Undergraduate Journal of Mathematical Modeling: One + Two

Volume 2 | 2010 Spring

Issue 2 | Article 3

Calculating the Time Constant of an RC Circuit

Sean Dunford
University of South Florida

Advisors:
Arcadii Grinshpan, Mathematics and Statistics
Gerald Woods, Physics

Problem Suggested By: Gerald Woods

Recommended Citation

Dunford, Sean (2010) "Calculating the Time Constant of an RC Circuit," *Undergraduate Journal of Mathematical Modeling: One + Two*: Vol. 2: Iss. 2, Article 3.
DOI: http://dx.doi.org/10.5038/2326-3652.2.2.3
Available at: https://scholarcommons.usf.edu/ujmm/vol2/iss2/3

Calculating the Time Constant of an RC Circuit

Abstract

In this experiment, a capacitor was charged to its full capacitance then discharged through a resistor. By timing how long it took the capacitor to fully discharge through the resistor, we can determine the RC time constant using calculus.

Keywords

Time Constant, RC circuit, Electronics

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PROBLEM STATEMENT

Using calculus, determine the time constant τ of an RC circuit for a recorded time with an initial charge on the capacitor of $7\mu f$, and a voltage of 30 volts.

MOTIVATION

This project and derivation is designed to acknowledge the value of a circuit's RC time constant. Knowing the time constant of an RC circuit can allow it to be used as a hardware filter. It can be utilized to only react to certain changes within the circuit. For instance windshield wiper speed settings in modern cars are controlled by RC circuits. They allow a lower voltage to reach the windshield wipers which makes them move slower. Electronic instruments, washing machines, children's toys, and many other pieces of technology all contain RC circuits. (Koehler,1)(Sheets. 1)

MATHEMATICAL DESCRIPTION AND SOLUTION APPROACH

After setting up the circuit to match the schematic and picture of circuit, we place a max charge on the circuit plate that is equal to that of the power supply (30 volts). The capacitance is supplied by a decade box and is given as $7\mu F$ (microfarads). The resistance is given by the DMM resistor as $10M\Omega$ (megaohms). Our resistor is also used to measure voltages at specific variable times. Charging the capacitor takes less than a second. We can determine the charge and current equations by considering Kirchhoff's voltage law. It states that the summation of the charges on all the pieces within a loop will always equal zero at any time t (Serway,785), i.e.

$$\varepsilon + \Delta V_{resistor} + \Delta V_{capacitor} = 0. \tag{1}$$

DOI: http://dx.doi.org/10.5038/2326-3652.2.2.3

The emf ε of the battery is taken to be 30 volts, the current at some instant is represented by I, the potential drop across the resistor is $-IR$, the magnitude of the charge on the capacitor at some instant is Q, the potential drop across the capacitor plates is shown as $-Q/C$ (Serway, 745/767). We can now rewrite (1) as

$$\varepsilon - IR - \frac{Q}{C} = 0. \tag{2}$$

Any time before the capacitor has a charge, $Q = 0$ and (2) becomes

$$I = \frac{\varepsilon}{R}. \tag{3}$$

Once the capacitor has reached the full voltage of the power supply $I = 0$ since charge no longer flows in the circuit. When this happens, (2) becomes

$$Q = C\,\varepsilon. \tag{4}$$

Current I is the change in charge over the change in time (Serway,753), i.e.

$$I = \frac{dQ}{dt}. \tag{5}$$

By substituting (5) into (2) we have,

$$\varepsilon - \left(\frac{dQ}{dt}\right) R - \frac{Q}{C} = 0. \tag{6}$$

and implies that

$$\frac{dQ}{\varepsilon\, C - Q} = \frac{dt}{RC}. \tag{7}$$

If we integrate both sides, we achieve

$$\int_{Q(t_0)=0}^{Q(t_f)} \left(\frac{1}{\varepsilon\, C - Q(t)}\right) dQ(t) = \frac{1}{RC}\int_{t_0=0}^{t_f} dt$$

which evaluates to

$$\ln\left|\varepsilon\, C - Q_f\right| - \ln|\varepsilon\, C| = \ln\left|\frac{\varepsilon\, C - Q_f}{\varepsilon\, C}\right| = -\frac{t_f}{R\, C}. \tag{8}$$

