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Developing Mathematics Teachers' Attention to Quantitative Reasoning in Task Design: A Modeling Approach

Abstract
This study examines how a models-and-modeling perspective affected teachers' attention to quantitative reasoning in task design. A Model-Eliciting Activity (MEA) was implemented with 21 teachers over four weeks, challenging teachers to design a quantitative reasoning task for their students. Teachers' initial quantitative reasoning tasks did not incorporate quantities or quantitative relationships, two essential components of quantitative reasoning. As teachers revised their tasks through the MEA, most teachers began attending to these components. This article details how a modeling approach to teacher education provided a method to describe and support teachers to incorporate quantitative reasoning in their classroom tasks, though attending to quantitative reasoning within high school concepts remains particularly challenging.

Keywords
mathematics teachers, models and modeling perspective, quantitative reasoning

Cover Page Footnote
David Glassmeyer is an associate professor of mathematics education at Kennesaw State University. He focuses on developing mathematics teachers' content knowledge at the middle and secondary level and researches how to best support quantitative and covariational reasoning when teaching mathematics, technology, engineering, and science concepts.

Introduction

In the last decade, mathematics education researchers have increasingly demonstrated the importance of quantitative reasoning for teachers' mathematical thinking and advocated for quantitative reasoning to be incorporated in all levels of mathematics education. Aspects of quantitative reasoning, such as identifying quantities and understanding the relationships between quantities, are vital for teachers to include in mathematics instruction ranging from the elementary to university level (Moore and Carlson 2012). Unfortunately, research suggests teachers struggle to reason quantitatively, which can have a negative effect on their students' quantitative reasoning (Moore et al. 2014b; Smith III and Thompson 2017). More research is needed to better understand how teachers attend to quantitative reasoning and the kinds of interventions that can help them develop quantitative reasoning for the benefit of their classrooms (Stump, 2017).

This study aims to provide insight into how to develop teachers' quantitative reasoning by asking the research question: how does a models-and-modeling approach affect teachers' attention to quantitative reasoning in task design? A models-and-modeling perspective was used to guide the study by documenting and developing teachers' attention to quantitative reasoning. Specifically, a four-week-long Model-Eliciting Activity was used to challenge teachers to develop a quantitative reasoning task for their students, thus providing qualitative data that revealed how teachers attended to quantitative reasoning and how their thinking changed. The following sections detail the context of the study, how the models-and-modeling perspective was applied, and how that approach affected teachers' attention to quantitative reasoning.

Literature Review

Given this study's research question, the relevant literature includes quantitative reasoning, teacher knowledge, and a models-and-modeling perspective. The following sections detail each area of research.

Quantitative Reasoning

As Vacher (2014) and Karaali et al. (2016) point out, numeracy, quantitative literacy, and quantitative reasoning are terms often used interchangeably, so care must be taken to precisely define the intended meaning in use. In this article, quantitative reasoning will be defined as attending to and identifying quantities, identifying and representing relationships between quantities, and constructing new quantities (Moore et al. 2009; Thompson 2011). Therefore, clearly defining quantities is an essential part of quantitative reasoning, worthy of researchers' and teachers' attention.

A quantity is a mental construction resulting from a person completing an act of quantification, defined as "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit" (Thompson 2011, 37). Thus, quantities are a cognitive object composed of four components: (a) an object, (b) a measurable attribute of the object, (c) a unit of measurement for the attribute, and (d) a conceivable numerical value, or values, associated through a proportional relationship with the unit of measurement. For example, the water in a vase can be an object with many attributes associated with it: height of water, volume of water, etc. Each attribute has multiple units of measurement: the height of the water could be measured as vertical distance from the bottom of the vase to the top of the water in inches, centimeters, or a non-standard unit of measure such as toothpicks. A quantity can be considered to have a constant value (a single, unchanging value such as 4.5) or varying values (a range of possible values such as all positive real numbers).

Quantities can be related through a quantitative operation, which is the conception of two quantities being taken to produce a new quantity (Thompson 2011). Quantitative operations differ from numerical operations, which deal only with numbers. "Quantitative and numerical operations are certainly related developmentally, but in any particular moment they are not the same even though in very simple situations children (and teachers) can confound them unproblematically" (Thompson 2011, 42). Ellis (2011, 216) offered an example of this, saying "one might compare quantities additively, by comparing how much taller one person is to another, or multiplicatively, by asking how many times bigger one object is than another. The associated arithmetic operations would be subtraction and division [respectively]." When a person conceives of two quantities being joined through a quantitative operation to create a third quantity, Thompson calls this a quantitative relationship.

When learners do not attend to the quantities in a problem, their mathematical understandings of concepts and ability to use problem solving can be negatively affected (Clement 1982; Thompson and Carlson 2017). Elementary and middle school students who did not conceptualize quantities in a word problem had no basis for "constructing function rules or graphs or interpreting what graphs convey over an interval of a function's domain" (Madison et al. 2015, 55). At the high school level, researchers have closely examined the impact of reasoning about quantities on students' thinking about function concepts. Findings from these studies "suggest that curriculum and instruction should attend to the emergent nature of students' images of problem contexts by frequently prompting them to reason about the quantities in a problem's context and how they change together" (Moore and Carlson 2012, 58). These findings corroborate other research advocating increased emphasis on quantitative reasoning to support students' and teachers' mathematical thinking (Thompson 2011; Smith III and Thompson 2017).

Two factors can be attributed to learners' lack of quantitative reasoning. First, Thompson and Carlson (2017) find that US curriculum standards and US textbooks do not emphasize or support students' attention to quantities or how quantities covary. Mathematics texts often describe quantities as a variable representing a single unknown value. These authors suggest standards and curricula that regularly emphasize quantities and examining the relationship between quantities (such as Japan's texts) would support students' quantitative reasoning throughout all grades.

Second, a lack of teachers' quantitative reasoning can affect how students understand mathematics (Moore and Carlson 2012). As Moore et al. (2014b, 141) point out, "if neither students nor teachers are receiving sufficient opportunities to develop their ability to reason about relationships between quantities . . . should we expect teachers to teach for these same understandings and reasoning abilities?" Similarly, Smith III and Thompson (2017) express difficulty imagining how students could develop quantitative reasoning without focused curricula and instruction on quantities and relationships between quantities.

