University of South Florida
Scholar Commons

Graduate Theses and Dissertations | Graduate School

2010

Domination in graphs

Jennifer M. Tarr
University of South Florida

Follow this and additional works at: http://scholarcommons.usf.edu/etd

Part of the American Studies Commons

Scholar Commons Citation
Tarr, Jennifer M., "Domination in graphs" (2010). *Graduate Theses and Dissertations.*
http://scholarcommons.usf.edu/etd/1786

This Thesis is brought to you for free and open access by the Graduate School at Scholar Commons. It has been accepted for inclusion in Graduate Theses and Dissertations by an authorized administrator of Scholar Commons. For more information, please contact scholarcommons@usf.edu.

Domination in Graphs

by

Jennifer M. Tarr

A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Stephen Suen, Ph.D.
Nataša Jonoska, Ph.D.
Brendan Nagle, Ph.D.

Date of Approval:
May 19th, 2010

Keywords: Domination, Fair Domination, Edge-Critical Graphs, Vizing's Conjecture, Graph Theory

© Copyright 2010, Jennifer M. Tarr

Acknowledgments

First and foremost, I am indebted to Dr. Stephen Suen for his advice, encouragement and support during the last year. He devoted countless hours to assisting me with this thesis and I am truly grateful for that. I would also like to thank my other committee members, Dr. Nataša Jonoska and Dr. Brendan Nagle. Finally, I appreciate the tremendous amount of support received from all my family and friends throughout this process.

Table of Contents

List of Tables ii

List of Figures iii

Abstract iv

Chapter 1 Introduction 1
- 1.1 History 1
- 1.2 Graph-Theoretic Definitions 3
- 1.3 Domination in Graphs 6

Chapter 2 Vizing's Conjecture 15
- 2.1 Classes of Graphs Satisfying Vizing's Conjecture 15
- 2.2 Vizing-Like Conjectures for Other Domination Types 22
- 2.3 Clark-Suen Inequality and Improvement 30

Chapter 3 Fair Domination 34
- 3.1 Definition and General Results 34
- 3.2 Edge Critical Graphs 37

Chapter 4 Conclusion 43

References 46

List of Tables

Table 1 Symbols 45

List of Figures

Figure 1 The first image depicts a standard 8×8 chessboard. The second image has a queen placed in the upper right corner. If we represent every square on the board by a vertex in a graph, then we would draw an edge from the queen to every vertex representing one of the shaded squares. 4

Figure 2 Cycles C_4 and C_5 5

Figure 3 A tree T and the star $K_{1,4}$ 5

Figure 4 Complete graphs K_4 and K_5 5

Figure 5 Graphs G and H, and the corona $G \circ H$ 9

Figure 6 Coronas $K_1 \circ K_1$ and $K_2 \circ K_1$ and cycle C_4. 10

Figure 7 Family $\mathcal{A}$ 10

Figure 8 Family $\mathcal{B}$ 10

Figure 9 Independent domination in non-claw-free and claw-free graphs 12

Figure 10 An example of equality in domination and total domination 13

Figure 11 An example of equality in domination, total domination, connected domination, and clique domination 14

Figure 12 A graph G with $\gamma(G) = \rho_2(G)$ and a decomposable graph H formed by adding edges to G. 20

Figure 13 Example of a Type $\mathcal{X}$ graph with a special clique 22

Figure 14 The daisy with 3 petals 27

Figure 15 Partitions $\Pi_{i\bullet}$ and the sets $D_{\bullet v}, S_{\bullet v}$, and $C_{i\bullet}$; and partitions $\Pi_{\bullet j}$ and the sets $D_{u\bullet}, S_{u\bullet}$, and $C_{\bullet j}$ 33

Figure 16 Example of a graph with $\gamma(G) = \gamma_F(G) + 1$ 37

Figure 17 Examples of 3-edge-critical graphs 39

Figure 18 Fair domination in 3-edge-critical graphs: In each graph, let S_1 = the vertices that are blue, S_2 = set of green vertices, and S_3 = red vertices. These sets form a fair reception of each graph of size 3. 40

Domination in Graphs

Jennifer M. Tarr

ABSTRACT

Vizing conjectured in 1963 that the domination number of the Cartesian product of two graphs is at least the product of their domination numbers; this remains one of the biggest open problems in the study of domination in graphs. Several partial results have been proven, but the conjecture has yet to be proven in general. The purpose of this thesis was to study Vizing's conjecture, related results, and open problems related to the conjecture. We give a survey of classes of graphs that are known to satisfy the conjecture, and of Vizing-like inequalities and conjectures for different types of domination and graph products. We also give an improvement of the Clark-Suen inequality [17]. Some partial results about fair domination are presented, and we summarize some open problems related to Vizing's conjecture.

Chapter 1
Introduction

Mathematical study of domination in graphs began around 1960. The following is a brief history of domination in graphs; in particular we discuss results related to Vizing's conjecture. We then provide some basic definitions about graph theory in general, followed by a discussion of domination in graphs.

1.1 History

Although mathematical study of domination in graphs began around 1960, there are some references to domination-related problems about 100 years prior. In 1862, de Jaenisch [21] attempted to determine the minimum number of queens required to cover an $n \times n$ chess board. In 1892, W. W. Rouse Ball [42] reported three basic types of problems that chess players studied during this time. These include the following:

1. **Covering:** Determine the minimum number of chess pieces of a given type that are necessary to cover (attack) every square of an $n \times n$ chess board.

