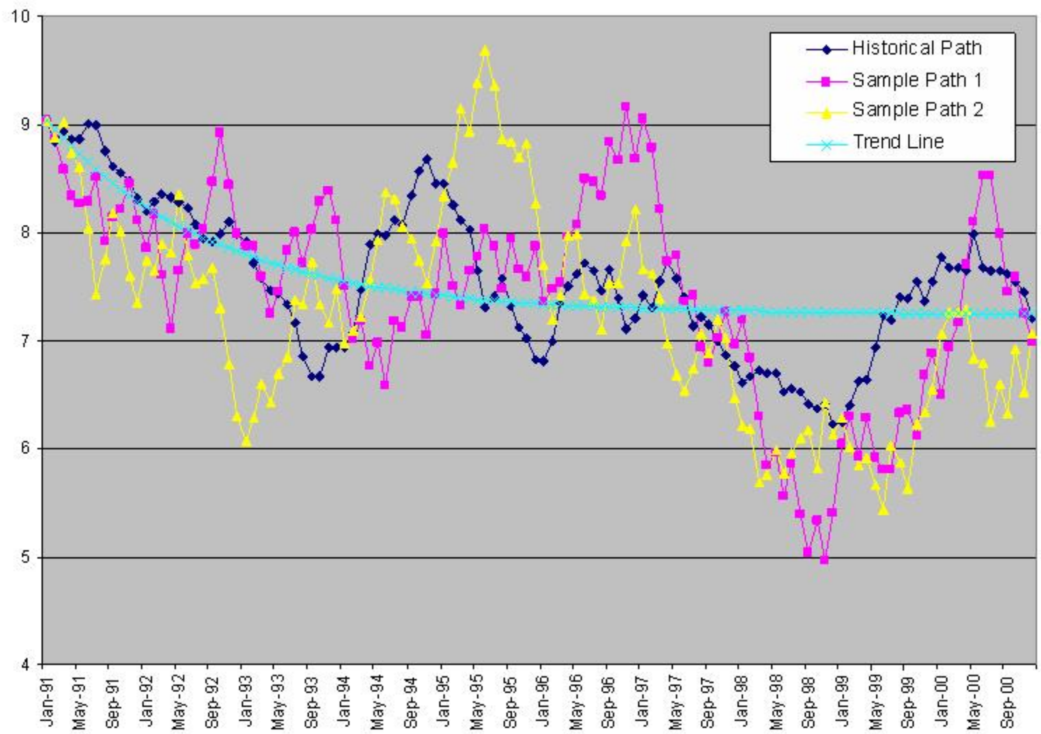

Figure 3.: Historical and Sample Paths of Risk-free Interest Rate

Now Table 5 shows empirical survival rate of the corporate bonds by the original credit quality constructed from historical bond default data for 1991-2000.

Each number in Table 5 gives the probability that the bond survives till the time (in years) elapsed. We use the data for BBB and BB. First, using the curve-fitting software DataFit (available at http://www.curvefitting.com/index.html), the function $p(t)$ without jump was modeled. Data was regressed to several twice differentiable functions, i.e., polynomials of degree two through four, exponential functions, reciprocal functions of polynomials of degree one and two, and the following result was obtained for BBB-rated companies with $R^2 = .9971$:

Table 5: Empirical Survival Rate by Original Credit Quality

Years	AAA	AA	A	BBB	BB	B	CCC
1	1.0000	1.0000	1.0000	0.9988	0.9904	0.9840	0.9565
2	1.0000	1.0000	1.0000	0.9940	0.9741	0.9354	0.8297
3	1.0000	0.9965	0.9998	0.9886	0.9350	0.8797	0.6900
4	1.0000	0.9946	0.9991	0.9827	0.9288	0.8215	0.6338
5	0.9997	0.9946	0.9988	0.9772	0.9088	0.7727	0.6147
6	0.9997	0.9946	0.9980	0.9715	0.9002	0.7406	0.5585
7	0.9997	0.9946	0.9975	0.9645	0.8853	0.7175	0.5330
8	0.9997	0.9946	0.9966	0.9630	0.8813	0.7024	0.5156
9	0.9997	0.9943	0.9960	0.9625	0.8659	0.6908	0.5156
10	0.9997	0.9941	0.9960	0.9602	0.8334	0.6849	0.4942