Researchers have noted the difficulties learners face when engaging in quantitative reasoning, whether students in K–12 classrooms (Ellis 2007), undergraduates (Moore and Carlson 2012), prospective teachers in certification programs (Moore et al. 2014a), or in-service teachers in professional development experiences (Smith III and Thompson 2017). Thompson and Carlson (2017) state that US teachers need support to develop and apply quantitative reasoning in their classrooms, but this support is largely unavailable. Additional research identifying the barriers to teachers developing quantitative reasoning is recommended.

Developing Teacher Knowledge Using a Models-and-Modeling Perspective

Lesh and Doerr (2003) developed a models-in-modeling perspective in part to develop teachers' knowledge. This approach provides a framework to facilitate teacher education in a way that affects teacher practice and, ultimately, how students learn mathematics. In this perspective, a model is considered to be "a way to describe, explain, construct, or manipulate an experience or a complex series of experiences . . . According to this perspective, all teachers have 'models' for teaching and learning mathematics" (Schorr and Koellner-Clark 2003, 197–198). These researchers have found teachers' models for teaching and learning mathematics were built around their experiences as learners and teachers. These models can be robust, but providing teachers time and multiple opportunities to develop their model is vital for creating changes to their classroom practice. Enacting more than surface-level changes to teachers' classroom practice often requires a concentrated effort to alter their models (Lesh 2003; Schorr and Lesh

2003), often involving "a series [of] iterative testing and revision cycles in which competing interpretations are gradually sorted out or integrated or both – and in which promising trial descriptions and explanations are gradually revised, refined, or rejected" (Lesh and Lehrer 2003, 109). Researchers find this approach "has proven to be especially effective in helping teachers build new models for the teaching and learning of mathematics" (Schorr and Lesh 2003, 145).

One method to develop teachers' models is through Model-Eliciting Activities (MEAs) designed for teachers. MEAs designed for teachers require them to document their thinking about a mathematical idea, create materials they can use in their own classrooms and share with other teachers, consider students' reasoning and learning, and revise ideas through iterative cycles of feedback (Doerr and Lesh 2003). This series of documents produces a trail of thinking that can be qualitatively analyzed to generate themes in individual and group thinking, how that thinking developed, and the factors promoting changes in thinking. Educators of mathematics teachers have successfully designed and used MEAs for teachers to promote deeper thinking about student thinking, engage in mathematics, reflect on prior beliefs about problem solving, and support teacher thinking about content in ways connected to their classroom practice (Schorr and Lesh 2003).

Conducting Research Using a Models-and-Modeling Perspective

A models-and-modeling perspective was also developed to explain conceptual systems within realistically complex problem-solving situations, including how teachers think about their practice. This perspective can qualitatively document teachers' understanding of particular mathematics topics, views of students' thinking, and beliefs about how mathematics teaching and learning occurs. This perspective provides detailed principles to guide the creation of MEAs that document teachers' models (Doerr and Lesh 2003). This careful creation not only allows teachers to test and revise their ways of thinking, but also gives researchers the opportunity to observe how teachers' ways of thinking develop throughout the revisions. Mathematics education researchers have successfully used MEAs to investigate and improve teachers' models within educational problem-solving situations (Koellner Clark and Lesh 2003). Design principles for MEAs specify how teachers' models can be elicited in observable ways (Lesh et al. 2003). These principles were used to develop an MEA that aimed to document and develop teachers' models of quantitative reasoning. The models-and-modeling perspective provided an analytical framework for understanding teachers' models and their development (Hjalmarson 2008; Sriraman and English 2010).

Methods

To determine how a models-and-modeling approach affects teachers' attention to quantitative reasoning in task design, I examined 21 secondary mathematics teachers in a graduate-level course. Therefore, this study is best categorized as a case study. At the same time, the group of 21 teachers in this study may not be representative of the larger population of mathematics teachers, a notion explored in the concluding section.

Setting

Teachers were in a two-year master's program in mathematics, where they took a combination of mathematics and mathematics education courses. The study focused on a summer mathematics education course called Quantitative Reasoning in Secondary Mathematics. This course met synchronously online four times a week for four weeks. During these meetings, live audio and video feeds were used for interaction, and a whiteboard was used as a tool for sharing written texts, such as PowerPoint slides, between the instructor and the teachers. Additionally, virtual spaces where small groups of teachers could interact, called breakout rooms, were used to facilitate small-group discussions. The instructor of the course was a mathematics educator who had designed and taught numerous secondary mathematics and science courses for pre- and in-service teachers. This was the instructor's first time teaching a pedagogy course on a quantitative reasoning topic for teachers and his first time teaching an online course for teachers. The author of this study was only a researcher associated with the course whose responsibility was facilitating data collection.

Participants

The 21 teachers in the course taught grades 6–12 mathematics, and all agreed to be participants in this study. The teachers were in the master's program for at least one year prior to taking the course. The teachers had experienced the online software and were familiar with their peers in the program. The requirements for admittance into the program ensured all teachers had taught for at least two years and were currently teaching mathematics between grades 6 and 12. The 21 participants had taught a mean of 8.5 years, with a range of 3 to 20 years of experience teaching K–12 mathematics. Eleven women and 10 men participated in the study, with 14 of them teaching high school grades (9–12), four of them teaching middle school grades (6–8), and three of them teaching both middle and high school grades.

Course Description

A main course objective was that teachers would understand ideas such as the meaning of quantities, quantitative relationships, and quantitative reasoning. Additional goals included teachers identifying these ideas in secondary mathematics curriculum and deepening their understanding of secondary mathematics content. The instructor provided opportunities for teachers to learn about quantitative reasoning and MEAs by assigning readings and reflection questions for homework, selecting tasks for teachers to engage in during class, and structuring class discussions to reinforce ideas about quantitative reasoning and MEAs introduced in these readings and tasks. Teachers were asked to read chapters and articles about quantitative reasoning (Carlson et al. 2010; Thompson 1994; Moore et al. 2009; Common Core State Standards Initiative 2010) and a models-and-modeling perspective (Lesh et al. 2000). These readings were assigned to provide teachers opportunities to consider quantitative reasoning in light of Thompson's quantitative reasoning framework and promote teachers' creation of sharable and reusable quantitative reasoning tasks that revealed students' mathematical thinking.