2. **Independent Covering:** Determine the smallest number of mutually nonattacking chess pieces of a given type that are necessary to dominate every square of an $n \times n$ board.

3. **Independence:** Determine the maximum number of chess pieces of a given type that can be placed on an $n \times n$ chess board such that no two pieces attack each other. Note that if the chess piece being considered is the queen, this type of problem is commonly known as the N-queens Problem.

The study of domination in graphs was further developed in the late 1950's and 1960's, beginning with Claude Berge [5] in 1958. Berge wrote a book on graph theory, in which he introduced the

"coefficient of external stability," which is now known as the domination number of a graph. Oystein Ore [39] introduced the terms "dominating set" and "domination number" in his book on graph theory which was published in 1962. The problems described above were studied in more detail around 1964 by brothers Yaglom and Yaglom [48]. Their studies resulted in solutions to some of these problems for rooks, knights, kings, and bishops. A decade later, Cockayne and Hedetniemi [16] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic.

Vizing's conjecture is perhaps the biggest open problem in the field of domination theory in graphs. Vizing [45] in 1963 first posed a question about the domination number of the Cartesian product of two graphs, defined in section 1.2. Vizing stated his conjecture that for any graphs G and H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ in 1968 [46].

This problem did not receive much immediate attention after being conjectured; however, since the late 1970s, several results have been published. These results establish the truth of Vizing's conjecture for certain classes of graphs, and for graphs that meet certain criteria. Note that we say a graph G satisfies Vizing's conjecture if, for any graph H, the conjectured inequality holds. The first major result related to Vizing's conjecture was a theorem from Barcalkin and German [4] in 1979. They studied what is referred to as decomposable graphs and established a class of graphs known as BG-graphs for which Vizing's conjecture holds. A corollary of this result is that Vizing's conjecture holds for all graphs with domination number equal to 2, graphs with domination number equal to 2-packing number, and trees. The result that Vizing's conjecture is true for trees was also proved separately by Faudree, Schelp and Shreve [22], and Chen, Piotrowski and Shreve [13].

Hartnell and Rall [27] in 1995 established Vizing's conjecture for a larger class of graphs. They found a new way of partitioning the vertices of a graph that is slightly different from the way Barcalkin and German partitioned the vertices in decomposable graphs. The Type $\mathcal{X}$ class of graphs that resulted from Hartnell and Rall's work is an extension of the class of BG-graphs.

Another approach to Vizing's conjecture is to find a constant $c > 0$ such that $\gamma(G\Box H) \geq c\gamma(G)\gamma(H)$. In 2000, Clark and Suen [17] were able to prove this inequality for $c = 1/2$. They used what is commonly referred to as the double projection method in their proof. As will be proven, this result can be improved to $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}$.

One of the most recent results related to Vizing's conjecture deals with the new concept of fair reception, which was first defined by Brešar and Rall [11] in 2009. They defined the fair domination number of a graph G, denoted $\gamma_F(G)$, and proved that $\gamma(G\Box H) \geq \max\{\gamma(G)\gamma_F(H), \gamma_F(G)\gamma(H)\}$. Thus, for any graph G having $\gamma(G) = \gamma_F(G)$, Vizing's conjecture holds. Brešar and Rall showed that the class of such graphs is an extension of the BG-graphs distinct from Type $\mathcal{X}$ graphs.

1.2 Graph-Theoretic Definitions

The study of domination in graphs came about partially as a result of the study of games and recreational mathematics. In particular, mathematicians studied how chess pieces of a particular type could be placed on a chessboard in such a way that they would attack, or dominate, every square on the board. With this in mind, graph theoretical definitions will be related to the game of chess where applicable.

A *graph* $G = (V, E)$ consists of a set V of vertices and a set E of edges. We shall only consider simple graphs, which contain no loops and no repeated edges. That is, E is a set of unordered pairs $\{u, v\}$ of distinct elements from V. The *order* of G is $|V(G)| = n$, and the $size$ of G is $|E(G)| = m$. If $e = \{v_i, v_j\} \in E(G)$, then v_i and v_j are *adjacent*. Vertex v_i and edge e are said to be *incident*.

Envision a standard 8×8 chessboard, as can be seen in Figure 1. Each square can be represented by a vertex in a graph G. Consider placing several queens on the board. A queen may move any number of spaces vertically, horizontally, or diagonally. Any square (or vertex) to which a queen is able to move is adjacent to the square containing the queen. Therefore, there is an edge between those two squares, or vertices of the graph G. Since the chessboard is 8×8, with each square reprented by a vertex of the graph G, the order of G is 64. The size of G depends on the number, type, and placement of chess pieces on the board.

We call the set of vertices adjacent to a vertex v in a graph G the *open neighborhood* $N(v)$ of v. The open neighborhood of a set of vertices $S \subset V(G)$ is $N(S) = \bigcup_{v \in S} N(v)$. The *closed neighborhood* $N[v]$ of v is $N(v) \cup \{v\}$, and the closed neighborhood of a set of vertices $S \subset V(G)$ is $N[S] = N(S) \cup S$.

