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On a Desert Island with Unit Sticks, Continued Fractions and Lagrange

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On a Desert Island with Unit Sticks, Continued Fractions and Lagrange

Abstract

GLY 4866, Computational Geology, provides an opportunity, welcomed by our faculty, to teach quantitative literacy to geology majors at USF. The course continues to evolve although the second author has been teaching it for some 20 years. This paper describes our experiences with a new lab activity that we are developing on the core issue of measurement and units. The activity is inspired by a passage in the 2008 publication of lectures that Joseph Louis Lagrange delivered at the *Ecole Normale* in 1795. The activity envisions that young scientists are faced with the need to determine the dimensions of a rectangle with no measuring device other than an unruled stick of unknown length – to hundredths of a stick length. Following Lagrange, the students use the stick to measure the lengths with continued fractions, and then they reduce the continued fractions and convert them to decimal form. In the process, these student veterans of calculus instruction learn that as a group they are not very good at the arithmetic of fractions, which they thought they learned in the fifth grade. The group score on a continued fraction item improved from 44% on the pre-course test to 84% on the post-course test in the first semester in which the new lab was included (Fall 2015).

Keywords

measurement, units of measure, quantitative literacy, continued fractions, arithmetic of fractions, measurement error, accuracy vs. precision, geoscience education

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Cover Page Footnote

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This notes is available in Numeracy: https://scholarcommons.usf.edu/numeracy/vol9/iss2/art8

Introduction

The concept of continued fractions has a long and distinguished history. The list in Wikipedia, for example, begins with Euclid's *Elements* (300 BC)[1] and includes such milestones as John Wallis's *Opera Mathematica* (1695),[2] which introduced the term; Leonhard Euler's *De fractionibus continuis dissertatio* (1737)[3] for an account that included the first proof of the irrationality of e; and an application by Johann Lambert that was the first proof of the irrationality of π (1761).[4] However, the question of interest to most impatient scientists—geologists in the case of this paper's authors and their students—is what practical use can a seemingly esoteric mathematical concept such as continued fractions possibly be to everyday concepts in our own field. For the answer to this question we can turn to the great Joseph Louis Lagrange, another 18th century mathematician.

Specifically, we were drawn to the following passage on the usage of continued fractions in *Lectures in Elementary Mathematics* (Lagrange 1795, 2–3):

> … Suppose, for example, that you have a given length, and that you wish to measure it. The unit of measure is given, and you wish to know how many times it is contained in the length. You first lay out the measure as many times as you can on the given length, and that gives you a certain whole number of measures. If there is no remainder, your operation is finished. But if there be a remainder, that remainder is still to be evaluated….
>
> If you have a remainder, since that is less than the measure, naturally you will seek to find how many times your remainder is contained in this measure. Let us say two times, and a remainder is still left. Lay this remainder on the preceding remainder. Since it is necessarily smaller, it will still be contained a certain number of times in the preceding remainder, say three times, and there will be another remainder or there will not; and so on….

As former field geologists, we conjure up an image of determining lengths, areas, and volumes while stranded at a field site (such as the proverbial desert island) without a tape measure, or a manufactured measuring device of any kind. What to do? Lagrange and a *unit stick* provide a method and, at the same time, drives home the notion of unit of measure, an everyday concept that underscores the transition from elements in our students' mathematics classrooms (numbers) to elements in ours on quantitative literacy (quantities, which—for us—are numbers with units).

[1] Ref, e.g., Euclid; Heath (2006).
[2] Ref, e.g., Wallis; Scriba (1972)
[3] See Wyman (1985)
[4] Ref, Lambert (1762)

It is worth noting in passing that Lagrange, of course, is not a random name from the history of mathematics for geologists and geology students. Our students are aware of Lagrange's role in the creation of the metric system (particularly relevant to our topic of the "unit stick"). In subsequent hydrogeology courses, the students learn the crucial difference between the Eulerian view and the Lagrangian view for conceptualizing fluid flow as a prelude to modeling it. Some of our more fortunate students who continue with their calculus make good use of Lagrange multipliers. His is a name they should recognize and continue to encounter, along with Laplace and Poisson.

At the same time, from our experience with teaching continued fractions in a lab about measurement, we are convinced that nothing is better than continued fractions at ferreting out student inability to manipulate fractions. How dependent they are on their calculators!