Teachers engaged in quantitative reasoning tasks coming from the *Pathways to Calculus* materials (Carlson et al. 2010) and quantitative reasoning problems posed in mathematics education literature (Clement 1982; Johnson 2011). For example, teachers completed the bottle-filling task (Fig. 1) during the first week of the course, which asked them to relate the quantity *height of the water* (from the bottom of the bottle to the top of the water, in centimeters) to the quantity *volume of the water* (within the bottle, in cubic centimeters) using a graph.[1] Teachers had both individual and group time to work on the task before the instructor led a whole-class discussion. During this discussion, the instructor discussed errors that often occur in thinking about the problem and why attention to quantities and quantitative relationships is important in this problem and in other mathematical contexts. Similar discussions occurred after other quantitative reasoning tasks and based on what teachers learned from the readings.

In addition to the readings and tasks, the teachers completed a quantitative reasoning MEA. This MEA constituted 50% of the course grade and was the primary method of collecting data in this study, as detailed in the following section.

[1] For a detailed analysis of this task, see Carlson et al. (2002).

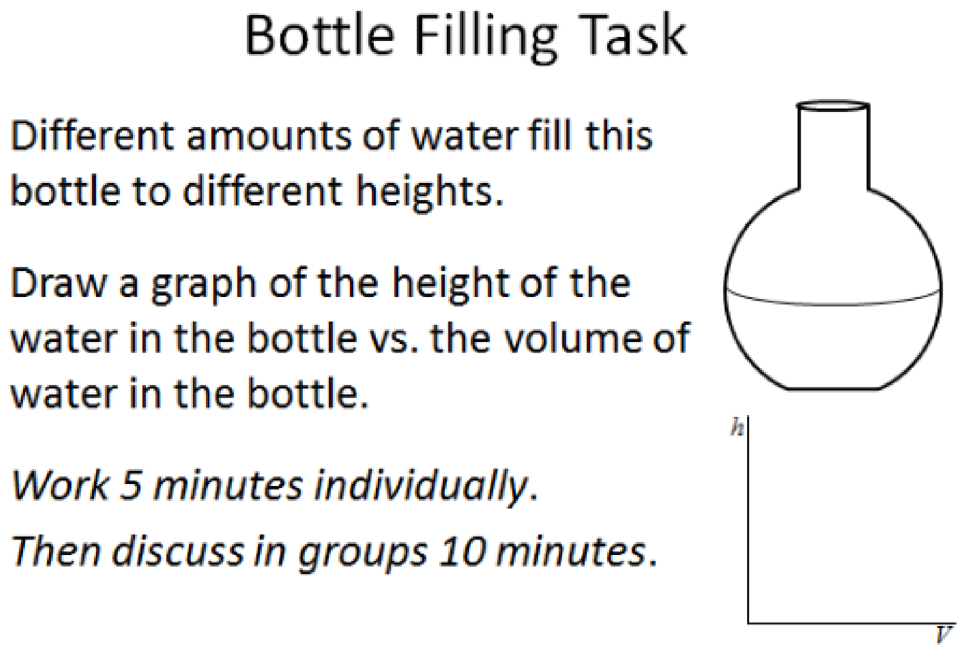


Figure 1. Quantitative reasoning task used in the course.
Adapted from Carlson et al. (2002) and Carlson et al. 2010.

Data Collection

The primary source of study data was the written documents generated as the 21 teachers completed the MEA in the course. Video recording software was also used to capture the small- and whole-group activity during the class sessions to supplement the MEA documents and contribute to data triangulation. This choice of data collection aligned with a models-and-modeling approach and allowed the researcher to assess the way teachers attended to quantitative reasoning in task design, which was the focus of the MEA. At the time of this study, no existing MEAs for teaching could be found focusing on quantitative reasoning, so a group of mathematics education researchers, teacher educators, and mathematicians collaborated to create a new MEA by attending to the five MEA design principles identified in Doerr and Lesh (2003): the reality, multilevel, multiple context, sharing, and self-evaluation principles (Table 1). The MEA was tailored for the online summer format of the four-week course to accommodate the fact that teachers could not meet face-to-face or have access to their own students. The 21 teachers worked in six groups to complete the quantitative reasoning MEA. Groups 1 and 2 were middle school teachers, Group 3 was a mixture of middle and high school teachers, and Groups 4, 5, and 6 were high school teachers.

Table 1
How the MEA Addressed Principles Outlined by Doerr and Lesh (2003)

MEA principle	The MEA addressed the principle by asking teachers to:
reality principle	create a quantitative reasoning task for their classroom.
multilevel principle	create a Facilitator Instructions and Assessment Guidelines document detailing implementation strategies, evaluation methods, and anticipated student difficulties.
multiple contexts principle	work in groups of 3–4, provide peer feedback to each other, consider student learning and misconceptions in the Assessment Guidelines.
sharing principle	create a Decision Log document detailing the evaluation, revisions, and rationale throughout the MEA; the Facilitator Instructions and Assessment Guidelines prompted teachers to create the document for another educator to use.
self-evaluation principle	reflect before and after each MEA feedback iteration.

The central components of this quantitative reasoning MEA included an individual Pre-Assignment, four group documents, and an individual Post-Assessment. The Pre-Assignment asked each teacher to define quantitative reasoning and identify quantitative reasoning tasks for students. This assignment provided data on individual teachers' attention to quantitative reasoning in a task suitable for their students. The course group documents included (1) the Quantitative Reasoning Task, (2) Facilitator Instructions, (3) Assessment Guidelines, and (4) a Decision Log.