The *degree* of a vertex v, denoted $\deg(v)$ is the number of edges incident with v. Alternatively, we can define $\deg(v) = |N(v)|$. The minimum and maximum degrees of vertices in $V(G)$ are

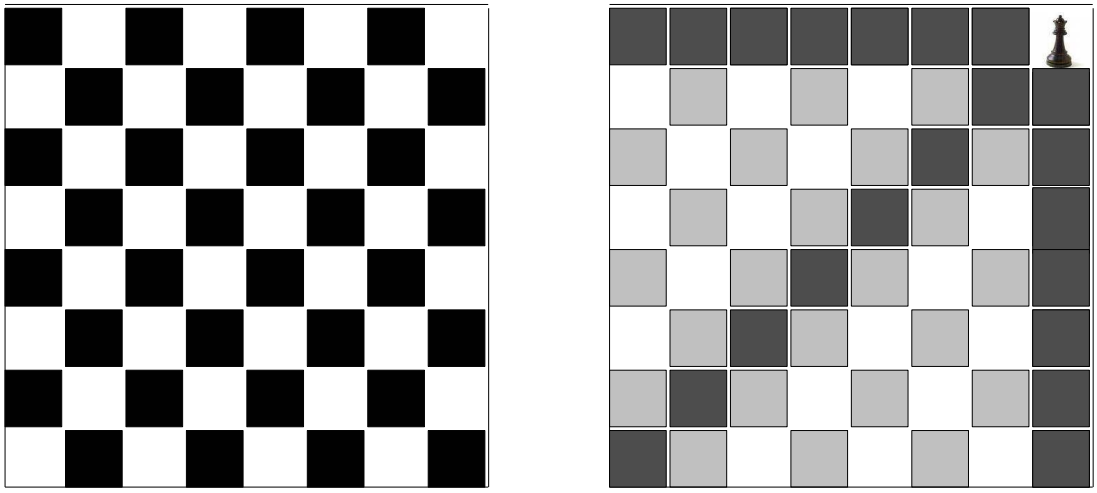

Figure 1.: The first image depicts a standard 8×8 chessboard. The second image has a queen placed in the upper right corner. If we represent every square on the board by a vertex in a graph, then we would draw an edge from the queen to every vertex representing one of the shaded squares.

denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $\delta(G) = \Delta(G) = r$, then the graph G is *regular* of degree r, or *r-regular*.

Consider, once again, placing several queens on a chessboard. Assume the space occupied by one of the queens is denoted by vertex v. Then the number of possible moves for the queen occupying that space, including those occupied by other queens, is equal to $\deg(v)$. If we count the number of possible spaces to which the queen in Figure 1 can move, we see that it has 21 possible moves. Thus, if we represent that chessboard by a graph and denote the space containing the queen as vertex v, we have $\deg(v) = 21$.

A *walk* of length k is a sequence $w = v_0, v_1, v_2, \ldots, v_k$ of vertices where v_i is adjacent to v_{i+1} for $i = 0, 1, \ldots, k-1$. A walk consisting of $k+1$ distinct vertices $v_0, v_1, ..., v_k$ is a *path*, and if $v_o = v_k$ then these vertices form a *cycle*. A graph G is *connected* if for every pair of vertices v and x in $V(G)$, there is a v-x path. Otherwise, G is *disconnected*. A *component* of G is a connected subgraph of G which is not properly contained in any other connected subgraph.

If there is at least one v-x walk in the graph G then the *distance* $d(v, x)$ is the minumum length of a v-x walk. If no v-x walk exists, we say that $d(v, x) = \infty$.

We now consider a few different types of graphs. The *cycle* C_n of order $n \geq 3$ has size $m = n$, is connected and 2-regular. See Figure 2 for the graphs C_4 and C_5. A $tree\ T$ is a connected graph

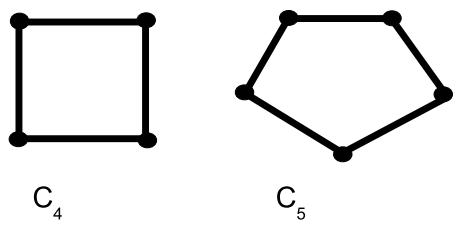

Figure 2.: Cycles C_4 and C_5

with no cycles. Every tree T with n vertices has $m = n - 1$ edges. The *star* $K_{1,n-1}$ has one vertex of degree $n-1$ and $n-1$ vertices of degree 1. Observe that a star is a type of tree. Refer to Figure 3 for examples of a tree and a star.

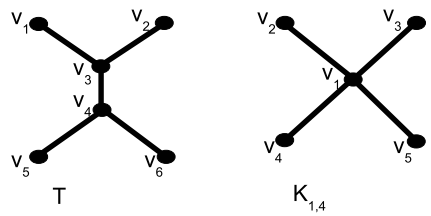

Figure 3.: A tree T and the star $K_{1,4}$

In any graph a vertex of degree one is an *endvertex*. An edge incident with an endvertex is a *pendant edge*. We can see that the graphs T and $K_{1,4}$ in Figure 3 each have four pendant edges and four endvertices. Specifically, in T, the endvertices are v_1, v_2, v_5, and v_6, and pendant edges are $\{v_1, v_3\}, \{v_2, v_3\}, \{v_4, v_5\}$, and $\{v_4, v_6\}$.