Background

The verbal description in the quotation from Lagrange translates to the following continued fraction,

$$3+\cfrac{1}{2+\frac{1}{3}}$$

assuming that the "unit of measure" of the quotation went into the "given length" three times with a remainder, and ignoring the "and so on." Using the rules for the arithmetic of fractions, the continued fraction is equivalent to ("reduces to") 24/7.

This arithmetic was introduced as a learning concept in a laboratory activity in GLY 4866—Computational Geology—at the University of South Florida (USF) in the fall 2015 semester, and then again in the spring semester in an amended version. This note describes the laboratory activity (see Appendix) and the results of our study of it. Student results have been collected from the two semesters and compared ad hoc to show what happened during the implementation of this activity. It should be noted that the study was not conceived as a rigorous educational research project. It was designed as a laboratory activity to assist students in reaching course learning objectives. Data collected were only those that were normal to the educational process.

The methods of the study were submitted to the USF Institutional Review Board (IRB), as required, for approval prior to submission of any papers for publication involving students as subjects. The USFIRB did not consider this work to be research under their purview because all students were required to participate as part of the normal activities of the course.

Method

Students in the fall section of the course were given a pre-course test on the first day of the course, and the test included a continued fraction to reduce/solve. During the semester, as laboratory activity #4, all students participated in a participation-graded lab consisting of a pre-activity quiz followed by the lab activity itself. Students were then given a question on their first midterm exam which, in text form, explained a similar scenario to that found in the lab (very similar to the example given by Lagrange); the exam question asked the students to organize the measurements as a terminated continued fraction (that is, a continuing fraction that is finite in its number of continuations, as opposed to a reduced form of that fraction), and give a decimal equivalent to a certain number of decimal places (credit for the question was solely for the decimal answer). At the last class session before the final exam review, students were given the same pre-course test from the first session of the semester, but as a post-course test.

In the spring semester, the pre- and post-course tests were not used, but the lab was largely similar; the class size was much smaller (27 students in fall versus 8 in spring). The pre-activity quiz was used unchanged in the lab. Students were given a similar test item on their first midterm exam, with separate answer areas for laying out a continued fraction, reducing it, and giving a decimal equivalent. After largely successful results (see discussion of results below) an additional, ratcheted-up test item was included in the second midterm exam. This additional question asked the students to work *backward* from a reduced fraction to determine the equivalent terminated continued fraction (with numerators all equal to one) and express in words the measurements and remainders that would have produced it.

The pre-activity quiz was a simple, terminated continued fraction that students were asked to "evaluate and solve." Students were graded for participation (making an attempt, even if they did not know what to do, resulted in points, but students who came late to class and missed the pre-quiz did not receive points). The primary purpose of this pre-activity quiz was to formatively assess student knowledge at the start of the lab activity (as this information would allow the authors/instructors to determine to what level it was necessary to cover or re-cover the skills for reducing the fractions themselves, versus the practical use of the fractions in measurement as used in the lab).

The lab activity itself was relatively simple. Students were organized into small groups and provided with an unruled plastic device—in this case actually a portion of a logarithmic slide rule that was used in a later lab, but without the scale labels—and asked to measure the length and width of their desk using this device. During initial runs of the lab, the length unit was referred to as the "Denis" in honor of Dr. Denis Voytenko, former graduate teaching assistant of the course, now a postdoctoral fellow at the Center for Atmosphere Ocean Science, Courant Institute of Mathematical Sciences (New York University). Future iterations of this lab will use something such as disposable chopsticks as the measuring device—the previous devices all now have labels for their use as slide rules—and will go by the simple name of "sticks." For purposes of simplicity in reporting this lab, the units are referred to as sticks and abbreviated "stx". For example, in terms of the unit stick, the decimal form of the continued fraction in the quotation from Lagrange is 3.43 stx.[5,6]

Students were asked to provide the length and width of their table—and calculate the area—to the hundredths place. All the tabletops used were of uniform size (for a given semester), as were the measurement devices (the units were produced on the USF campus using a "3D printer" to pre-set measurements of 184 mm length, although students did not know this value at the time of the lab). Fall and spring cohorts met in different classrooms. In other words, the tabletops being measured are different in size across the two semesters, while the "stick" measuring standards were the same. In the initial lab, students were asked to measure each value once. After comparison of student group results (see below), this requirement was modified for the spring semester, and student groups were asked to measure length and width three times each.