$$p(t) = p_0 + \int_0^t a_p(p,t)dt + \int_0^t s_p(p,t)dW_2 \tag{6.4}$$

with

$$\begin{aligned} p_0 &= 0, \\ a_p(p,t) &= 0.0000674957t^2 - 0.001170141t + 0.003576423 \\ s_p(p,t) &= 0.001506268. \end{aligned} \tag{6.5}$$

Figure 4 shows some samples paths with jumps of the intensity p_t, each represented by the sequence of squares, pluses, and diamonds, computed from (6.4) and (6.5). Here, for the seasonal jumps U_j on coupon payment dates (at the end of multiples of 6 months), we assume that it is constant $U_j = -.1$.

In the same manner, the parameters of the default intensity for BB-rated companies was found to be

$$\begin{aligned} p_0 &= 0, \\ a_p(p,t) &= -0.004114407675t^2 - 0.03025322658t \\ s_p(p,t) &= 0.174409213 \end{aligned} \tag{6.6}$$

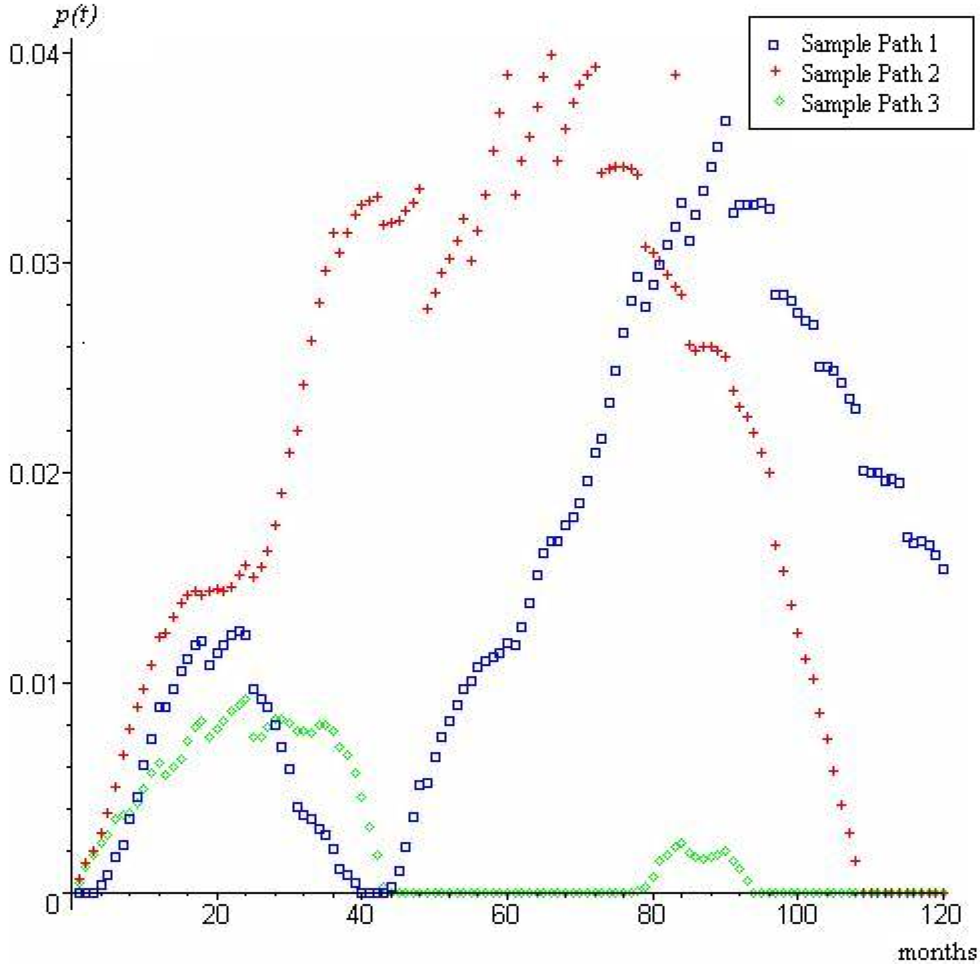

Figure 4.: Sample Paths of the Intensity p

with $R^2 = .9207$.

