

Analysis of the Resistor-Inductor-Capacitor Circuit

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Abstract. Starting with the basics of what a Resistor-Inductor-Capacitor circuit (RLC) is, i.e. its fundamental components, and ending with practical applications using advanced calculus to aid in predetermining the results and circuit design, this paper analyzes the RLC circuit via an advanced calculus based approach.

Keywords. circuit design, resistor, inductor, capacitor, advanced calculus

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Problem Statement

Define a Resistor-Inductor-Capacitor (RLC) circuit, express it with advanced calculus and provide the examples of how it can be applied.

Motivation

This paper is designed to layout the fundamentals of an RLC circuit and provide assistance in the practical application in a complex circuit design. A better understanding of what an RLC circuit is, how it functions, how it can be calculated, and how it interacts with a complex circuit, will aid to better the electrical/electronic design as well as minimize the time required for a component selection and layout. The RLC circuit is used in both AC and DC circuits as filters, oscillators, voltage multipliers and pulse discharge circuits (Young, 1036).

MATHEMATICAL DESCRIPTION AND SOLUTION

APPROACH

An RLC circuit is comprised of three electrical devices: resistor, inductor and capacitor. A resistor is used to reduce the amount of current (i) in an electrical system much like a valve can reduce the amount of water flow through a pipe. This reduction can be called resistance (R), impedance/admittance (Z), or attenuation (α). Resistors are measured in Ohms (Ω). A capacitor is “an electrical component consisting of two or more conductive plates separated by insulation and used to store energy in an electrostatic field.” (Terrel, 10). This means an electric charge is stored in an electric field in the volume between two terminals. It is determined by the distance between the terminals: the greater the distance is the greater the potential energy that can be stored. This potential to store voltage is called capacitance (C) and it is measured in farads (F). An inductor is a component that stores electrical energy (q) within an electromagnetic field (Terrel, 10) that is produced when a current is flowed through a wire wound around a magnet. An inductor opposes the changes in the direction of the current flow. This magnetic field is measured in Henries (H). Capacitors and Inductors have impedance as well, which is usually specified by the manufacturer’s data sheet. The values of these components, and how they are configured determine how a RLC circuit functions.

Note: all the circuit diagrams were created using **National Instruments Multisim Blue 14.0.**

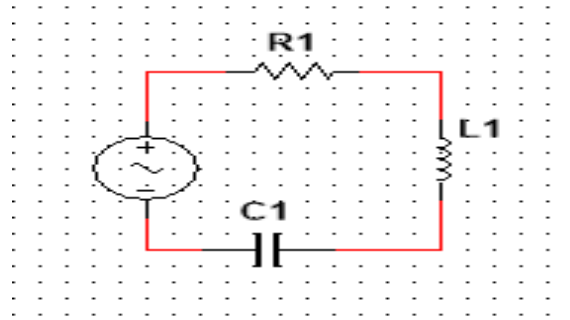


Image 1. An RLC circuit configured in series. Notice that the current, I , must flow from one to the next to the next. The configuration can be mixed with R , L and C in any order.

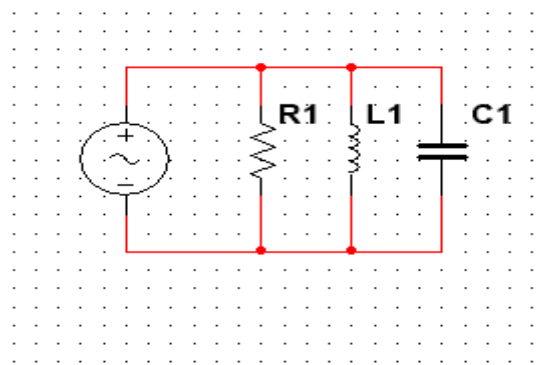


Image 2. An RLC circuit configured in parallel. Notice that the current can flow through all the components at once. Circuits can also be a mixture of both.

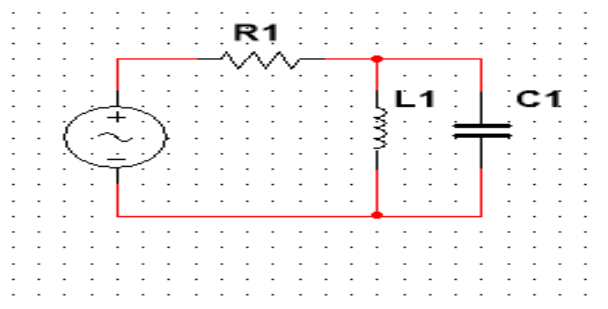


Image 3. An RLC circuit configured with the resistor in series with a capacitor and inductor that are in parallel.