In detail, in order to complete the lab properly, students would measure how many times the unit stick measured against the length of the desk. Students would count how many whole times the stick would fit, and then, assuming the number was not evenly divisible—that is, there was a remainder left over, which was indeed the case for all iterations of the lab on the first length and width measurement—they would note the length of this remainder by some means such as a mark on a blank piece of paper. This remainder (r_1) was then compared against the length of the stick. The remainder would, much like the original length, fit a certain number of whole times, with perhaps a second remainder (r_2). The length of r_2 would then be compared to the length of r_1—if necessary by marking the length of each on a blank piece of paper to see whether this was evenly divisible, or a third remainder (r_3) was present. If so, students would have to compare r_3 versus the length of r_2, but as this work was all done by hand and without magnification, the limits of human measurement made it unlikely that the process would continue beyond r_3 (although theoretically it could continue for quite some time). The results were arranged into a simple continued fraction (that is, a continued fraction where the numerator is always one).

[5] Or would 3.4 or 3.429 stx be better? We hardly broached the subject of measurement uncertainty and will develop it further in future iterations of the course.

[6] We look forward to word problems involving about kilostx and millistx. We already use word problems involving sq stx.

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After measurement, the continued fraction was reduced/solved, and converted to a decimal equivalent (students were permitted to use a calculator for this lab, although prior labs in this course did require pencil-and-paper and mental-only calculation methods to refresh those skills), rounded to the hundredths place. This procedure was then repeated for the width (and in the case of the spring semester version of the lab, it was performed three times for each dimension). Students were also asked to provide error for their values, but they were not provided with written instructions for how to determine error.

Student groups then provided their measurements to the class at large. Measurements were compared against each other. The first author went at a later time and measured the tabletop surface using a standard ruled meter stick to determine the length and width in SI units, which he converted to “sticks” so that a “correct” (Gold Standard) value would be available for comparison. Group values were compared for accuracy and precision to determine whether a hundredths place measurement was practical. In other words, accuracy was measured by determining whether the student-reported value range contained the actual Gold Standard value for the unknowns. Precision was measured by the repeatability of measurements across groups (or in the spring semester, by repeat measurements by the same group), testing whether the measured range was smaller than 0.01 times the base unit, or alternately by how small a range the values or their reported errors included.

Students were later given one question on their first midterm exam in the fall semester and one question on each of the two midterms in spring that re-created the experience of the lab, and thus required them to set up a continued fraction from a text description of the lab measurements, reduce/solve the continued fraction in fraction form, and give a decimal approximation to a certain number of significant figures (the second midterm in spring had students work backwards from the solution to determine the original measurements). Student responses in the fall semester on the continued fraction item from the pre-course test, the lab pre-activity quiz, the first midterm, and the post-course test were compared (as aggregate percentages per class, to protect student privacy) to determine whether, and to what extent, gains were made on the skill/task over the semester. Spring semester students were given a pre-activity quiz and two midterm exam items regarding the skill.

Results

Student Lab Group Responses

Table 1 indicates student responses for the fall semester; there were 27 students total, with one absent on the day of the lab. Table 2 indicates student responses for spring. As students were grouped into groups of up to four for each semester, there were two groups in the spring (Team Hutton and Team Steno,[7] 8 students total). It is therefore unreasonable to draw any quantitative conclusions from the spring student responses other than to report them.

First, we look at length measurements from Table 1. The length measurement provided by group 8 is a clear outlier, compared to the others. When removed, the mean becomes 8.22 stx and the standard deviation 0.05 stx. This standard deviation indicates a reasonable precision to the tenths place (two significant figures, in this case). However, students were asked to give values to the hundredths place (three significant figures). Clearly the measurements taken using this method did not meet this level of precision. It is possible that repeated measurements would have given more precise figures. The Gold Standard value was obtained using a meter stick and dividing the value by the known length of the stick from its 3D printing programming. The actual value was about 8.23 stx, and the average of the measured lengths was very close to this value if the outlying value for group 8 is removed (8.22 stx). The width measurements had a smaller standard deviation, even when compared to the size of the measurements. The average width (3.22 stx) and the actual value (3.26 stx) varied, but within the standard deviation (0.09 stx). Hundredth place precision is also not indicated for this measurement.