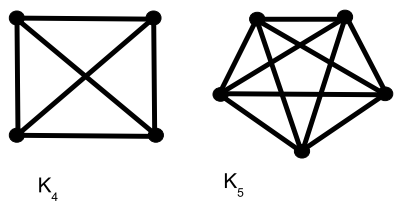

Figure 4.: Complete graphs K_4 and K_5

The *complete graph* K_n has the maximum possible edges $n(n-1)/2$. See Figure 4 for the graphs of K_4 and K_5. The *complement* $\bar{G}$ of a graph G has $V(\bar{G}) = V(G)$ and $\{u, v\} \in E(G)$ if and only if $\{u, v\} \notin E(\bar{G})$. Thus, the complement of a complete graph is the empty graph.

A *bipartite graph* is one that can be partitioned as $V = V_1 \cup V_2$ with no two adjacent vertices in the same V_i. We define the *chromatic number* of a graph G to be the minimum k such that $V(G)$ can be partitioned into sets $S_1, S_2, \ldots, S_k$ and each S_i is independent. That is, for each i, no two vertices in S_i are adjacent. Denote the chromatic number of G by $\chi(G)$. If $\chi(G) = k$, then G is k-colorable which means we can color the vertices of G with k colors in such a way that no two adjacent vertices are the same color. Observe that a graph is 2-colorable if and only if it is a bipartite graph.

The graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H satisfies the property that for every pair of vertices u and v in $V(H)$, the edge $\{u, v\}$ is in $E(H)$ if and only if $\{u, v\} \in E(G)$ then H is an *induced subgraph* of G. The induced subgraph H with $S = V(H)$ is called the *subgraph induced by* S. This is denoted by $G[S]$.

There are several different products of graphs G and H; we shall define the Cartesian product, strong direct product, and categorical product. All three of these products have vertex set $V(G) \times V(H)$. The *Cartesian product* of G and H, denoted by $G \square H$, has edge set

$$E(G \square H) = \{\{(u_1, v_1), (u_2, v_2)\} \mid u_1 = u_2 \text{ and } \{v_1, v_2\} \in E(H);$$

$$\text{or } \{u_1, u_2\} \in E(G) \text{ and } v_1 = v_2\}.$$

The *strong direct product* of G and H has edge set

$$E(G \square H) \cup \{\{(u_1, v_1), (u_2, v_2)\} \mid \{u_1, u_2\} \in E(G) \text{ and } \{v_1, v_2\} \in E(H)\}$$

and is denoted by $G \boxtimes H$. The *categorical product*, denoted by $G \times H$, has edge set

$$E(G \times H) = \{\{(u_1, v_1), (u_2, v_2)\} \mid \{u_1, u_2\} \in E(G) \text{ and } \{v_1, v_2\} \in E(H)\}.$$

1.3 Domination in Graphs

We now introduce the concept of dominating sets in graphs. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is a *dominating set* if every vertex $v \in V$ is an element of S or adjacent to an element of S. Alternatively, we can say that $S \subseteq V$ is a dominating set of G if $N[S] = V(G)$. A dominating set S is a *minimal dominating set* if no proper subset $S' \subset S$ is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. We call such a set a γ-set of G.

For a graph $G = (V, E)$ and $S \subseteq V$ a vertex $v \in S$ is an *enclave* of S if $N[v] \subseteq S$. For $S \subseteq V$ a vertex $v \in S$ is an *isolate* of S if $N(v) \subseteq V - S$. We say that a set is *enclaveless* if it does not contain any enclaves. Note that S is a dominating set of a graph $G = (V, E)$ if and only if $V - S$ is enclaveless.

Theorem 1.1 *[39] A dominating set S of a graph G is a minimal dominating set if and only if for any $u \in S$,*

1. *u is an isolate of S, or*

2. *There is $v \in V - S$ for which $N[v] \cap S = \{u\}$.*

Proof. [39] Let S be a γ-set of G. Then for every vertex $u \in S$, $S - \{u\}$ is not a dominating set of G. Thus, there is a vertex $v \in (V - S) \cup \{u\}$ that is not dominated by any vertex in $S - \{u\}$. Now, either $v = u$, which implies u is an isolate of S; or $v \in V - S$, in which case v is not dominated by $S - \{u\}$, and is dominated by S. This shows that $N[v] \cap S = \{u\}$.

In order to prove the converse, we assume S is a dominating set and for all $u \in S$, either u is an isolate of S or there is $v \in V - S$ for which $N[v] \cap S = \{u\}$. We assume to the contrary that S is not a γ-set of G. Thus, there is a vertex $u \in S$ such that $S - \{u\}$ is a dominating set of G. Hence, u is adjacent to at least one vertex in $S - \{u\}$, so condition (1) does not hold. Also, if $S - \{u\}$ is a dominating set, then every vertex in $V - S$ is adjacent to at least one vertex in $S - \{u\}$, so condition (2) does not hold for u. Therefore, neither (1) nor (2) holds, contradicting our assumption. □

Theorem 1.2 *[39] Let G be a graph with no isolated vertices. If D is a γ-set of G, then $V(G) - D$ is also a dominating set.*