If $x = y$, then $e^x = e^y$ and (8) becomes

$$e^{\ln\left|\frac{\varepsilon C - Q_f}{\varepsilon C}\right|} = \frac{\varepsilon C - Q_f}{\varepsilon C} = e^{\left(-\frac{t}{RC}\right)} \tag{9}$$

and simplifies to

$$Q(t) = Q_f = \varepsilon\, C\, \left(1 - e^{-\frac{t}{RC}}\right). \tag{10}$$

Note that we may substitute (10) into (5) to get an equation for current:

$$I = \frac{dQ}{dt} = \frac{d}{dt}\left[\varepsilon\, C\, \left(1 - e^{-\frac{t}{R\,C}}\right)\right] = \frac{\varepsilon}{R} e^{-\frac{t}{RC}}. \tag{11}$$

Our main goal is to determine the exact value of the time constant $\tau = RC$. At time $t = \tau$, equation (10) tells us that

$$Q(\tau) = Q(RC) = \varepsilon\, C\left(1 - e^{-\frac{RC}{RC}}\right) = \varepsilon\, C\left(1 - \frac{1}{e}\right). \tag{12}$$

We can determine the time constant more accurately by considering similar equations for the discharging process. At time $t = \tau$ the emf ε from the battery is present on the plates of the capacitor. After the battery is disconnected from the circuit, the emf on the plates begins to dissipate. This discharging equation for Q is similar to (2) and is given by

$$IR - \frac{Q}{C} = 0 \tag{13}$$

which via (11) reduces to

$$Q(t) = RC\left(\frac{\varepsilon}{R} e^{-\frac{t}{RC}}\right) = \varepsilon\, C\, e^{-\frac{t}{RC}}. \tag{14}$$

During this discharge process ε is constant as the battery is disconnected from the circuit. Also, note that the current switches direction away from the capacitor becoming negative. This transforms (5) into

$$I = -\frac{dQ}{dt}. \tag{15}$$

In light of equation (14) and (15) we still get the same positive current equation as in (11)

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\left[\varepsilon\, C\, e^{-\frac{t}{RC}}\right] = \frac{\varepsilon}{R} e^{-t/RC}. \tag{16}$$

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DOI: http://dx.doi.org/10.5038/2326-3652.2.2.3

During the discharge process, voltages were recorded at various time intervals (see Table 1.2 in the Appendix). Graphing the voltage versus time yielded an exponentially decaying graph. To determine the time constant, we must establish a linear relationship ($y = mx + b$) of voltage and time. If we consider that relationship between charge and voltage

$$V = \frac{Q}{C} \tag{17}$$

where V is voltage, Q is charge and C is capacitance, (14) becomes

$$V(t) = \frac{Q(t)}{C} = \varepsilon\, e^{-\frac{t}{RC}}. \tag{18}$$

By taking the natural logarithm of both sides of (18), we establish a linear relationship between $\log[V(t)]$ and time t:

$$\log[V(t)] = \log\left[\varepsilon e^{-\frac{t}{RC}}\right] = -\,\frac{1}{RC}\,t + \log\varepsilon. \tag{19}$$

Using linear regression implemented in Excel (Graph 2.1), we find that

$$\log[V(t)] \approx -0.0205342\; t + 3.39561$$

which means

$$\tau = RC \approx 48.6993 \qquad \text{and} \qquad \varepsilon \approx 29.8329. \tag{20}$$

From (18) we know that the voltage is $e^{-1} \approx 37\%$ of the initial charge during the charging process. Since we know the initial value is equal to $\varepsilon = 30v$, the emf of the battery the capacitor at time τ will read

$$30(e^{-1}) \approx 11.0364 \text{ volts.} \tag{21}$$

Charging the capacitor and timing how long it took to reach our target value of 11.04 volts yielded the experimental value of RC.

DISCUSSION

This project found that the RC time constant of our circuit was 48.7 seconds. The objective of the project was met. According to our theoretic results, the voltage at the time should be 11.04 volts which was shown in Graph 1.1 to have occurred (see Appendix). With this time constant we could direct certain frequencies of voltages through our circuit and control a multitude of different objects.