Table 1
Fall 2015 Student Measurement Data

GROUP	Length (stx)	+/-	Width (stx)	+/-	Area (stx^2)	+/-
1	8.20	0.005	3.20	0.005	26.24	0.057
2	8.32	0.005	3.21	0.005	27.5	0.01
3	8.21	0.02	3.23	0.02	26.52	0.11
4	8.25	0.05	3.32	0.050	29.39	0.786
5	8.167	0.005	3.2	0.05	26.133	0.43
6	8.17	0.085	3.03	0.015	27.18	0.09
7	8.24	0.005	3.25	0.005	26.78	0.06
8	9.2	0.05	3.2	0.05	29.44	0.62
MEAN	8.34	0.03	3.22	0.03	27.40	0.27
MEDIAN	8.23	0.01	3.22	0.02	26.98	0.10
ST. DEV.	0.35	0.03	0.09	0.02	1.33	0.30

All measurements and errors are self-reported by student groups, and are shown exactly as provided by the groups. All means, medians, and standard deviations were rounded to the hundredths place for simplicity. *N*=26

In Table 2 (spring semester) the width measurement for both groups was accurate to the hundredths place (3.22 stx). Length for “Team Steno” was also very closely replicated by repeat measurements, but not so for “Team Hutton.” That team’s length range was accurate (included correct value, 9.95 stx), but unreasonably wide. Given the small number of groups, it is difficult to draw much meaning from these numbers. However, it is quite troubling that Team Hutton failed so profoundly in measuring the table’s length. The first author was

[7] James Hutton (1726–1797) and Nicolas Steno (1638–1686) were important figures in the history of geology—contemporaries of, e.g., Euler (1707–1783) and Leibniz (1646–1716), respectively.

instructing the class during the lab session and was with this group during part of this time. The group appeared to be performing the measurement as one would expect—making a measure, lightly marking the slate table with a pencil, then measuring from there, and so forth. How the team then got values of 11.05, 10.88, and 9.74 stx is befuddling. It did appear that each measurement was made by a different member of the group, and if the 9.74 stx value was a simple blunder, the other two values do at least round to the same integer value, 11 stx.

Table 2
Spring 2016 Student Measurement Data

GROUP	L1 (stx)	L2 (stx)	L3 (stx)	LAvg (stx)	+/-	W1 (stx)	W2 (sx)	W3 (stx)	Avg (stx)	+/-	Area (stx^2)	+/-
HUTTON	11.06	10.88	9.74	10.56	1.10	3.25	3.33	3.33	3.30	0.05	34.85	1.78
STENO	9.14	9.11	9.14	9.13	0.03	3.33	3.33	3.33	3.33	0.00	30.40	0.10
MEAN	10.10	10.00	9.44	9.85	0.57	3.29	3.33	3.33	3.32	0.03	32.63	0.94
ST. DEV.	1.36	1.25	0.42	1.01	0.76	0.06	0.00	0.00	0.02	0.04	3.15	1.19

Measurements reported by student groups. Statistics are based on only two groups. Tables were of a different size than in fall semester, but stick units were the same size. *N*=8.

One item of interest regards error in both the fall and spring groups. Both were asked to an estimate of measurement error (specifically "+/-") but not instructed how to obtain this value. This issue was a secondary concern in this lab activity, and the question was included to determine whether there was a need for more time and instruction on that concept/skill set. Fall groups mostly gave error ranges that were much smaller than the measurements warranted, with the exception of groups 3 and 6. Group 3 correctly surmised that the smallest fractional remainder was the proper basis for determining the precision of the measurement. (Group 6 did not indicate in their work any particular reason for their range of error being so large.) Spring-semester groups tended to assume that the "+/-" included the maximum and minimum range limits of the three measurements rather than inherent error in the measurement method. In both cohorts these results indicate misunderstanding of measurement error on a fundamental level.

Task Ability Changes

Figure 1 indicates the progression of student assessment results in fall and spring semesters. For each portion of the figure, assessment items are shown in time progression along the *x*-axis (note that the spacing is not "to scale" in terms of the amount of time between assessments; the figure just shows the order in which things happened). Each of the different assessment items were rated for rigor along a relative difficulty scale (*y*-axis) from low to very high. The low-difficulty item was the pre-lab quiz, which required students to reduce an already-provided terminating continued fraction. The medium-difficulty level included the pre- and post-course test items (identical) which gave a terminated continued fraction and asked for a percentage equivalent. The high-difficulty items were the first exam in each semester. Each required students to set up a continued fraction from text description, reduce it, and give a decimal equivalent (in the fall, this last item was the graded item, while in the spring semester each section received separate credit). The exam 2 item in the spring semester gave a reduced fraction and required students to work backwards to determine the measurement remainders using continued fractions techniques; the difficulty of this item was rated as very high.