A RLC circuit functions by creating a harmonic oscillator for current and resonates respectively (Young, 1009). The resonance of the circuit creates an alternating current (AC) signal. It is this AC signal that is the desired effect of the circuit. As later shown, the signal can be used to provide timing, signal filtration/reduction/cancelation, or signal tuning. The

application of calculus to the circuit design, and its desired effects, is a second-order differentiation.

The first thing to know is that all the RLC circuits operate at a frequency (f). It is the values of the components that determine the value. The frequency at which the circuit resonates is called its resonance frequency (f_0). As frequency is measured in hertz and mathematics prefers to measure it in radians (ω), ω_0 is defined as

$$\omega_0 = 2\pi f_0$$

(Stewart, 15). The resonance occurs due to the energy being stored in a capacitor and inductor. As the capacitor charges, current is being “consumed” by the capacitor as it charges its electric field. The capacitor will discharge, and current will then flow through to the inductor. The inductor will then “consume” the current as it creates an electromagnetic field. The inductor's field will discharge and current will flow back into the capacitor. Current can flow back and forth between those two since it is stored within their respective energy fields. The resistor is in the circuit to reduce the rate of current flow as well as reduce the amount of current. If a circuit could have zero resistance/impedance, then it would continue to oscillate for an indefinite time even after the initial energy source is removed!

All the LC circuits have what is called a natural frequency. This is the frequency at which a circuit oscillates, it is called the driven frequency (Young, 1036), and it has a value

Equations: 1, 2, 3, and 4

$$\omega_0 = \sqrt{LC} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\alpha = \frac{R}{2L} \text{ and } \alpha = \frac{1}{2RC}$$

Notice R is not present. This is to infer that the signal frequency into the circuit is not attenuated and is allowed to oscillate at maximum amplitude. When resistance/impedance is introduced into the circuit, the signal frequency will be attenuated or damped. Since all the electrical devices have some resistance/impedance, all the LC circuits can be treated as RLC circuits. Damping's attenuation (α) is measured in nepers per second. But trying to use nepers (Np) to calculate the damping is quite difficult. It is best to use a unit-less damping factor of zeta (ζ) to calculate the attenuation.

Both α and ω_0 are units of angular frequency. α is a measurement of how fast the transient response will fade after the removal of stimulus from the circuit (nepers is the SI unit) which is called attenuation and ω_0 is defined as the angular resonance frequency. They have different values depending on if the circuit is series or parallel.

Zeta is the actual dampening (c) divided by the critical dampening (c_c).

$$\zeta = \frac{c}{c_c}$$

Equations: 5 and 6

$$\zeta = \frac{c}{c_c}$$

When ζ is equal to 1, then the circuit is said to be critically damped and will just be short of oscillating. This means that if the correct frequency is applied to a RLC circuit it will fail to oscillate, but just so. Any variation in the frequency will disturb the balance.

The practical application of this phenomenon is using the frequency as a filter. The RLC circuits have a range of frequencies at which they start to become effective. This is called the bandwidth of the circuit. The bandwidth work starts at 3 dB points or when the signal is halved. The signal can be halved below (ω_1) and above (ω_2) the resonance frequency such that

Equation: 7

$$\Delta\omega = \omega_2 - \omega_1$$

A generalization of the bandwidth is the bandwidth being presented as a ratio of the resonant frequency.

Equation: 8

$$\frac{R}{2\pi L} = BW$$

Using the above, the Q factor can be expressed.

Equation: 9

$$Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} = \frac{1}{R} \sqrt{L/C}$$

The Q factor is defined “as the quality of the second-order filter stages.” (Dorf, 822) Wide bandwidth circuits will have a low Q and will dampen the signal. Narrow bandwidth circuits have a high Q and are under damped.