6.2 Bond Price

In this section, using the data obtained in the previous section and the models derived in subsections 2.1.2 and 4.1.2, the price of nonsecured coupon bonds of Ford Motor Credit Company with unexpected default only was computed. The terms and conditions of the relative bonds are

Issue date: November 21, 1999

Issue price: 99.812%

Coupon: 7.375% s.a. (due on February and October)

Maturity: October 28, 2009

The actual market price range of these bonds in Year 2002 was 88.45% and 104.28%, and the 3-month T-bill was 1.76% during February 2002. Table 6 shows the bond price computed with r to be constant.

Table 6: Ford Motor Credit Bond Price with Constant r

$r(\%)$	Price
1.76	133.89%
4.82	112.29%

As it can be easily seen, the price computed using the model with $r = 1.76\%$, which is the short term interest rate as of February 2002, is overpriced compared to the actual market price. Even when the average short term rate over 1991 through 2001, which is $r = 4.82\%$, it is still overpriced. At least a couple of reasons for this can be considered. First, even though the short term rate from February 2002 was used, the prospect of the short-term rate was unseen at this point. Actually, the short term rate drastically dropped at the beginning of Year 2002, after which it gradually came back. (Therefore, the stochastic model is preferred.) Second, even the average yield of AAA corporate bonds during the same period was $200 - 300pb$ higher than the short terms rate. It should be also mentioned that the long term rate is more stable than short term rate, which implies that the long term bond price is less sensitive to the change in short term rate.

Since the default intensity gives the default probability over the default free bonds, if we compute the bond price using the yield of AAA bonds as risk-free interest rate, instead of short term rate. For $r = 7.55\%$, which is the average over 1991-2000, we obtained the price of 96.32%; and for $r = 6.51\%$, which is the value as of February 2002, the price was estimated to be 102.08%, both of which are more realistic than using the rate of 3-month T-bill.

Next, the price of the same corporate bonds was computed using stochastic process for r. However, as seen in the pricing with constant r, using short term rate overprices the bonds, giving 124.99%.

We ran the linear regression on the yields of AAA corporate bonds from January 1991 to December

2000 and obtained the following result.

Table 7: Parameters implied from the regression estimates - AAA corporate bonds

β_1	-0.048039321	Mean reversion rate	θ	0.576471853
β_0	0.003470349	Reverting mean	ν	7.22%
Standard error	0.001648158	Volatility	s_r	0.165%

Therefore, we estimate the yield of AAA corporate bonds by the following equation, and Figure 5 shows the historical and sample path of the yield of AAA corporate bonds.