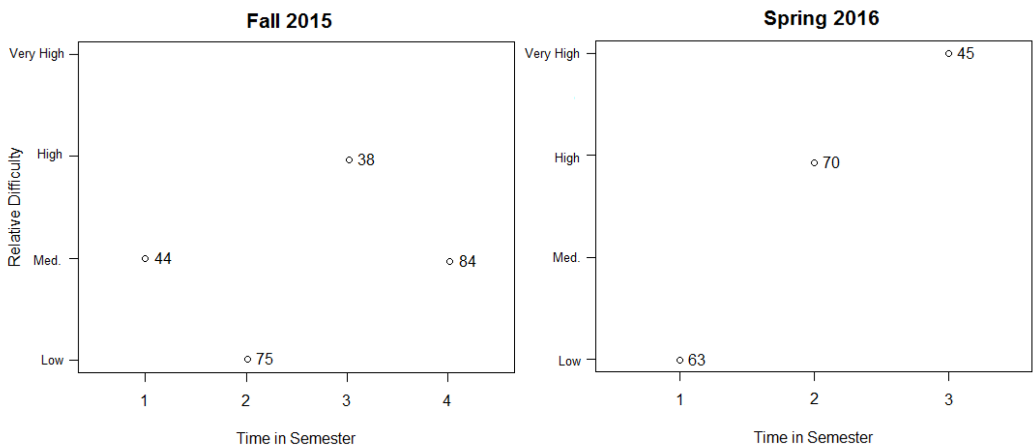

Figure 1. Student Assessment Results. Relative difficulty is selected by the authors to indicate the level of rigor of the question for each assessment. Time in semester shows the progression of assessments, with the fall sequence being 1-pre-course test, 2-pre-lab quiz, 3-exam 1, 4-post-course test. Spring sequence was 1-pre-lab quiz, 2-exam 1, 3-exam 2. Student average scores are shown beside each data point. *N*=27 in fall and *N*=8 in spring, with some occasional absences.

Taking the difficulty into account, it becomes possible to compare like kinds of fruit, so to speak. Students in the fall cohort improved by 40 percentage points on the same question between the pre-course and post-course test. Students in the spring semester showed an improvement of 7 percentage points between the pre-lab quiz and the first exam, despite the significant increase in rigor. There were no other activities in that semester utilizing this skill, therefore it seems reasonable to infer that this lab activity was helpful to the spring students. The later drop of 35 percentage points is attributable to the rigor of the question (the thought process of the author being that if 70% of the students can already do the item correctly, an increase in rigor is warranted). However, the improvement from the lab activity shown on the first exam was not noted in the fall – in fact, there was a drop of 37 percentage points from the pre-lab quiz to the first exam during that semester, despite essentially the same activity.

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Discussion: Collateral Benefits

This unit sticks–continued fractions exercise is rich with teaching and learning opportunities. One of the benefits of the activity is that the arithmetic can reveal for some advanced students that there are hazards with a seemingly elementary task that they are apt to take for granted because it is taught in the fifth grade. The task is *adding fractions*.[8] The insidiousness of this common deficiency is well known to the quantitative literacy (QL) community (e.g., Tucker 2008; Schield 2008).[9] For example, quoting Tucker (2008, 75):

> … It is in the transition from whole number arithmetic to fractions that too many students fall off the ladder of mathematical learning. They continue their education and become adults without ever understanding fractions.
>
> Consider the following question on the TIMSS 8th grade test:
>
> Find the approximate value, to the closest integer, of the sum 19/20 + 23/25.
>
> Possible answers were a) 1, b) 2, c) 42, d) 45. (Answer: b) The majority of U.S. students chose c) or d). These students did not think of a fraction as a number. When asked to add two fractions and get an integer answer, they added the numerators or the denominators of the two fractions. The only numbers that they knew about were counting numbers (whole numbers). A fraction to them was some combination of two whole numbers. To be fair, *fractions are a sophisticated mathematical concept compared to whole numbers* (emphasis added).

Fast forward to our senior-level, geology-major QL course populated with students who, for the most part, have had at least one semester of calculus. Following is the continued-fraction question on the midterm for the fall semester class.