Series RLC Circuits

Equation: 10 $\alpha = \frac{R}{2L}$

Equation: 11 $\omega_0 = \sqrt{LC}$

and

Equation: 12 $\zeta = \frac{\alpha}{\omega_0}$

Both α and ω_0 are considered units of angular frequency. α is a neper, or is a measurement of how fast the transient response of a circuit will fade after the stimulus being applied has been removed. This is called attenuation. ω_0 is the angular resonance frequency. ζ can be used as a way to describe the ratio or factor of the circuit such that

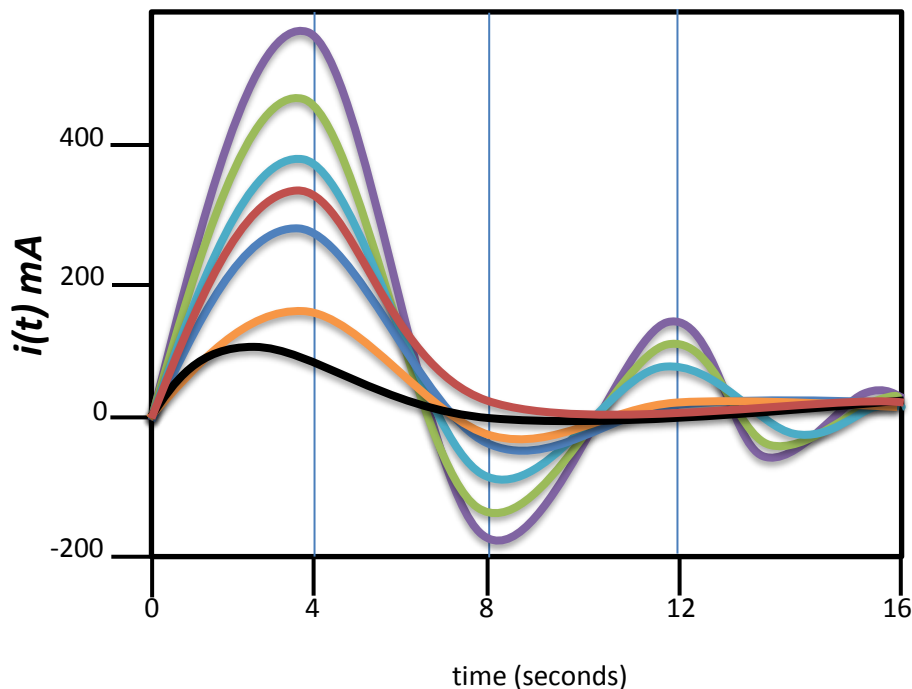


Image 4. If $\zeta < 1$, then the circuit is under damped (purple, green and light blue lines). If $\zeta > 1$, then the circuit is over damped (dark blue, orange and black lines). When $\zeta=1$, then the circuit is said to be critically damped (red line).

The total voltage of the serial circuit is defined as (t = sources time varying voltage)

Equation: 13 $V(t) = V_R(t) + V_L(t) + V_C(t)$

(when $t = 0$, treat the inductor as a short circuit and the capacitor as an open one). Rewrite as

$$\text{Equation: 14} \quad V_R(t) = IR, V_L(t) = L \frac{dI}{dt}, V_C(t) = \frac{Q}{C}$$

and it leads to

$$\text{Equation: 15} \quad L \frac{dI}{dt} + IR + \frac{Q}{C} = v_{(0)} \sin(\omega t + \varphi)$$

where

$$\text{Equation: 16} \quad I = \frac{+dQ}{dt}$$

and divide by L, the second order differentiation becomes

$$\text{Equation: 17} \quad \frac{d^2i}{dt} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = \frac{\omega v_{(0)}}{L} \cos(\omega t + \varphi)$$

And the circuit's current amplitude is

$$\text{Equation: 18} \quad I_0 = \frac{v_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

To calculate the reactance of the circuit, the individual reactance of the components will be added together. The reactance of the inductor is

$$\text{Equation: 19} \quad X_L = j\omega L$$

The reactance of the capacitor is

$$\text{Equation: 20} \quad X_C = \frac{1}{j\omega C}$$

The phase difference of the circuit is described as

$$\text{Equation: 21} \quad \varphi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

Total impedance is

$$\text{Equation: 22} \quad |Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