First, the Quantitative Reasoning Task asked teachers to create a quantitative reasoning task for their students that (a) captured students' quantitative reasoning, (b) was tailored to a grade and mathematical subject they taught, (c) had students working in groups of 2–4, (d) could be completed within 90 minutes of class time, and (e) could be implemented by another mathematics teacher. Second, the Facilitator Instructions asked teachers to create a document explaining how another educator could implement their quantitative reasoning task, including preliminary information, prompting questions, and anticipated student responses. Third, the Assessment Guidelines asked teachers to create a document suitable for someone else to evaluate student responses to the task by establishing some kind of criteria for assessing student responses to the quantitative reasoning task. Fourth, the Decision Log asked teachers to create a document articulating the refinements made while designing and revising their documents. The Post-Assessment was an individual entry from each group member detailing how they defined quantitative reasoning, how their task incorporated quantitative reasoning, how the task changed during the feedback iterations, and how their thinking about quantitative reasoning changed throughout the feedback iterations.

Revisions to these documents were prompted by required interaction cycles prompted by instructor, peer, and undergraduate feedback. These iterations gave teachers the opportunity to test, revise, and refine the four documents. After creating the initial documents, called Version 1, the instructor provided feedback to focus the teachers on the occurrence of quantitative reasoning in the activity, and the teachers revised their documents to create Version 2. Each group gave feedback on another group's Version 2, after which groups again revised their documents to create Version 3.

Because teachers did not have access to their own students during the summer, the author implemented the teachers' Version 3 Quantitative Reasoning Task with three or four undergraduate students in a summer liberal arts mathematics course or business calculus course. The work was then returned back to the teachers, where each group evaluated the work and then revised their documents to create Version 4. These four versions all took place during the four-week course. In the fall, after the course ended, funding was provided as an incentive for the teachers to implement a final Version 5 of their activity with their own students. Four teachers submitted reflections on their implementations by the end of the fall.

Data Analysis

To analyze the data, I first used content analysis (Merriam 1998) on all the MEA documents to identify statements that attended to quantitative reasoning in the teachers' task design. Given the vast amount of text data generated by the MEA, the content analysis allowed for the classification of the data relevant to this study's research question. Second, I used Hjalmarson's (2008) analytical tool to compare statements about quantitative reasoning across iterations of the MEA documents to draw inferences about how teachers attended to quantitative reasoning in the task design. Each group's MEA documents were coded based on the four components in Hjalmarson's analytical tool: conceptual systems, purpose and goals, pedagogical framework, and mathematical content. This tool helped identify the multiple ways teachers communicated quantitative reasoning in the documents. In addition to each group's submission of MEA iterations, the same analysis was conducted on individual teacher's documents, such as the Pre-Assignment, Post-Assessment, and Version 5. This analytical tool aligns with a models-and-modeling perspective (Hjalmarson 2008) and provided a way to identify the ways teachers attended to quantitative reasoning in each task iteration.

Using Hjalmarson's (2008) analytical tool can be classified as a comparative analysis (Doerr and English 2003; Corbin et al. 2014). For example, Group 1's

Version 1 was compared to Group 1's Version 2; similarly an individual's response to the Pre-Assignment was compared with that same individual's response in the Post-Assessment. These comparisons provided insight into how teachers attended to quantitative reasoning in the task design and how their thinking developed over time either within a single group or in an individual teacher (Thomas and Hart 2010).

To describe the patterns emerging from the data analysis, language was adopted from the research literature along with some terms developed specifically for this study. I used Thompson's (2011) definition of quantity when teachers attended to all four components of a quantity: object, measurable attribute of the object, unit of measurement for the attribute, and conceivable numerical value(s) associated through a proportional relationship with the unit of measurement. No common terms existed in the research literature to contrast quantity, so the word "pseudo-quantity" was adopted in this study. The term pseudo-quantities is used to characterize statements made by a teacher that attended to numerical values, variables, unknowns, or other features of a contextual setting where the teacher did not fully distinguish the object, attribute of the object, and units of the attribute being considered. Addressing some aspects of a concept but not others is typical of emerging understanding (Gilmore and Papadatou-Pastou 2009), and the term pseudo-quantity is used to indicate when a teacher has not described all four parts of a quantity. Examples of psuedo-quantities as compared to quantities will be shared in the next section.

Thompson's definition of quantitative relationship was used when teachers attended to two quantities being joined through a quantitative operation to create a third quantity. "Those three quantities in relation to one another constitute a quantitative relationship" (Thompson 1994, 188). To contrast quantitative relationship, I used Thompson's definition of numerical relationship when teachers related two pseudo-quantities through arithmetic or algebraic operations to compute a new pseudo-quantity. In other words, numerical relationships use arithmetic or algebraic operations between numbers, variables, or unknowns to create or compute a new number, variable, or unknown in a problem context. As Thompson (2011, 42) discusses, "quantitative and numerical operations are certainly related developmentally, but in any particular moment they are not the same."

Results

To answer the research question on how a models-and-modeling approach affected teachers' attention to quantitative reasoning in task design, the findings are structured in three sections: (1) how teachers initially attended to quantitative reasoning at the onset of the MEA, (2) changes occurring in teachers' attention to quantitative reasoning as they completed the MEA, and (3) reasons teachers gave for their changes in thinking with regard to how they attended to quantitative reasoning. An overview of each group's task and grade level is given in Figure 2.

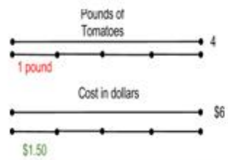
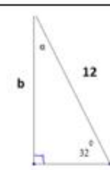
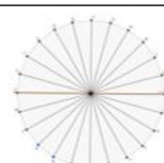
Group	Participant Pseudonym	Task Title	Task Picture	Grade Level Associated with Task Content
1	Brandon Charles Carol Glen	Proportional Reasoning Across Multiple Representations		Middle school
2	Julie Charlotte Samantha Rose	Fundraiser Profit Presentation		Middle school
3	Tiffany Alice Allie Penny	Modeling Scenarios Graphically		Middle school
4	Jack Darium Dylan	Right Triangle Problem Classification		High school
5	Gary Ken Byron	Ferris Wheel Rescue Plan		High school
6	Nicholas Joyce Percy	Introduction to Logarithms		High school

Figure 2. Summary of groups and the tasks they developed.
Note that three of the four teachers in Group 3 were high school teachers, but this group's task focused on middle school content.