Proof. [39] Let D be a γ-set of the graph G and assume $V(G) - D$ is not a dominating set of G. This means that for some vertex $v \in D$, there is no edge from v to any vertex in $V(G) - D$. But then the set $D - v$ would be a dominating set, contradicting the minimality of D. We conclude that $V(G) - D$ is a dominating set of G. □

Theorem 1.3 *[39] If a graph G has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

Proof. Let G be a graph with no isolated vertices and let D be a γ-set of G. Assume to the contrary that $\gamma(G) > \frac{n}{2}$. By Theorem 1.2, $V(G) - D$ is a dominating set of G. But $|V(G) - D| < n - \frac{n}{2}$, contradicting the minimality of $\gamma(G)$. We conclude that $\gamma(G) \leq \frac{n}{2}$. □

Theorem 1.4 *[36] For any graph G,*

$$\gamma(G) + \gamma(\bar{G}) \leq n + 1 \tag{1.1}$$

$$\gamma(G)\gamma(\bar{G}) \leq n \tag{1.2}$$

Proof. [36] We show (1.1) first. If the graphs G and $\bar{G}$ have no isolated vertices, then Theorem 1.3 implies $\gamma(G)+\gamma(\bar{G}) \leq n$. If G has an isolated vertex, then $\gamma(G) \leq n$ and $\gamma(\bar{G}) = 1$. Then we have $\gamma(G) + \gamma(\bar{G}) \leq n + 1$. Similarly, if $\bar{G}$ has an isolated vertex, we have $\gamma(\bar{G}) \leq n$ and $\gamma(G) = 1$, which implies $\gamma(G) + \gamma(\bar{G}) \leq n + 1$.

Now we prove (1.2). Define for $X \subseteq V(G)$ the following sets:

$$D_0(X) = \{u \in V(G) - X \mid \{u, v\} \in E(G) \text{ for all } v \in X\},$$

and

$$D_1(X) = \{u \in X \mid \{u, v\} \in E(G) \text{ for all } v \in X\}.$$

Now, let $D = \{v_1, v_2, ..., v_{\gamma(G)}\}$ be a γ-set of G and partition the vertices of $V(G)$ into sets Π_i such that $v_i \in \Pi_i$ for each $i = 1, 2, ..., \gamma(G)$ and if $v \in \Pi_i$ then $v = v_i$ or $\{v, v_i\} \in E(G)$. Choose this partition in such a way that $\sum_{i=1}^{\gamma(G)} |D_1(\Pi_i)|$ is a maximum.

Suppose $|D_0(\Pi_j)| \geq 1$ for some j. Then there is a vertex $v \in \Pi_k$, for $k \neq j$, such that $\{u, v\} \in E(G)$ for all $u \in \Pi_j$.

If $v \in D_0(\Pi_k)$ then $(D - \{v_j, v_k\}) \cup \{v\}$ is a dominating set of G with cardinality smaller than $\gamma(G)$, a contradiction. Thus, $v \notin D_0(\Pi_k)$.

Now we can re-partition the vertices of G so that $\Pi'_l = \Pi_l$ for $l \neq j$ and $l \neq k$, $\Pi'_j = \Pi_j \cup \{v\}$ and $\Pi'_k = \Pi_k - \{v\}$. But then $|D_1(\Pi'_l)| = |D_1(\Pi_l)|$, $|D_1(\Pi'_j)| = |D_1(\Pi_j)| + 1$, and $|D_1(\Pi'_k)| \geq |D_1(\Pi_k)|$. This contradicts the choice of our original partition of G.

We conclude that $|D_0(\Pi_i)| = 0$ for all $i = 1, 2, \ldots, \gamma(G)$. As any set X with $|D_0(X)| = 0$ dominates $\bar{G}$, each set Π_i dominates $\bar{G}$ and so $\gamma(\bar{G}) \leq |\Pi_i|$. Therefore, we have

$$n = \sum_{i=1}^{\gamma(G)} |\Pi_i| \geq \gamma(G)\gamma(\bar{G}).$$

□

We define the corona G of graphs G_1 and G_2 as follows. The *corona* $G = G_1 \circ G_2$ is the graph formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. Refer to Figure 5 for an example of a corona of two graphs. We take the original graph G and, as $|V(G)| = 4$, we have four copies of H. Both vertices in the ith copy of H are adjacent to the ith vertex in G for each $i = 1, ..., 4$.

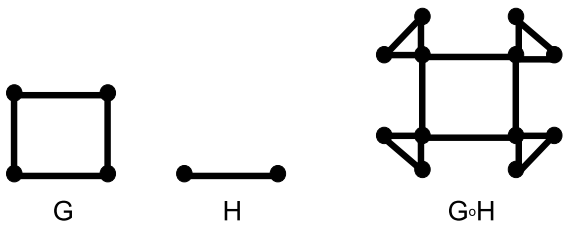

Figure 5.: Graphs G and H, and the corona $G \circ H$

The following theorem, which was proved independently by Payan and Xuong and by Fink, Jacobson, Kinch and Roberts, tells us which graphs have domination number equal to $\frac{n}{2}$. Thus, we can use this result to find extremal examples of graphs which achieve the upper bound in Theorem 1.3.