Transferring the equation from the frequency domain to the time domain

$$\text{Equation: 23} \quad f = \frac{1}{T} \rightarrow T = \frac{1}{f}$$

The differential equation takes on a characteristic equation. For example

$$\text{Equation: 24} \quad s^2 + 2\alpha s + \omega_0^2 = 0$$

S has roots for over damped

Equation: 25
$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

when

$$R > 2 \sqrt{\frac{L}{C}}$$

critically damped

Equation: 26
$$s_1 = s_2 = -\alpha$$

when

$$R = 2 \sqrt{\frac{L}{C}}$$

and under damped

Equation: 27
$$s_1, s_2 = -\alpha \pm j \sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j \omega_d$$

when

$$R < 2 \sqrt{\frac{L}{C}}$$

The differential equation is a general solution of an exponential in either root or a linear superposition of both. The arbitrary constants **A₁** and **A₂** are determined by boundary conditions. They can also be considered as the coefficients determined by the voltage and current once transience has started and the value they are expected to settle to after infinite time.

When the circuit is over damped, that is $\zeta > 1$,

then

Equation: 28
$$i_{(t)} = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

and the response of the circuit will be a delay of the transient current without oscillation.

When the circuit is under damped, that is $\zeta < 1$,

Then

$$\text{Equation: 29} \quad i_{(t)} = A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t)$$

Using trigonometric identities, the function can be expressed as

$$\text{Equation: 30} \quad i_{(t)} = A_3 e^{-\alpha t} \sin(\omega_d t + \varphi)$$

and the response of the circuit will be a decaying oscillation at frequency ω_d . The rate of decay in the oscillation is determined by the attenuation of α . The exponential in α describes the upper and lower wave, also called the envelope of the oscillation. The arbitrary constants B_1 and B_2 (also B_3 and the phase shift φ) are determined by the boundary conditions. ω_d is defined by

$$\text{Equation: 31} \quad \omega_d = j \sqrt{\omega_0^2 - \alpha^2}$$

And lastly, the critically damped response, that is $\zeta = 1$,

$$\text{Equation: 32} \quad i_{(t)} = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$$

Critically damped circuits respond with a decay as fast as possible without oscillation.

Using the definitions from above and the Laplace transform for admittance we obtain

$$\text{Equation: 33} \quad Y_{(s)} = \frac{i_{(s)}}{V_{(s)}} = \frac{s}{L(s^2 + \frac{R}{L}s + \frac{1}{LC})}$$

$$\text{Equation: 34} \quad Y_{(s)} = \frac{i_{(s)}}{V_{(s)}} = \frac{s}{L(s^2 + 2\alpha s + \omega_0^2)}$$

to find the values of s such that $Y_{(s)} = 0$ or the zeros of the function

$$\text{Equation: 35} \quad s = 0 \text{ and } |s| \rightarrow \infty$$

and the values of \mathcal{S} such that $\mathbf{Y}(\mathcal{S}) \rightarrow \infty$ or the poles of the function. The roots of \mathbf{S}_1 and \mathbf{S}_2 of the characteristic polynomial of the equation above are identical to the poles of $\mathbf{Y}(\mathcal{S})$.

The circuit will have a natural response (\mathbf{x}_n)

When

$$\omega_0 < \alpha$$

and the circuit is over damped

Equation: 36
$$\mathbf{x}_n(t) = A_1 t e^{s_1 t} + A_2 e^{s_2 t}$$

when

$$\omega_0 = \alpha$$

and the circuit is critically damped

Equation: 37
$$\mathbf{x}_n(t) = (A_1 + A_2 t) e^{-\alpha t}$$

when

$$\omega_0 > \alpha$$

and the circuit is under damped

Equation: 38
$$\mathbf{x}_n(t) = (A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)) e^{-\alpha t}$$

A circuit in series will result in having a minimum impedance at resonance, making it a resonator

Parallel RLC Circuits

Circuits that have the components in parallel require a different set of values and equations. In a parallel circuit

Equations: 39, 40, 41, 42 and 43
$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\zeta = \frac{1}{2R} \sqrt{\frac{L}{C}}$$

$$F_b = \frac{1}{R} \sqrt{\frac{L}{C}}$$

$$Q = R \sqrt{\frac{C}{L}}$$

Note: Q in a parallel circuit is inverse to that of a series circuit, component values that are a wide band filter in a series circuit will become a narrow band filter in a parallel circuit and vice versa

Equation: 44
$$\omega_d = \sqrt{\alpha^2 - \omega_0^2}$$

In the frequency domain, the admittance of the circuit is found by adding the individual components admittance together