> Suppose that you are conducting a measurement, much as you did in Vic's Lab 4, using an unruled Denis stick (symbol Ж). You measure the width of your computer screen and find that it measures 2 Ж with a remainder. The remainder, when compared to the Denis stick, fits in 2 times, with a second remainder. The second remainder fits into the first remainder 4 times, with a third remainder. The third remainder fits exactly into the second remainder 2 times (no remainder). What is the width of your computer screen? Give your answer to hundredths of a Denis. (Hint: use the method of continued fractions).

The answer is 2.45 Ж, which can be obtained by:

$$length = 2\,\text{Ж} + \frac{1}{2+\frac{1}{4+\frac{1}{2}}}\text{Ж} = 2\,\text{Ж} + \frac{1}{2+\frac{2}{9}}\text{Ж} = 2\,\text{Ж} + \frac{9}{20}\text{Ж} = 2.45\,\text{Ж}.$$

About 40% of the students gave the correct answer. One of the incorrect answers was 2.41 Ж. The result was obtained by

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[8] Coincidentally, another paper in this issue of *Numeracy* deals explicitly with a group of journalism students and their experience with adding fractions. See Harrison, "Journalists, Numeracy and Cultural Capital."
[9] See also Devlin 2005, 230–233).

$$length = 2\,Ж + \frac{1}{2+\frac{1}{4+\frac{1}{2}}}Ж = 2\,Ж + \frac{1}{2+\frac{4}{9}}Ж = 2\,Ж + \frac{9}{22}Ж = 2.41\,Ж\,,$$

where the greyed expression shows the incorrect step. The point in the steps internal to that shaded incorrect step were

$$2\,Ж + \frac{1}{2+\frac{1}{4+\frac{1}{2}}}Ж = 2\,Ж + \frac{1}{2+\frac{1}{\frac{8}{2}+\frac{1}{2}}}Ж = 2\,Ж + \frac{1}{2+\frac{1}{9/4}}Ж = 2\,Ж + \frac{1}{2+\frac{4}{9}}Ж\,,$$

where again the grey shading shows the misstep. The mistake is 8/2 + 1/2 = 9/4; in other words, the addition was done by adding the numerators and the denominators.

Surely our senior geology students know that 4 + 1/2, which is obviously larger than 4, is not equal to 9/4, which is only a little larger than 2, but in the "heat of the moment" one can lapse into bad technique when focusing intently on another part of a calculation, such as, in this case, the steps involving reciprocals of fractions. Thus the calculation itself provides teaching opportunities. An obvious one here is Step 4 of Polya's *How to Solve It* classic (Polya 1945): Looking back (Check your work!).

The calculation also provides the opportunity for brief, beneficial digressions, which help enliven interactive exchanges available in a comfortable lab environment. For example, there is the concept of mediant

$$mediant\left(\frac{a}{b},\frac{c}{d}\right) = \frac{a+c}{b+d}\,,$$

and the quotation from Conway and Guy (1996, 153), in connection with Farey fractions (noting *en passant*, and in context appropriate to our class, that John Farey, 1766–1826, was an early geologist[10]),

> Warning: forming the mediant is not the way to add fractions, unless you're calculating batting averages!

The Conway and Guy quotation then provides a nice segue to "baseball math." For example, suppose a batter with 10 hits in 40 at bats (batting average 0.250) has a good day with 4 hits in 4 at bats. What is the batter's new batting average? The numerical answer is 14/44 = 0.318. However, the students know perfectly well that 10/40 + 4/4 is 1.25. From a QL in Computational Geology perspective—where the concept of units is paramount—the discrepancy is immediately cleared up with *units* and a reminder of the schema for the weighted arithmetic mean, one of the stalwart and recurring subjects of the course. For

[10] James Hutton (see footnote 3) is widely regarded as the "father" of geology with publication of his *Theory of the Earth* in 1788. Thus Farey was roughly the age of our students when geology was born, and the Earth was seen to be unimaginably old.

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here, the question is asking for the average of two ratios, 0.250 hits/at bats and 1.000 hits/at bats, weighted by 40 at bats and 4 at bats, respectively. Thus

$$\frac{(40\ at\ bats)\left(0.25\,\frac{hits}{at\ bats}\right)+(4\ at\ bats)\left(1.00\,\frac{hits}{at\ bats}\right)}{(40\ at\ bats)+(4\ at\ bats)}=\frac{14\ hits}{44\ at\ bats}$$

Finally, the mistake in context provides a convenient point on our side for holding firm on no partial credit. While there may be a tendency for some students to think that 2.41 Ж is close enough to 2.45 Ж to warrant some partial credit, those students are easily convinced that size of a numerical difference is not proportional to the magnitude of the error in terms of their own fundamental arithmetic skills. Not even the most recalcitrant, point-counting student would argue that quantitative thought equivalent to $\frac{1}{2}+\frac{1}{2}$ equals $\frac{2}{4}$ should be in any way acceptable.