Theorem 1.5 *[23] [40] For a graph G with even order n and no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

Proof. [40] It can easily be verified that if the components of a graph G are C_4 or the corona $H \circ K_1$ for a connected graph H, then $\gamma(G) = \frac{n}{2}$.

Now we assume that $\gamma(G) = \frac{n}{2}$. We may assume that G is connected. Let $C = \{S_1, S_2, \ldots, S_p\}$ be a minimal set of stars which cover all vertices of G. Since $\gamma(G) = \frac{n}{2}$, C must be a maximal matching of $p = \frac{n}{2}$ edges. For each $S_i \in C$, let $S_i = \{x_i, y_i\}$. We consider two cases.

If $p \geq 3$ then for every i, either x_i or y_i has degree 1. If not, there is i such that $\deg(x_i) \geq 2$ and $\deg(y_i) \geq 2$. But then we can find a dominating set of G with cardinality less than $\frac{n}{2}$. This implies G is a corona $H \circ K_1$ for some connected graph H.

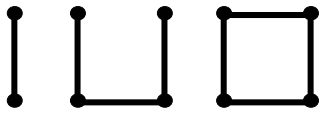

Figure 6.: Coronas $K_1 \circ K_1$ and $K_2 \circ K_1$ and cycle C_4.

If $p \leq 2$ then G is isomorphic to one of the graphs in Figure 6. Note that the first two graphs are coronas and the third is the cycle C_4.

We conclude that $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ where H is a connected graph. $\square$

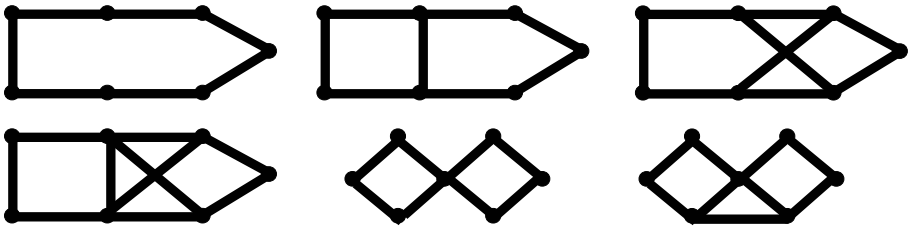

Figure 7.: Family $\mathcal{A}$

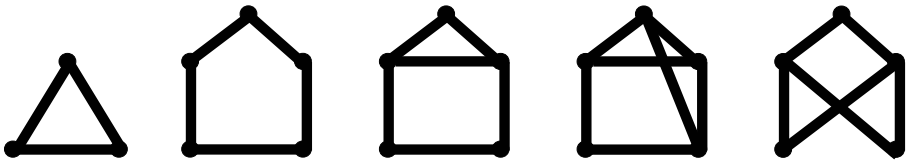

Figure 8.: Family $\mathcal{B}$

We now characterize connected graphs with $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ by defining the following six classes of graphs. These results were proved independently by Cockayne, Haynes and Hedetniemi [15] and by Randerath and Volkmann [41].

1. $G_1 = \{C_4\} \cup \{G \mid G = H \circ K_1$ where H is connected$\}$.

2. $G_2 = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are the families of graphs depicted in Figure 7 and Figure 8.

3. $G_3 = \bigcup_H S(H)$ where $S(H)$ denotes the set of connected graphs, each of which can be formed from $H \circ K_1$ by adding a new vertex x and edges joining x to at least one vertex in H.

4. $G_4 = \{\Theta(G) \mid G \in G_3\}$ where $y \in V(C_4)$ and for $G \in G_3$, $\Theta(G)$ is obtained by joining G to C_4 with the single edge $\{x, y\}$, where x is the new vertex added in forming G.

5. $G_5 = \bigcup_H \mathcal{P}(H)$ where u, v, w is a vertex sequence of a path P_3. For any graph H, $\mathcal{P}(H)$ is the set of connected graphs which may be formed from $H \circ K_1$ by joining each of u and w to one or more vertices of H.

6. $G_6 = \bigcup_{H,X} \mathcal{R}(H, X)$ where H is a graph, $X \in \mathcal{B}$, and $\mathcal{R}(H, X)$ is the set of connected graphs obtained from $H \circ K_1$ by joining each vertex of $U \subseteq V(X)$ to one or more vertices of H such that no set with fewer than $\gamma(X)$ vertices of X dominates $V(X) - U$.

Theorem 1.6 *[15] [41] A connected graph G satisfies $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $G \in \mathcal{G} = \bigcup_{i=1}^{6} G_i$.*

As a result of Theorem 1.5 and Theorem 1.6, we can completely classify graphs with domination number $\gamma(G) = \lfloor \frac{n}{2} \rfloor$.

We now define several additional types of domination in graphs. We shall show Vizing-like inequalities and conjectures for these types of domination in Section 2.2.

Let $f : V(G) \to [0, 1]$ be a function defined on the vertices of a graph G; this is a *fractional-dominating function* if the sum of the values of f over any closed neighborhood in G is at least 1. The *fractional domination number* of a graph G is denoted $\gamma_f(G)$ and is the minimum weight of a fractional-dominating function, where the weight of the function is the sum over all vertices of its values. A similar type of domination is integer domination. Let $k \geq 1$ and let $f : V(G) \to \{0, 1, \ldots, k\}$ be a function defined on the vertices of a graph G. This is a $\{k\}$*-dominating function* if the sum of the function values over any closed neighborhood of G is at least k. As with fractional domination, the weight of a $\{k\}$-dominating function is the sum of its function values over all vertices. We define the $\{k\}$*-domination number* of G to be the minimum weight of a $\{k\}$-dominating function of G. This is denoted by $\gamma_{\{k\}}(G)$.