CONCLUSION AND RECOMMENDATIONS

This project exploited the use of calculus to determine the RC time constant of a circuit. The time constant is a time in which it takes the capacitor to lose 63% of its initial charge. With a maximum initial voltage across a set of capacitor plates, a capacitance of $7\mu F$ and a $10 M\Omega$ resistor, we calculated the time constant to be around 48.7 seconds. Our theoretical results were compared with the experimental quantities. We measured the time constant by using a stop watch to record the time it takes for the voltage to drop from $30V$ to a target voltage of $11.04V$.

DOI: http://dx.doi.org/10.5038/2326-3652.2.2.3

NOMENCLATURE

Symbol	Name	Measured Units
τ	Time Constant	Seconds (sec)
C	Capacitance	Farads (f) or Microfarads (µf)
V	Voltage	Volts (V)
R	Resistance	Megaohms (MΩ)
ε	Emf of Battery	Volts (V)
Q	Charge	Coulombs (C)
T	Time	Seconds (sec)

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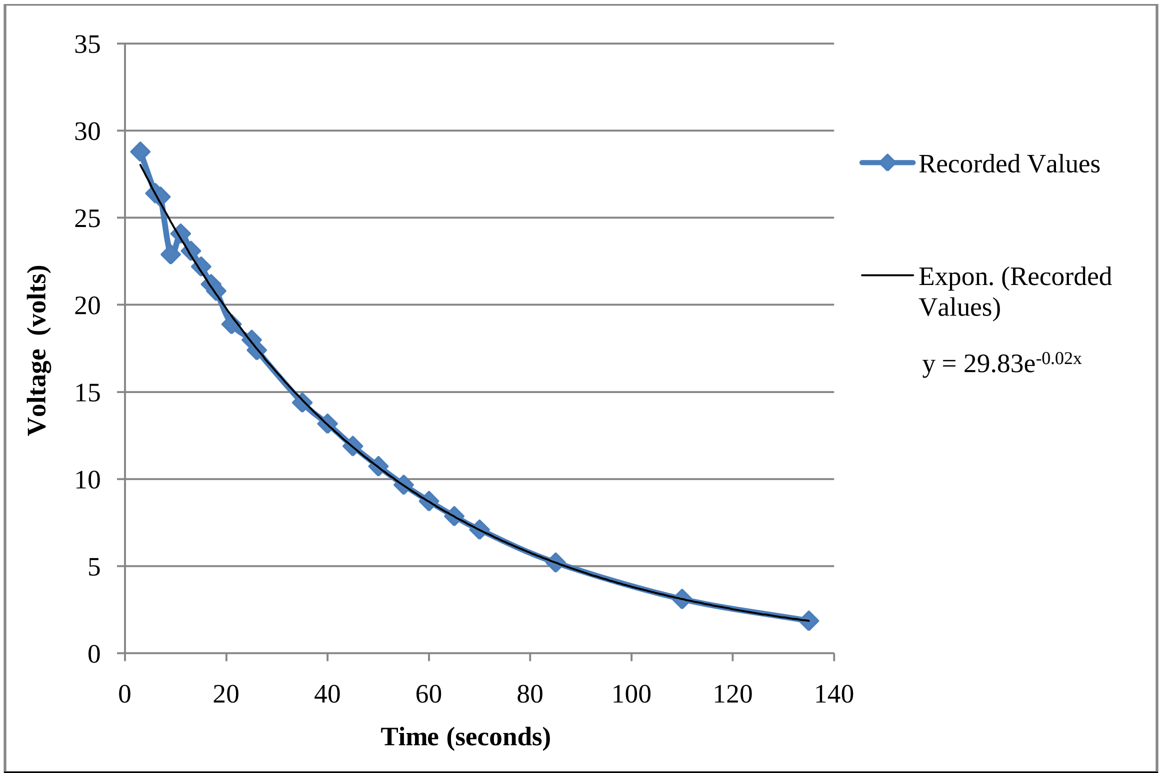
APPENDIX