$$dr_{AAA} = 0.576471853(.07224 - r)dt + 0.001648158dW. \tag{6.7}$$

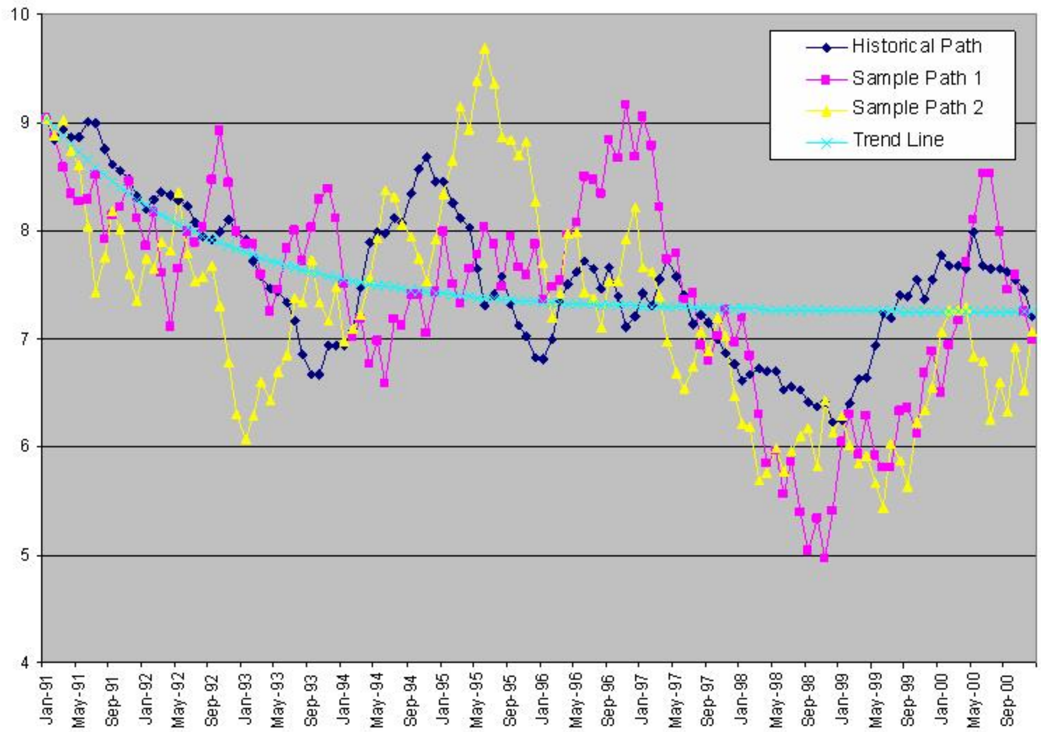

Figure 5.: Historical and Sample Paths of the Yield of AAA Corporate-Bonds

However, using this equation results in the bond price of 120.33%. Considering using the constant r resulted in a more realistic value, it can be assessed that the problem lies in the analysis of the stochastic process of the yield of the AAA corporate bonds. It is obvious that the yield does not rise forever or fall until it gets zero. So it is plausible to consider the mean reverting model and using the linear model for the variable ν. However, the long term average of the yield in this stochastic process was 7.22% compared to 7.54%, the arithmetic mean for the same period, reflecting the historical path. However, we do not know the sentiment of the market just from this data, which might predict the rise in the long-term yield.

6.3 Credit Default Swaption Price

In this subsection, we shall compute the price of the credit default swaption of Ford Motor Credit Company on the following terms, based on the pricing of the bonds computed in the previous subsection. Since using the 3-month T-bill for the risk free interest rates gives overpricing of the bonds, we shall use AAA-bond yield as the risk free interest rate.

Reference entity: Ford Motor Credit Company

Onset of the swaption: March, 2004

Expiration of the swaption: September, 2004

Duration of the underlying CDS: 1 year, 3 years, 5 years

Table 8 gives the summary of the pricing:

Table 8: Ford Motor Credit - CDS Swaption Price

Risk free interest rate	1 Year		3 Years		5 Years	
%	z^*=0bp	z^*=50bp	z^*=0bp	z^*=50bp	z^*=0 bp	z^*=50bp
7.22 (AAA yield average on 1991-2000)	30.22bp	15.11bp	110.63bp	55.32bp	163.88bp	81.94bp
5.33 (AAA yield as of March 2004)	31.26bp	15.63bp	116.78bp	58.38bp	175.43bp	87.72
Stochastic model	30.28bp	15.14bp	113.68bp	56.84bp	172.53bp	86.27bp

For example, the right column under 1 year gives the price of CDS swaption, which entitles its holder to enter a CDS agreement for paying $z^* = 50bp$ every 6 months. Based on the stochastic model, such price is $15.14bp$.