The maximum cardinality of a minimal dominating set of a graph G is called the *upper domination number* and is denoted by $\Gamma(G)$. We say that a set $S \subset V(G)$ is *independent* if for all u and v in S, $\{u, v\} \notin E(G)$. The maximum cardinality of a maximal independent set in G is the *independence number* $\alpha(G)$, and the minimum cardinality of a maximal independent set is the *lower independence number* $i(G)$. Note that the lower independence number is also often referred to as the *independent domination number*.

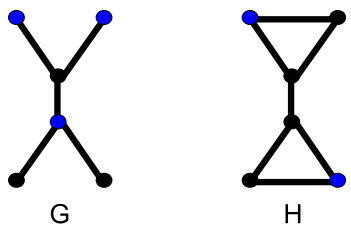

Figure 9.: Independent domination in non-claw-free and claw-free graphs

Observe that claw-free graphs, or graphs that do not contain a copy of $K_{1,3}$ as an induced subgraph, have $\gamma(G) = i(G)$. This result was proved by Allan and Laskar in 1978 [3]. Refer to Figure 9. It can easily be verified that the graphs G and H both have domination number equal to 2. The graph G is not claw-free and $i(G) = 3$; an example of a minimal independent dominating set of G is indicated by the blue vertices. The graph H, on the other hand, is claw-free and has $i(H) = 2$. We can see that the blue vertices in H form an independent dominating set.

A set $S \subseteq V(G)$ is a *total dominating set* of G if $N(S) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set. Note that a dominating set S is a total dominating set if $G[S]$, the subgraph induced by S has no isolated vertices. The *upper total domination number* of G, denoted by $\Gamma_t(G)$, is the maximum cardinality of a minimal total dominating set of a graph G. The function $f : V(G) \to \{0, 1, \ldots, k\}$ is a *total* $\{k\}$*-dominating function* if the sum of its function values over any open neighborhood is at least k. The *total* $\{k\}$*-domination number* $\gamma_t^{\{k\}}$ of a graph G is the minimum weight of a total $\{\mathrm{k}\}$-dominating function of G.

The above defined parameters of a graph G are related by the following lemma.

Lemma 1.1 *[38] For any graph G, $\gamma_f(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$. If G has no isolated vertices, then $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$.*

For any graph G, a matching is a set of independent edges in G, and a perfect matching of G is one which matches every vertex in G. The set $D \subseteq V(G)$ is a *paired dominating set* of G if D dominates G and the induced subgraph $G[D]$ has a perfect matching. We denote the *paired domination number*, or the minimum cardinality of a paired dominating set, by $\gamma_{pr}(G)$.

The *independence domination number* of a graph G, denoted by $\gamma^i(G)$, is the maximum, over all independent sets I in G, of the minimum number of vertices required to dominate I. Note that this is different from the independent domination number.

There are several other types of domination, defined below, for which we will not present further Vizing-like results.

Let $G = (V, E)$ be a bipartite graph, with partite sets V_1 and V_2. If a set of vertices $S \subseteq V_1$ dominates V_2, we say that S is a *bipartite dominating set* of G.

A *connected dominating set* is a dominating set that induces a connected subgraph of the graph G. We denote by $\gamma_c(G)$ the *connected domination number*, or the minimum cardinality of a dominating set S such that $G[S]$ is connected. Clearly, $\gamma(G) \leq \gamma_c(G)$.

Observe that when $\gamma(G) = 1$, $\gamma(G) = \gamma_c(G) = i(G) = 1$. This implies that if G is a complete graph or a star, the domination number, connected domination number, and independent domination number all equal 1. Also, since a connected dominating set of G is also a total dominating set of G, we have $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$. An example of the sharpness of this bound can be seen in the complete bipartite graph $K_{r,s}$, in which $\gamma(K_{r,s}) = \gamma_t(K_{r,s}) = \gamma_c(K_{r,s}) = 2$. See Figure 10, which depicts the graph $K_{2,3}$. The blue vertices form both a minimal dominating set and a total dominating set.

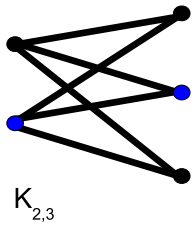

Figure 10.: An example of equality in domination and total domination

If D is a dominating set of G and $G[D]$ is complete, then we call D a *dominating clique*. The minimum cardinality of a dominating clique is the *clique domination number*, denoted $\gamma_{cl}(G)$. Not every graph has a dominating clique; for example, any cycle C_n where $n \geq 5$ does not contain a dominating clique. Clearly, if $\gamma(G) = 1$, then $\gamma(G) = \gamma_c(G) = \gamma_{cl}(G) = 1$. If G has a dominating clique and $\gamma(G) \geq 2$ then $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq \gamma_{cl}(G)$. An example of the sharpness of these bounds can be seen in the corona $K_p \circ K_1$, which has $\gamma(K_p \circ K_1) = \gamma_t(K_p \circ K_1) = \gamma_c(K_p \circ K_1) = \gamma_{cl}(K_p \circ K_1) = p$. The blue vertices in the graph of the corona $K_3 \circ K_1$ in Figure 11 form a minimal dominating set which is also a total dominating set, connected dominating set, and a dominating clique.