GRAPH 1.1



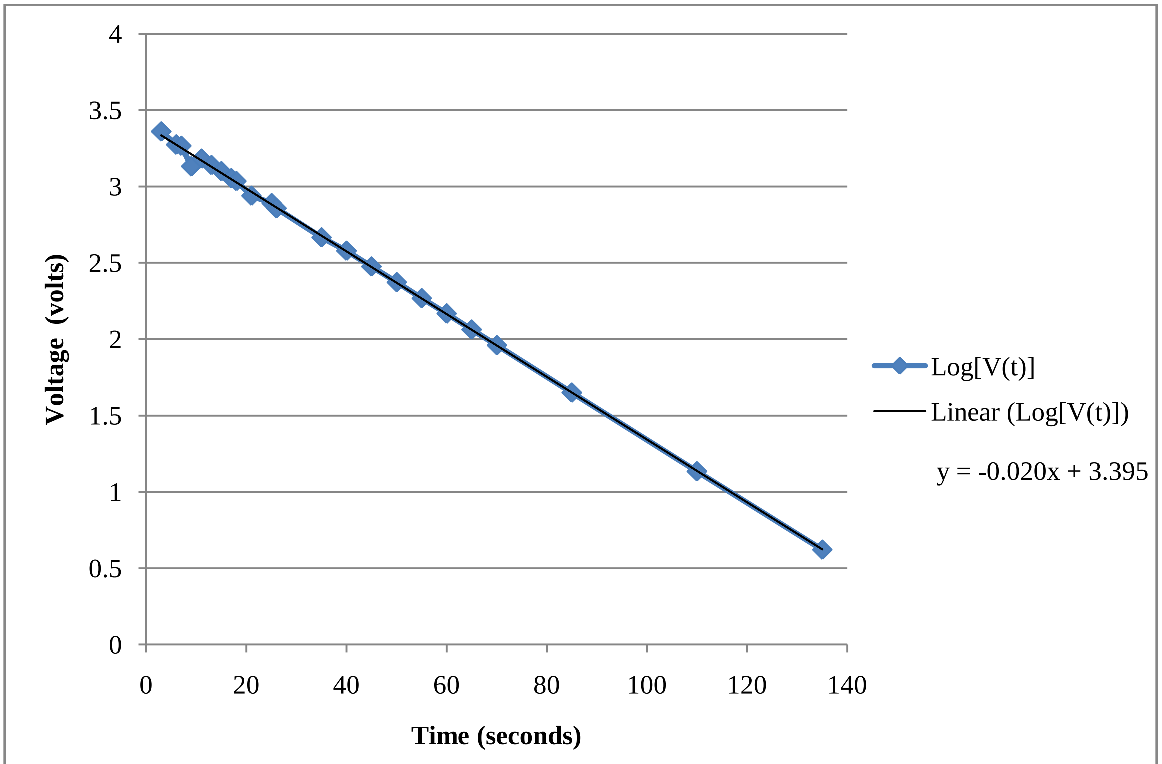
Graph 1.1 – Voltage recorded at the capacitor once the $30V$ power source was removed.

TABLE 1.2

Time	Voltage
3	28.79984
6	26.39976
7	26.19992
9	22.90024
11	24.10016
13	23.09989
15	22.19985
17	21.2002
18	20.79984
21	18.89992
25	17.9999
26	17.40003

Time	Voltage
35	14.39992
40	13.18014
45	11.88995
50	10.73001
55	9.670079
60	8.729915
65	7.869914
70	7.100027
85	5.210027
110	3.109997
135	1.859987

GRAPH 2.1



Graph 2.1 – Natural logarithm of the voltages achieved in Graph 1.1 versus time.

TABLE 2.2

Time	Voltage
3	3.36037
6	3.273355
7	3.265756
9	3.131147
11	3.182218
13	3.139828
15	3.100086
17	3.054011
18	3.034945
21	2.939158
25	2.890366
26	2.856472

Time	Voltage
35	2.667223
40	2.578711
45	2.475694
50	2.373044
55	2.269036
60	2.166756
65	2.063047
70	1.960099
85	1.650585
110	1.134622
135	0.620569

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