Note that if $z^* = 0bp$, the price of the swaption is theoretically the same as that of the forward swaption. Since there is no market for forward CDS option or CDS swaption, we shall compare the price of the swaption with $z^* = 0bp$, and the price of CDS forward option computed using the actual CDS price (midprice; see Figure 6), discounted using the discount rate based on the computation in the previous section. Table 9 below shows the comparison between the forward CDS option price based on our model and that based on the actual CDS price.

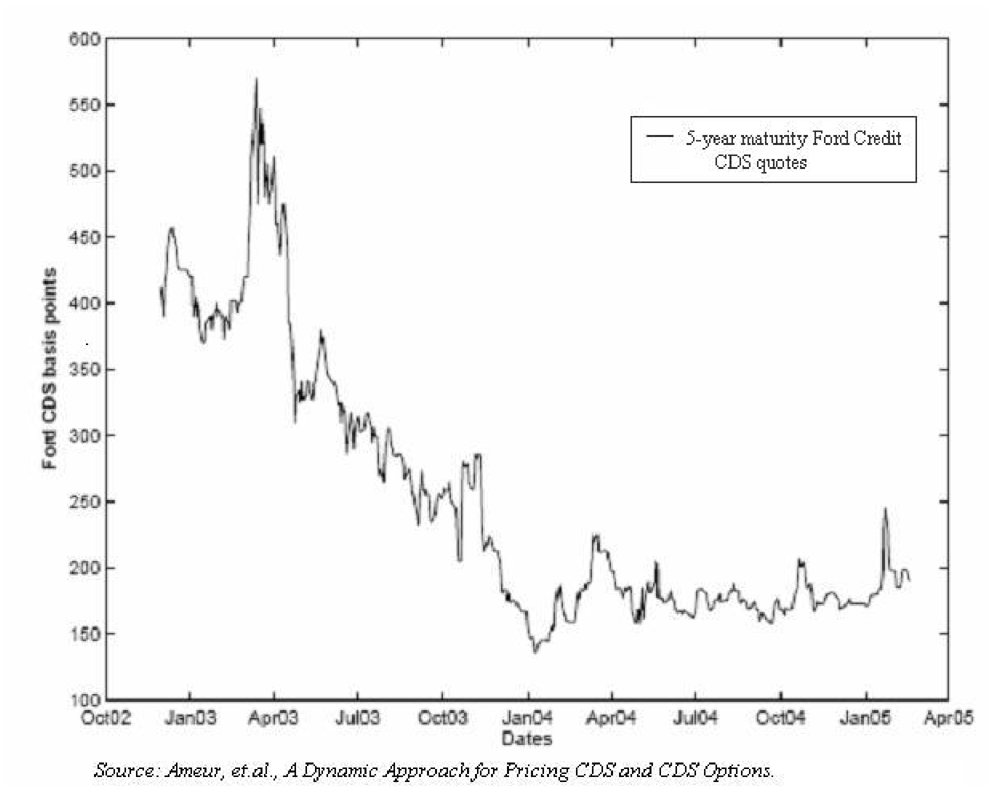

Figure 6.: 5-Year-Maturity FORD Credit CDS Quotes

In using our model, the coupon payment date of the bonds of the reference entity should technically match the onset and maturity of the swaption, and maturity of the underlying forward CDS. However, in computing the above, since the matching CDS rate was not available, we could not match them (there is one month gap between them).

Table 9: Ford Motor Credit - CDS Forward Price Comparison

Risk free interest rate %	1 Year		3 Years		5 Years	
	Our Model	Estimated from CDS price	Our Model	Estimated from CDS price	Our Model	Estimated from CDS price
7.22 (AAA yield average on 1991-2000)	30.22bp	36.00bp	110.63bp	126.39bp	163.88bp	163.59bp
5.33 (AAA yield as of March 2004)	31.26bp	36.30bp	116.78bp	127.43bp	175.43bp	164.94
Stochastic model	30.28bp	36.73bp	113.68bp	128.94bp	172.53bp	166.89bp

We can see that that the forward CDS option price obtained from our model is relatively close to that obtained from the market rate.