Equation: 45
$$\frac{1}{Z} = \frac{1}{Z_L} + \frac{1}{Z_C} + \frac{1}{Z_R} = \frac{1}{j\omega L} + j\omega C + \frac{1}{R}$$

and

Equation: 46
$$|Z| = \frac{1}{\sqrt{\frac{1}{R^2} + (\frac{1}{\omega L} - \omega C)^2}}$$

and the power factor is described as

Equation: 47
$$\varphi = \tan^{-1}\left(R\left(\frac{1}{\omega L} - \omega C\right)\right)$$

The differential equation is

Equation: 48
$$\frac{d^2}{dt^2} i(t) + 2\alpha \left(\frac{d}{dt}\right) i(t) + \omega_0^2 i(t) = 0$$

an example of the characteristic equation is

Equation: 49
$$s^2 + \frac{2s}{\alpha} + \omega_0^2$$

over damping is described as

Equation: 50
$$s_1, s_2 = -\alpha \pm \omega_d$$

when

$$R < \frac{1}{2} \sqrt{\frac{L}{C}}$$

critical damping is described as

Equation: 51
$$s_1 = s_2 = -\alpha$$

when

$$R = \frac{1}{2} \sqrt{\frac{L}{C}}$$

and under damping is described as

Equation: 52
$$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

when

$$R > \frac{1}{2} \sqrt{\frac{L}{C}}$$

A circuit in parallel will result in a peak in an impedance at resonance making the circuit an anti-resonator.

Mixed Configuration RLC Circuits

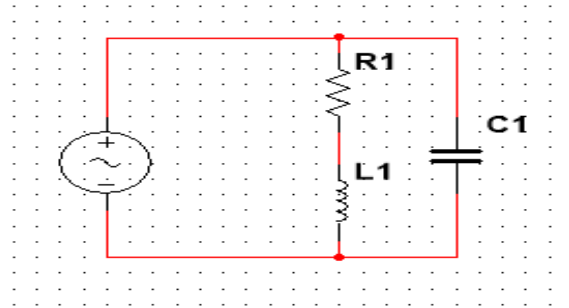


Image 5. The use of RLC circuits configured as bandpass filters is very common. The typical topology of the components in the circuit is a capacitor in parallel with a series resistor and inductor.

The resonant frequency of the circuit is described as

Equation: 53
$$\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

the characteristic equation is

Equation: 54
$$s^2 + 2\alpha s + \omega_0'^2 = 0$$

The above lends to the circuit having a natural undamped resonant frequency described as

Equation: 56
$$\omega_0' = \sqrt{\frac{1}{LC}}$$

To find the frequency at which the magnitude of impedance is at a maximum

Equation: 57
$$\omega_m = \omega_0' \sqrt{-\frac{1}{Q_L^2} + \sqrt{1 + \frac{2}{2Q_L^2}}}$$

$Q_L = \frac{\omega_0' L}{R}$ is a description of the quality factor (normally provided by the manufacturer) of the inductor. The maximum magnitude of the impedance can be found with

Equation: 58

$$|Z|_{Max} = RQ_L^2 \sqrt{\frac{1}{2Q_L \sqrt{Q_L^2 + 2 - 2Q_L^2 - 1}}}$$

This circuit configuration will result in a peak in impedance at resonance making the circuit an anti-resonator for a band of frequencies. The filter will “turn on” at a frequency, allowing a frequency range of signals to pass, then “turn off” for higher frequencies.