Teachers' Initial Attention to Quantitative Reasoning in Task Design

The Pre-Assignment part of the MEA captures teachers' initial attention to quantitative reasoning. In their Pre-Assignments, 18 of the 20 teachers[2] described quantitative reasoning as (a) attending to pseudo-quantities and/or (b) attending to numerical relationships when designing a quantitative reasoning task. For example, Penny gave the response that quantitative reasoning was "giving students a problem involving quantities where they have to determine a strategy for solving the problem" with no further statements about what was meant by "quantities." Six other teachers used the word "quantity" in their Pre-Assignment responses in ways that were either synonymous with "solution," "number," or "amount," or used this word in vague ways, and were thus coded as attending to pseudo-quantities. The other two teachers made statements about quantitative reasoning that attended to quantities and relationships between quantities.

Initially 16 of the 20 teachers gave Pre-Assignment responses that were coded as attending to numerical relationships. For example, Charles said quantitative reasoning is when students understand "how to write equations and functions" that model situations. He gave the following example of a task that involved quantitative reasoning:

> A simple task could be some sort of money saving problem. If you have $100, and make $40 per week mowing lawns this summer, define your variables and write a function modeling this situation. How long will it take you to have saved $500?

In this statement Charles focused on writing a function rule and then using algebra to evaluate the function given a specific input amount, $500. The components of the contextual problem included the initial amount of money, the amount of money increasing each week, the number of weeks, and the resulting total amount of money. These components were not clearly defined because Charles did not attend to what object, attribute, or in some cases what units were associated with each component. (For instance, if time is a variable, from what point is time measured, and in what units?) Thus, Charles' statement was coded as attending to pseudo-quantities. The type of interactions Charles described in this statement were arithmetic operations because after setting up an equation, algebraic operations were needed to solve for the number of weeks it takes to save $500. Charles' responses were coded as referring to numerical relationships because his task design attended to algebraic operations (subtractions, division) between pseudo-quantities (the initial amount of money, amount of money increasing each

[2] One teacher did not submit a Pre-Assignment, taking the total number of teachers down to 20 for this set of documents.

week, the total amount of money saved) to calculate a new pseudo-quantity (the number of weeks).

Only two teachers initially attended to quantities and quantitative relationships: Gary and Rose. Gary made statements attending to quantities by saying that reading the Common Core State Standards for Mathematics (CCSSM) and the materials from the first course meeting influenced his ways of thinking about quantitative reasoning.[3] Similarly, Rose explicitly attended to quantities in her Pre-Assignment responses by referencing the units involved and attended to the meaning of the quantities by stating:

> Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of the quantities, not just how to compute them . . . using symbols to represent different quantities in a problem and understanding exactly what the meaning of those quantities are *throughout* the problem, not just in the answer at the end. [emphasis in original]

Here Rose referenced the units involved and attended to the meaning of the quantities, indicating quantities were an aspect of quantitative reasoning. Like Gary, Rose made a comment suggesting the CCSSM standard for mathematical practice Reasoning Quantitatively and Abstractly gave her an idea about how to think about quantities and quantitative relationships.

When the groups of teachers began Version 1 of the MEA, all six groups of teachers incorporated quantitative reasoning in their task by attending to pseudo-quantities and attending to numerical relationships. For example, Group 6's initial task indicated "Richter scale and energy" were quantities, explaining:

> Our task involves the concept of logarithms. We have all taught the subject, however the students demonstrate poor or inadequate understanding of what logarithms are, and more importantly, what the quantities associated to a logarithmic function represent.

In this statement the teachers acknowledge students' inadequate understanding of what quantities are being represented in logarithmic relationships but do not specify the components of the quantity involved here or elsewhere in their Version 1 documents. For example, describing energy as a quantity involved assigning an object (earthquake), attribute (energy released), unit (e.g., Terajoules), and conceivable numerical values (e.g., $1x10^{\wedge-12}$ to $4x10^{\wedge 6}$). Richter scale as a quantity would involve identifying an object (earthquake), attribute (magnitude), unit (Richter scale units), and conceivable numerical values (e.g., 0–10). Describing energy and Richter scale as quantities is a complex task that Group 6 never discussed, and thus these teachers were coded as initially attending to pseudo-quantities in their task design.

[3] Gary was one of the few teachers to submit his Pre-Assignment after the first day of class. Most of the other teachers submitted this prior to the first class meeting.

An example of a group attending to numerical relationships can also be seen in Group 6's initial task, which asked, "Write an equation that relates the variables from the table on the previous page. What type of equation is this?" By using a table with numerical values relating these two pseudo-quantities (Richter scale and relative intensity), Group 6 asked students to create a new algebraic representation of this existing exponential relationship. Group 6's Assessment Guidelines stated the expected response from students was "Let $y = 10^x$; this is an exponential equation." Teachers expected students to create this equation by raising the number 10 to the power of the input pseudo-quantity, Richter scale. This equation generated the output pseudo-quantity of relative intensity. These statements attended to numerical relationships because students were asked to combine a pseudo-quantity (the number 10) with another pseudo-quantity (Richter scale) using an algebraic operation (exponentiation) in order to create a new pseudo-quantity (relative intensity). Group 6 did not attend to quantities in their Version 1 documents, and thus the operations they described between the pseudo-quantities were numerical and produced numerical relationships. Similar patterns of statements attending to pseudo-quantities and numerical relationships were found in the other groups' Version 1 documents.