Chapter 7

Conclusion

In this paper, we combined the intensity model and the structural model to find the price of the corporate coupon- bearing bonds. We formed the portfolio so that we could hedge the risk caused by default intensity and/or fluctuation of the asset value. We used the arbitrage principle and the Ito Lemma to derive the PDE with terminal and boundary conditions for pricing. We assumed that the solution would be in the exponential form. This is because the intensity model is basically derived from the hazard rate model, which has the solution in the exponential form. The Black-Scholes equation, which is a homogeneous parabolic equation with variable coefficients combined with the terminal condition and asymptotic boundary condition, admits the solution in the exponential form via exponential transformation. Further analysis will be needed to see whether a nonhomogeneous parabolic equation with variable coefficients, terminal condition, and asymptotic boundary condition yields a unique solution.

In data analysis, for unpredictable default occurrence, we merely depended on the historical data of default probability/intensity. The historical data we used was over the period 1991-2000 and across all the industrial sectors. In actual pricing, the default probability/ intensity needs to be computed by the industrial sector. To increase the accuracy of the estimate of the default probability, we need to take into the consideration other elements such as economic fluctuation (growth, recession, or depression), the size of the company, monetary policy and so on. Also, the size of the jump in the default intensity was arbitrary assigned in Section 6.2. Even though this is theoretically plausible, we need to collect empirical evidence and incorporate it in measuring the variable U_j.

In Chapters 4 and 5, we assumed that the risk-free interest rate follows the Vasicek model since this model is most used in the market. Applying this model, we used linear regression to estimate the parameters. The shortcoming of using the linear regression is that the long-term average is determined by the trend of the period from which the data come from. We need to take into the consideration the future trend of the relative interest rate to estimate the parameters.

In Sections 2.2, 3.2, 4.2 and 5.2, to predict expected default, we assumed that the firm assets value follows a geometric Brownian motion, with jumps at each coupon payment date, and that expected default occurs when the assets value hits the predetermined barrier. In Section 2.2, we applied the reflection principle to find the expected default probability, the probability that the total assets value hits the predetermined barrier.

As mentioned in the introduction, even though this model seems to make sense theoretically, there are still some shortcomings. The amount of total assets value does not necessarily determine the financial health of the company. A huge company with large assets can be unhealthy financially. Also, two companies with the same assets value can be considerably different in their financial conditions, which will lead to the different level of the barrier for default. We shall need to further investigate the quality of their assets.

In addition to these shortcomings, this model is difficult to implement since the required quantities are not readily observable; we can have an access to the firms' financial statements only quarterly for the best in most cases. We also need to incorporate the unforeseen factors in the future, such as market trend (Is the market growing or not?), overall economy trend (Is it in growth period, in recession, or depression?) and so on.

Schonbucher (2003b) suggests stock price and KMV for alternate parameters for predicting expected default. Stock price, even though we incorporated stock price as part of firm assets value, is too speculative, and therefore, does not reflect the company's financial condition in any better way.

The KMV model, marketed by Moody's, sets the default barrier somewhere between the face value of total liabilities and the face value of short-term liabilities. The idea behind this is that the company needs to refinance its short-term liabilities continuously to continue its daily operations while the long-term liabilities do not require refinancing until their maturities. This idea makes a better sense than using the firm assets value; however it will also require as much effort to collect the necessary data.

In Sections 2.2, 3.2, 4.2 and 5.2, in solving the problem, we assumed that endogenous default events and exogenous default events are uncorrelated, that is, the intensity rate is uncorrelated with the value of the firm assets, which is not realistic. In Sections 4.2 and 5.2, we also assumed that there is no correlation between the risk-free interest rate and the intensity rate, or between the risk-free rate and the firm assets value.

In most existing paper, the pricing model of credit default swaption is based on applying Black-Scholes Formula to the price of forward Credit Default Swap. However, there is no observable forward CDS market, which makes the existing approach less attractive. In this paper, the price of credit default swaption was computed directly from the bond price.

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