More examples of common filter configurations are

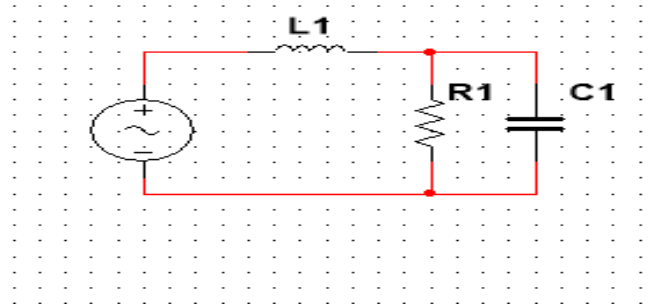


Image 6. RLC circuit is configured as a low pass filter

3dB points determined by

Equation: 59

$$\omega_c = \frac{1}{\sqrt{LC}}$$

and the damping factor is determined by

Equation: 60

$$\zeta = \frac{1}{2R_L} \sqrt{\frac{L}{C}}$$

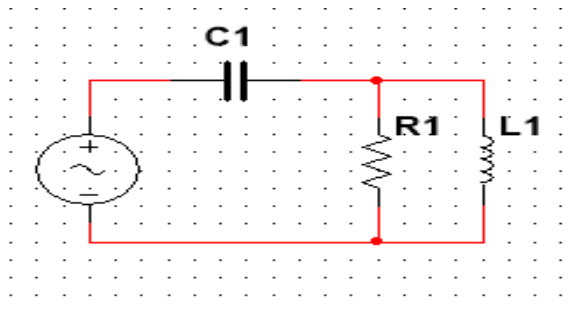


Image 7. RLC circuit is configure as a high pass filter with a stop band width determined by

Equation: 61
$$\omega_c = \frac{1}{\sqrt{LC}}$$

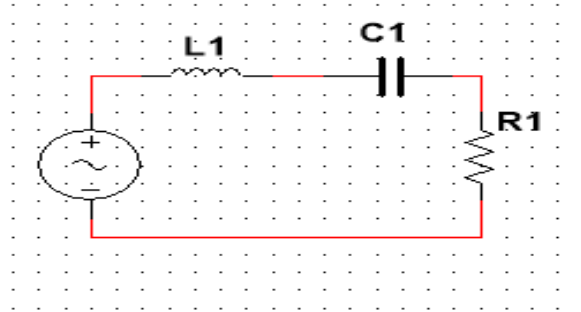


Image 8. RLC circuit is configured as a band pass series with a center frequency determined by

Equation: 62
$$\omega_c = \frac{1}{\sqrt{LC}}$$

and bandwidth of

Equation: 63
$$\Delta\omega = \frac{R_L}{L}$$

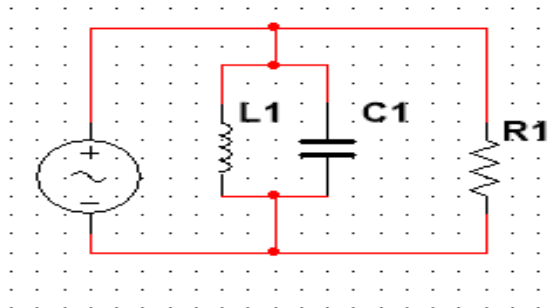


Image 9. RLC circuit is configured as a band pass parallel filter with a shunt across line the center frequency is the same as above but

Equation: 64
$$\Delta\omega = \frac{1}{CR_L}$$

Discussion

The analysis of the mathematical model of a RLC series circuit, configured to be a low pass filter, with a resistor of $4\ \Omega$, a capacitor of $160\ \mu\text{f}$, and an inductor of $1.5\ \text{mH}$, finds the theoretical response of the circuit.

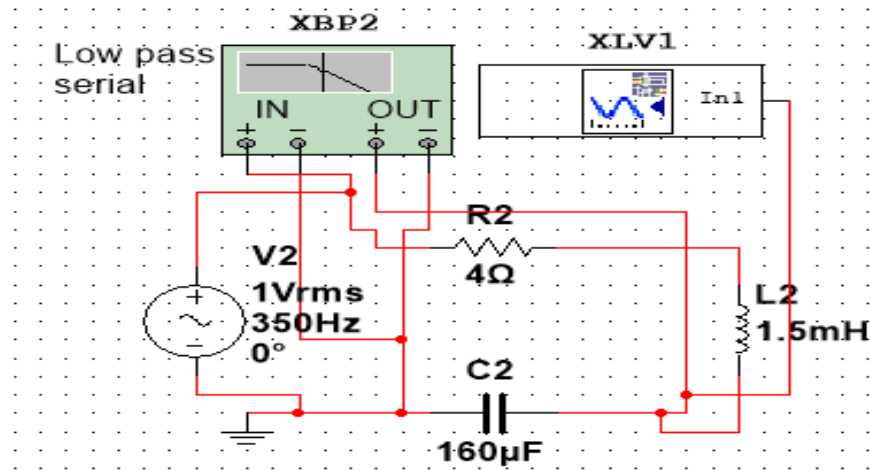


Image 10. Screen capture of circuit. XBP2 is a Bode plotter and XLV1 is a signal analyzer.