In addition to attending to pseudo-quantities and numerical relationships, Group 5 also included some statements attending to quantities in their Version 1 documents. Their task asked students to "devise a way to determine how far above or below the ground each seat will be and the horizontal position of each seat along the ground" for a Ferris wheel that "turns counter-clockwise at a rate of one revolution every two minutes." Group 5 was the only group to provide evidence of quantities being an aspect of quantitative reasoning, which they did by asking students to identify quantities relevant to the problem context, to explain why these quantities are important to the problem, and to identify how the quantity was represented. These goals were reflected in the Facilitator Instructions when they said the facilitator should:

> Begin by asking the students what quantities they see in the problem. Once you have a list, ask them what object each quantity is connected with, what attribute of the object the quantity is measuring, what units will be used, and what values they can expect to see for the quantity . . . make sure they include vertical distance from the ground to the seat, horizontal position . . . make sure the idea of rotation comes up in the discussion on quantities. If no one brings it up, ask how they will know where each seat is located, and try to lead them into the idea that they will need to know an angle of rotation (although they are not likely to use that terminology, and you don't need to give them that vocabulary yet) . . . [make sure] they are aware of these three quantities.

Group 5 gave details for how students should measure the quantities of vertical and horizontal distance and how these quantities change with respect to the rotation angle. While this group described the components of a quantity for vertical and horizontal distance, they did not do so for rotation angle. Rotation angle was described to have "degree measurements," but did not have an object, attribute, or unit associated with this variable.[4] Group 5 mentions the fixed quantity "rate of revolution" and its influence on rotation angle and hence vertical and horizontal distance, but did not mention the role of elapsed time influencing these quantities in any of their documents. Both the rate of revolution and the elapsed time were coded as attending to pseudo-quantities given the lack of description accompanying these terms. Thus, Group 5 attended to both quantities and pseudo-quantities in their Version 1 documents.

Changes in Teachers' Attention to Quantitative Reasoning in Task Design

By the end of the course, most groups and individual teachers changed how they attended to quantitative reasoning in task design. Five of the six groups changed how they attended to quantitative reasoning in their task design by focusing their task to include quantities rather than pseudo-quantities. Four of the six groups also attended to quantitative relationships in their tasks rather than numerical relationships. Similar to the groups' development, individual teachers began making statements attending to quantities and quantitative relationships, rather than their initial patterns of attending to pseudo-quantities and numerical relationships. At the end of the course, 12 teachers had changed the way they made statements about quantitative reasoning by attending to quantities and/or quantitative relationships. The other nine teachers continued to make statements about quantitative reasoning that were coded as attending to pseudo-quantities and/or numerical relationships.

For example, Group 2 incorporated a table "designed to help [students] think critically about what quantities would be present in fundraising situations." This table was in their Version 4 Quantitative Reasoning Task and had accompanying expectations in the Assessment Guidelines asking students to identify the object, attribute, and unit for "all of the varying and unvarying quantities that are present in a fundraising situation." These expectations indicated Group 2 attended to quantities in their task design. This group then asked students to create an equation that combined the quantity's unit price per item and number of items sold in order to create a new quantity, the profit. Thus, Group 2 made statements coded as attending to quantitative relationships because quantities were being taken together to form new quantities. Similar patterns of attending to quantities and quantitative relationships were found in the other two middle school groups (1 and 3).

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[4] An example of rotation angle being stated as a quantity would be to define the object to be an angle, the attribute to be openness, the unit of measure to be the fractional amount of a circle's circumference subtended by an angle (computed according to how much time the ride has been turning since loading the last seat in minutes), and the units to be radians.

Another example comes from Group 6, who changed their task by removing the Richter scale and energy question (which were coded as pseudo-quantities in Version 1) and switched to a savings account scenario where the student "invested $5,000 in a stock guaranteed to make 5.3% interest per year." Group 6 added statements to their task that were coded as attending to quantities by expecting the students to identify the variables in this situation as "Time in years since the investment was 1st made" and "Amount in dollars since the investment was 1st made." In their Version 4 Facilitator Instructions, this group said the facilitator should "be careful not to use the variables *x* and *y*, rather focus on the quantities, time in years, and amount of stock value." These statements specified the object (time), attribute (time since money was invested to now), and units (number of years) for the quantity and the object (stock value), attribute (amount of money presently in the stock), and units (dollars) for the quantity, and were thus coded as attending to quantities.

Twelve of the 21 teachers made statements in their Post-Assessment or Version 5 documents referring to quantities as an aspect of quantitative reasoning. These 12 teachers made statements depicting quantities as specifying objects, attributes, and units. For example, in his Post-Assessment Byron said:

> I understand quantitative reasoning to be sorting through a situation to identify measurable attributes, how they relate to each other, which are appropriate to work within a given task, and how to work with them . . . As we have worked through this project, I have shifted away from looking at the values of the measurements and looking more at the attributes themselves . . . the students must look for patterns between the quantities using actual values that will help them transition to looking at the general behavior of the quantities in relation to each other which should help the students see them as actual attributes as opposed to specific values at specific points in time.

Byron's description says quantities have measurable attributes that vary in accordance to the context students are using. The phrases "actual attributes" and "actual values" suggest Byron considered quantities as attributes of an object, and that the measurable values of the attribute most likely had units to make them meaningful in the context and thus were coded as attending to quantities. Byron also indicated that working in Group 5 influenced him to consider attributes of quantities and how they vary within the context of the problem. Byron's group mates, Gary and Ken, also expressed quantities in their Post-Assessment in similar ways that related to their group's task.

Nine of the 21 teachers made statements coded as referring to quantities and quantitative relationships as an aspect of quantitative reasoning by the course conclusion. An example of one teacher doing this was Charlotte, when she said in her Post-Assessment:

> It's essential for students to focus on recognizing relationships and having them write or explain their thought processes in how quantities relate to one another and showing they work together in a process not individually, as well as, constructing new quantities that are

> not given to form a conclusion . . . Our groups [sic] MEA relates to quantitative reasoning when we have students . . . creating visuals to identify relationships, having students explain what it means to have quantities co-vary, constructing general equations through these discoveries, and presenting their work to peers and teachers.

Charlotte was in Group 2, and the MEA to which she referred had questions that asked students to "identify two co-varying quantities in your fundraising situation and explain in detail how they are related to each other." In Group 2's MEA documents, the quantities "cost" and "income" were related in a linear equation to create the new quantity "profit." Charlotte's statement was coded as referring to quantitative relationships because she referenced her group's activity in a way that conveyed a quantitative relationship and covariation within that relationship.