Concluding Remarks

The "stick lab," having been piloted twice in the Computational Geology course at USF, had the learning objectives of helping geology students (a) wake up about pitfalls of adding fractions, and (b) think about terminated continued fractions in a practical way that has a possible (if unlikely) field usage, with a secondary observation objective (c) to determine student understanding of measurement error. Preliminary results indicate success at achievement of learning objectives (a, b) with the need for further study indicated due to relatively small class numbers, and poor understanding of measurement error observed in both classes regarding (c). Class sections beginning in fall 2016 will continue to perform the lab activity, with the spring 2016 modification of taking three sets of measurements for each dimension in place.

It is interesting that students did not show prior learning in scientific measurement techniques, an absence which indicates a need for an adjustment in future semesters. The listed topic of the lab (when given to the students) was "Measurement and Error". Student responses were not reasonable or consistent with standard scientific practices with regards to measurement error. The backgrounds of the students are generally varied, but as this is an upper-level geology course, it is expected that this activity is by no means the first time students have performed measurements or recorded uncertainty. However, student concepts of error, how to measure error, and how to report measurements containing error indicate a definite need to alter course instruction before this lab is introduced in fall 2016 so that students are familiar with basic scientific expectations and procedures for measurement and assessing measurement error.

Appendix: Student Handout (fall semester 2015)

Lab 4: Measurement and Error

This lab is going to operate a little differently than the previous labs, and probably a bit differently than the labs that came before. This time, you're going to be presented with a *problem* in the life context, and I'm not going to tell you how to solve it. Well, not much, anyway.

Materials: You will be given a piece of plastic without any markings in or on it. **Please do not do anything to this item!** This is part of the slide rule we're going to use later in class, and there was a lot of time and effort (and a reasonable amount of money) that went into their construction. We have no more of them! So please, don't mark them, don't cut them, and don't leave them where they might melt, like on the seat of a hot car (I learned that one the hard way with our prototype). You may **NOT** use any rulers or similar measurement devices, or apps that simulate them.

Scenario: You and a small group of coworkers are stranded after a crash. You have nothing with you that can be used for "known" distance measurements, like a meter stick or ruler. What do you do?

The basic answer to this is you take something nearby that is sturdy enough to serve as a basic unit. It could be a stick, or a piece of metal, or a length of PVC, as long as it is easy to tell where it starts and ends, and that doesn't change. You then measure things in terms of how many of this local unit they are in length (we can call it a "Denis" and give it the unit marker "Ж" (Dr. Denis Voytenko is the former TA of this course, and the Cyrillic character is in his honor). The easy part is measuring that, say, the width of the doorway in front of you is a bit more than 5 Ж. The hard part is, how much more than "a bit"? How do you go from "a bit more than 5" Ж to 5.24 Ж?

One solution is to use continued fractions. You have to mark out (on something other than my ruler piece!) what the remainder is, and then determine how many times that piece divides back into the original ruler. If there's a remainder left when you measure that out, you have to repeat the procedure!

Assignment: Given only the unmarked ruler, and adding to this only paper and marking equipment (i.e., pencil or pen), measure the width of one of the tables in the classroom (the ones you are all sitting at that are all the same size). For clarity here, assume that the length of the slide rule piece *the long way* is 1 Ж.

1. Working in groups of 2-4, measure the length and width of your tabletop in continued fractions of a Ж as precisely as you can, and then convert the continued fraction to two decimal points (i.e., 14.54 Ж); then calculate the area of the tabletop surface in $Ж^2$. For this exercise, assume the corners of the desk are perfectly square, rather than slightly rounded, so take your measurements slightly off the very edge.
2. Estimate the error of all three measurements (length, width, area). That is, your measurements are +/- how many Ж, $Ж^2$, or fractions/decimals thereof?
3. Report your solutions to the above to the class on the board before the end of class so that a distribution of class measurements can be determined.

Items 1 and 2 are due at 10:30, so we have time to complete the rest.

References

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