$$\alpha = \frac{R}{2L} = \mathbf{1333.33} \text{ nepers (damping coefficient)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \mathbf{326.6} \text{ Hz/s (resonant frequency)}$$

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}} = \mathbf{.7655}$$

$$\frac{R}{2\pi L} = \mathbf{BW} = \mathbf{424.41} \text{ Hz}$$

$$I_0 = \frac{v_0}{\sqrt{R^2 + (\omega_0 L - \frac{1}{\omega_0 C})^2}} = \mathbf{.5130A}$$

$$|Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} = \mathbf{.2463\ \Omega}$$

$$\varphi = \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) = \mathbf{.1211 \text{ rads}}$$

$$\zeta = \frac{\alpha}{\omega_0} = \mathbf{4.104} \text{ when } \zeta > \mathbf{0} \text{ and}$$

$$R > 2\sqrt{\frac{L}{C}} \text{ the circuit is over damped}$$

$$s^2 + 2\alpha s + \omega_0^2 = \mathbf{0}$$

$$\omega_d = j\sqrt{\omega_0^2 - \alpha^2} = \mathbf{408.89}$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = \mathbf{-40.18, -2626.5Hz}$$

(natural frequencies)

$$x_n(t) = i(t) = A_1 t e^{s_1 t} + A_2 e^{s_2 t} = \mathbf{.104 \text{ volts}}$$

$$t = \mathbf{.0006 \text{ s}}$$

$$A_1 = \mathbf{1.5}, A_2 = \mathbf{0.5}$$

At the frequency of ~325 Hz, this RLC is over damped. Verification can be performed using Multisim Blue.

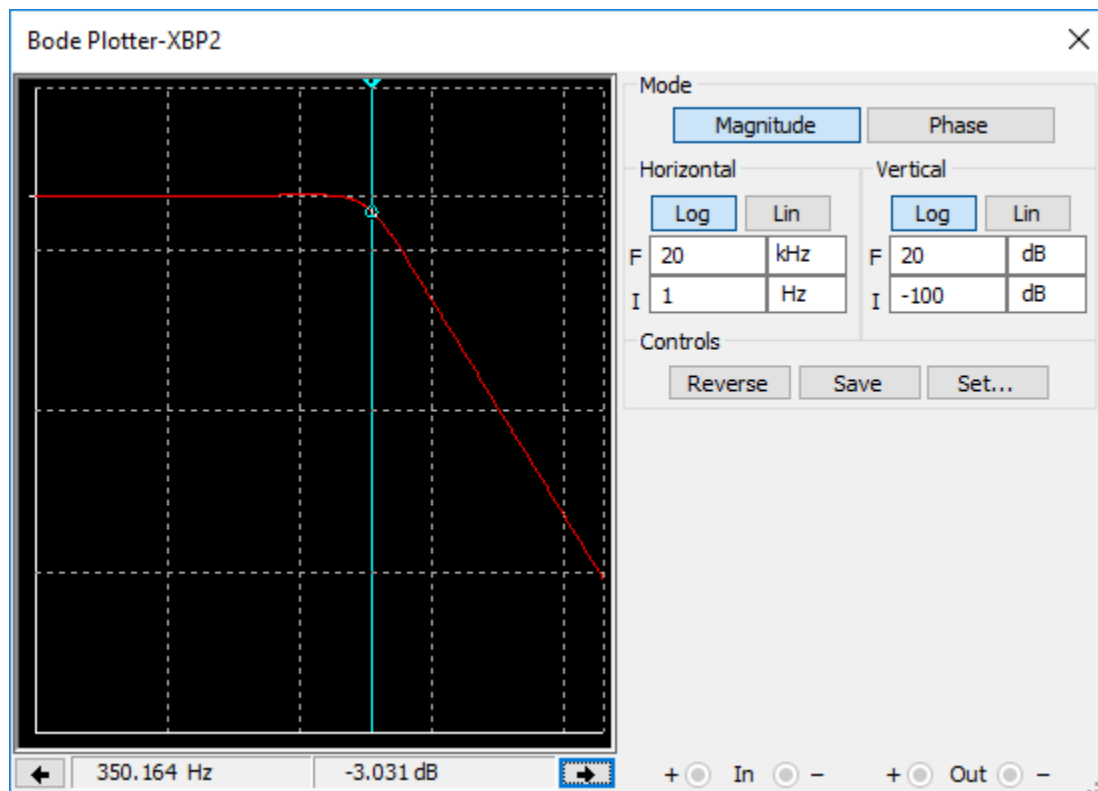


Image 11. The Bode plot shows the signal almost reaching -3.031dB at ~350hz.