Not all teachers changed the ways they individually attended to quantitative reasoning. Nine teachers continued attending to pseudo-quantities and/or numerical relationships as quantitative reasoning. Responses from these nine teachers included the word "quantity" in vague ways or used this word synonymously with numerical values, reflecting these teachers' initial responses that were coded as attending to pseudo-quantities. Five of the nine teachers were in groups that had made statements coded as attending to quantitative relationships. While these group responses were coded as attending to quantitative relationships, the individual teachers' Post-Assessments indicated teachers were thinking about quantitative reasoning differently. Thus, while the group may have made statements coded as referring to quantitative relationships, these five teachers did not provide evidence they shared their group's view that quantitative relationships were a characteristic of quantitative reasoning that was attended to in their task design.

More high school teachers continued to attend to pseudo-quantities and numerical relationships throughout the course. Eight middle school teachers attended to quantities and/or quantitative relationships as an aspect of quantitative reasoning by the course conclusion. Only four high school teachers did the same: Gary, Ken, and Byron (all in Group 5), and Joyce (Group 6). Joyce was one of the teachers to complete Version 5, where she commented on the role of quantities in her own students' work. After reviewing her students' work on her Quantitative Reasoning Task, she said, "When I discuss quantities in class, I need to move beyond saying, for example, '*x* represents time,' and say, '*x* represents the time in years since money was first invested in the account.'" Joyce clarified how quantities were a part of the quantitative reasoning task by stating students needed to include a way to assign values to attributes as well as units associated with this attribute. Her statements were coded as attending to quantities in her task, showing change from her initial statements. Joyce also referenced varying quantities within functions, which was interpreted as attending to how the input quantity relates to the output quantity. In this way Joyce considered the input quantities affecting the output quantity through covariation. Joyce's statements attended to quantitative relationships because Joyce described how quantities (such as function inputs) are taken to create a new quantity (function output) within a problem context.

Why Teachers Changed How They Attended to Quantitative Reasoning

All teachers commented on factors that influenced how they attended to quantitative reasoning in their tasks, particularly the reasons teachers shifted from attending to pseudo-quantities to attending to quantities. Three main factors account for this change: the student feedback MEA iterations (both undergraduate and K–12), course materials, and the peer feedback MEA iteration.

Groups 1 and 6 said that undergraduate student feedback prompted them to be more explicit about how quantities were included in their task. For example, Group 1 said one of the undergraduate students "used the word 'quantity' a few times but never said what that quantity was. (Perhaps we should include a more explicit definition of what 'quantity' means in terms of what we have talked about in class in the facilitator instructions?)." Similarly, Group 6 responded to student performance on their task by saying:

> Students articulated the general sense of the variables, but none of the students spent much time defining the variables and their units of measure. Certainly a point that needs to be addressed for Version 4 is the articulation of what we want the students to produce.

Both groups made changes in their Version 4 documents that aligned with the problems they identified from the undergraduate student feedback. These changes reflect the groups attending to quantities, rather than pseudo-quantities, in their task design.

Three of the four teachers who completed Version 5 said K–12 student feedback influenced how they incorporated quantitative reasoning in their classroom. For example, Joyce's Version 5 stated:

> As far as quantitative reasoning in my classroom, I still see it as something that helps students understand math concepts better. I need to discuss the ways that quantities affect each other so that students can move beyond superficial, symbolic understanding of problems. As far as what I have learned from looking over my students' work on this activity . . . I need to provide my students with opportunity for discussion about differences in how quantities vary/relate depending on what kind of function we are using. I need to make it more evident to my students that they can use their prior knowledge to support their conjectures about the way certain quantities vary and relate to each other.

Joyce's reflection shows the influence the K–12 student feedback had on prompting her to recognize how quantitative relationships could be emphasized in her classroom tasks.

One group and two teachers said the course materials influenced how they attended to quantitative reasoning in their task design. Group 2 stated the *Pathways* *to Calculus* materials (Carlson et al. 2010) influenced them to incorporate quantities in the task. For example, Group 2 stated in their Version 2 Decision Log:

> After doing our homework 6 we decided to offer the students a table to fill out to help organize their work. This table is designed to help them think critically about what quantities would be present in fundraising situations and how they might affect any decisions they'll need to make.

The instructions for Homework 6, which was due the day before Version 2 was due, asked teachers to complete three worksheets in the *Pathways to Calculus* materials (Carlson et al. 2010, Module 2, Worksheets 1–3). Group 2 included that table in subsequent Versions 3 and 4 by adding scaffolding, additional questions, and expectations related to quantities. Group 2 did not comment that instructor feedback was influential in their decision to incorporate the materials even though after Version 1 the instructor asked Group 2 to consider how students were "thinking about proportional reasoning and quantities based on their product."

The first course meeting may have influenced some teachers' attention to quantitative reasoning in their task design. Gary and Rose attended to quantitative relationships at the beginning of the MEA, and Gary made statements indicating the first course meeting influenced his view of quantities. In his Post-Assessment, Gary again stated how the course in general influenced the way he attended to quantitative reasoning in his task:

> My understanding of quantitative reasoning has evolved a great deal over the course of this class. Before this class I don't think I would have made a distinction between mathematical/arithmetic reasoning and quantitative reasoning. I probably equated the word "quantity" with the words "number" and "amount" and didn't stop to think that these are only part of the idea of "quantitity" [sic]. One of the greatest insights I developed was the idea that there are four parts to quantity: object, measurable attribute, unit, and number. Although I think I was aware of all of these aspects, I didn't always stop to consider them for each quantity, and I didn't realize how much that could help avoid mistakes and deepen understanding. I know that I will be focusing on these ideas in my teaching in the coming year.