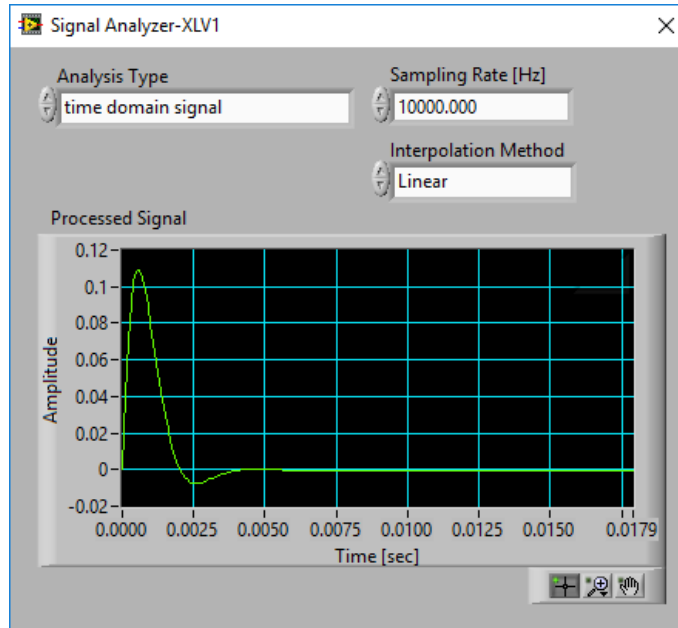


Image 13. The Signal analyzer shows the circuit is over damped.

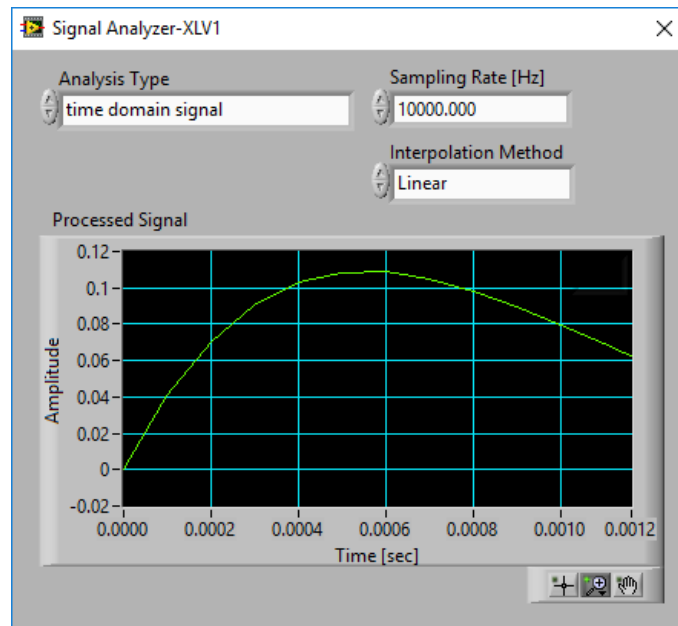


Image 13. Zooming in on the scale, the Signal analyzer shows that the attenuation equals the calculated within .01v.

The theoretical response of the example circuit is within 25 Hz and .01v of the actual response.

Conclusion and Recommendations

The analysis shows that calculating the resonant frequency and response of an RLC circuit is possible. The differences between the theoretical and actual are due to factors of: % ratings of components (most components have a plus or minus rating to the declared value), variances in trace material (not able to account for without measuring and adding to circuit), and the precision of the equipment measuring. However, an engineer can use calculus to predict the response of the RLC circuit to better aid in complete circuit design and analysis.

Nomenclature

Symbol	Name	Measured Unit
t	Time constant	Seconds (s)
R	Resistance/Impedance	Ohm (Ω)
L	Inductance	Henry H)
C	Capacitance	Farad (f)
I or i	Current	Amp (A)
V or v	Voltage	Volts (V)
f	Frequency	Hertz(Hz)
rad	Radian	Angle(rad)
Np	Neper	Ratios(Np)

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