While Gary was not specific about what part of the course influenced his thinking, the similarities between his definition and the Thompson (1994) definition of quantity presented in the first week of the course may be referenced here, especially because he referenced this first course meeting in his Pre-Assignment. Similarly, Darium referenced the Moore et al. (2009) article in his Post-Assessment as influencing his ways of thinking about quantities but did not give further details about how or why this occurred. Rose was the only teacher to make a statement suggesting the CCSSM affected how she considered quantities in her Pre-Assignment.

Another contributing factor to how teachers thought about quantities was peer feedback. Group 6 acknowledged that receiving and giving peer feedback influenced their thinking. Group 2, who gave feedback to Group 6, stated:

> The task asks for students to explain ideas to another student but does not explicitly imply the use of quantities . . . instead of just identifying variables, have them look at all of the quantities more in depth and how it will relate to the situation and the formula they're supposed to come up with.

Group 2 challenged Group 6 to consider quantities rather than pseudo-quantities in the task. This excerpt indicated how the peer feedback provided motivation for groups to consider an object, a measurable attribute of the object, a way to assign values to this measure, and an accompanying unit. In their Version 3 Decision Log, Group 6 made a comment about the effect of the peer feedback process:

> We also received feedback from our peers. They had some excellent suggestions concerning the quantitative reasoning task. In particular, they suggested questions that ask students to analyze the quantities involved with the stock problem in more detail. We added a little more to the directions in order to give the students an idea of what we wanted them to explore.

In this comment, Group 6 acknowledged the influence of peer feedback on how they attended to quantitative reasoning in their task design, particularly by identifying the attributes and units involved in the task's quantities. This comment suggests Group 6's shift toward attending to quantities in their task was promoted through the peer feedback process.

Group 6 was also influenced to consider quantities as an aspect of quantitative reasoning by providing peer feedback. In their feedback to Group 1, Group 6 commented on an "awesome list of four prompting questions . . . [for] investigating quantitative reasoning." Three of these questions referred to Group 1's questions about quantities, including: "What quantities should be represented in your explanation? How will you measure each of the quantities? (i.e., What kind of units?) What quantities are important to the situation?" Group 6 incorporated Group 1's questions into their Version 3 documents. While Group 6 did not directly acknowledge the effect Group 1 had on their thinking, the implication of Group 6's comment in the peer feedback process suggests the added questions came from Group 1. Thus Group 6's shift from attending to pseudo-quantities to attending to quantities in their task was affected by providing feedback to another group.

Conclusions

This study provides needed information on the effect a models-and-modeling perspective had on one group of mathematics teachers' attention to quantitative reasoning in task design. Twelve of the 21 teachers were affected by a models-and-modeling perspective to better attend to quantitative reasoning in their tasks.

Nine of the 21 teachers did not attend to quantities or quantitative relationships in their task design even after four weeks of reading articles, completing the MEA, receiving instructor feedback, and engaging in course materials.

The study highlights the need for continued efforts to support teachers' incorporation of quantitative reasoning in the classroom. This study found teachers' initial incorporation of quantitative reasoning in task design did not attend to quantities or quantitative relationships. Researchers note that not attending to quantities and quantitative relationships leads to conflation of components in a problem and hinders conceptual understanding of mathematical content (Clement 1982; Moore et al. 2014a). Therefore teachers' initial attention to quantitative reasoning in tasks did not uphold recommended practices for teachers and could be much improved (Thompson 2011).

This case study indicates that a models-and-modeling perspective was successful in promoting and documenting teachers' attention to quantitative reasoning in task design. This perspective guided how the MEA was created and implemented and how the resulting data were analyzed. A models-and-modeling perspective promotes communication and sharing across contexts (Lesh et al. 2003), allowing others to understand in-depth how this particular group of teachers designed tasks attending to quantitative reasoning (Merriam 1998). The findings from this case study may indicate trends in quantitative reasoning task design within similar educational settings involving middle and high school in-service mathematics teachers in the United States. Specifically this study suggests that without interventions, mathematics teachers are not fully attending to quantities and quantitative relationships when designing tasks for the students and without interventions will likely not change their classroom behaviors. Additional research is needed to (a) test the previous conjectures for why this occurred, (b) identify ways of supporting high school teachers who develop tasks involving more advanced content, and (c) examine the nature in which teachers' conception of a pseudo-quantity can be developed to a quantity. Since teachers are expected to demonstrate quantitative reasoning at whatever level of mathematics they teach (Thompson 2011, Conference Board of the Mathematical Sciences 2012), this call for further research is particularly important.

Three limitations of this study should be noted. First, the factors affecting teachers' attention to quantitative reasoning in task design mostly came from teacher comments themselves. Second, data came from 21 teachers within a single setting. Third, due to the course occurring during the summer, one iteration of task feedback came from undergraduate students rather than the teachers' own students. A future study could improve upon these limitations by empirically investigating the effect of various readings, feedback, and homework assignments on teachers' task design, gathering data from another setting or additional teachers to support generalization of this study's findings to other settings of a differing nature, and implementing the task with the teachers' own students.

An implication for mathematics teacher educators is that using an MEA for teachers can have positive effects on the ways teachers attend to quantitative reasoning in task design. Two particularly effective components of this MEA included having mathematics teachers develop a task and supporting documents for their own classroom practice and providing the opportunity for teachers to revise their documents after receiving various forms of feedback. For this study, teachers received feedback from the instructor, provided peer feedback, received feedback from students similar to their own, and in some cases implemented their Quantitative Reasoning Task to acquire feedback from their own students. The feedback iterations influenced how teachers thought to consider the aspects of quantities and quantitative relationships. Teacher educators can have teachers read selected articles (Carlson et al. 2010; Moore et al. 2009; Thompson 2011) and engage in carefully crafted activities in order to prompt revisions to the quantitative reasoning tasks they create. These readings and activities provide alternative ways of thinking about quantitative reasoning and give teachers examples of how to connect quantitative reasoning to their classroom practices. The design of the MEA using the models-and-modeling perspective supports the shareability of this activity to other teacher education settings. Teacher educators working with in-service teachers are invited to use and adapt this MEA and course materials to provide teachers opportunities to advance their thinking about quantitative reasoning in the context of their classrooms.

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