A *cycle dominating set* is a dominating set of G whose vertices form a cycle.

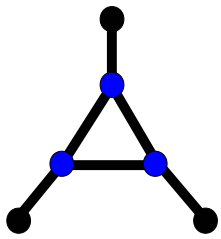

Figure 11.: An example of equality in domination, total domination, connected domination, and clique domination

Chapter 2
Vizing's Conjecture

Since Vizing's conjecture was first stated in the 1960s, several results have been published which establish the truth of the conjecture for classes of graphs satisfying certain criteria. As the problem has not yet been solved in general, researchers have also studied similar problems for different types of graph products and for other types of domination. Some of these similar problems also remain conjectures, while others have been proven. Here, we describe the classes of graphs which are known to satisfy Vizing's conjecture and provide a brief discussion of the similar Vizing-like conjectures which have also been studied. Another common approach to solving the conjecture is to find a constant c such that for any graphs G and H, $\gamma(G\Box H) \geq c\gamma(G)\gamma(H)$. As Clark and Suen [17] proved in 2000, this is true for $c = \frac{1}{2}$. We provide a slight improvement of this lower bound by tightening their arguments.

2.1 Classes of Graphs Satisfying Vizing's Conjecture

Vizing's conjecture is that for any two graphs, the domination number of the Cartesian product graph of G and H is greater than or equal to the product of the domination numbers of G and H. The conjecture is stated as follows:

Conjecture 2.1 *[46] For any graphs G and H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.*

Recall that the Cartesian product of graphs G and H has vertex set

$$V(G\Box H) = V(G) \times V(H) = \{(x, y) \mid x \in V(G) \text{ and } y \in V(H)\}$$

and it has edge set

$$E(G\Box H) = \{\{(x_1, y_1), (x_2, y_2)\} \mid x_1 = x_2 \text{ and } \{y_1, y_2\} \in E(H);$$

$$\text{or } \{x_1, x_2\} \in E(G) \text{ and } y_1 = y_2\}.$$

Define a *2-packing* of G as a set $X \subset V(G)$ of vertices such that $N[x] \cap N[y] = \emptyset$ for each pair of distinct vertices $x, y \in X$. Alternatively, we can define a 2-packing as a set X of vertices in G such that for any pair of vertices x and y in X, $d(x, y) > 2$. The maximum cardinality of a 2-packing of G is called the *2-packing number of* G and is denoted by $\rho_2(G)$.

Observe that for any graph G, $\rho_2(G) \leq \gamma(G)$. Let S be a maximal 2-packing of G. Then, as $d(u, v) > 2$ for every pair of vertices u and v in S, we need at least one vertex in $V(G)$ to dominate each vertex in S. Hence, the cardinality of a minimal dominating set is greater than or equal to the cardinality of a maximal 2-packing.

Note that we say a graph G satisfies Vizing's conjecture if, for any graph H, the conjectured inequality holds. Several results establish the truth of Vizing's conjecture for graphs satisfying certain criteria. The case where $\gamma(G) = 1$ is trivial. A corollary of Barcalkin and German's [4] proof that Vizing's conjecture holds for decomposable graphs is that Vizing's conjecture is true for any graph G with $\gamma(G) \leq 2$. In 2004, Sun [44] verified Vizing's conjecture holds for any graph G with $\gamma(G) \leq 3$.

We now consider classes of graphs that are proven to satisfy Vizing's conjecture.

Lemma 2.1 *[26] If G satisfies Vizing's conjecture and K is a spanning subgraph of G such that $\gamma(G) = \gamma(K)$, then K satisfies Vizing's conjecture.*

Proof. Let K be a spanning subgraph of G obtained by a finite sequence of edge removals which does not change the domination number. Since K is a subgraph of G, $K \Box H$ is a subgraph of $G \Box H$. Thus we have $\gamma(K \Box H) \geq \gamma(G \Box H) \geq \gamma(G)\gamma(H)$ by assumption on G. By assumption on K, we have $\gamma(G)\gamma(H) = \gamma(K)\gamma(H)$. We conclude that K satisfies Vizing's conjecture. $\Box$

Theorem 2.1 *[28] Let G be a graph and let $x \in V(G)$ such that $\gamma(G - x) < \gamma(G)$. Then if G satisfies Vizing's conjecture, the graph $G - x$ satisfies Vizing's conjecture.*

Proof. [28] Let G be a graph which satisfies Vizing's conjecture, and assume $\gamma(G - x) < \gamma(G)$ for some $x \in V(G)$. Then $\gamma(G - x) = \gamma(G) - 1$. Now assume there is a graph H such that $\gamma((G - x) \Box H) < \gamma(G - x)\gamma(H)$. Let A be a γ-set of $(G - x) \Box H$ and let B be a γ-set of H. Define $D = A \cup \{(x, b) \mid b \in B\}$. Clearly D is a dominating set of $G \Box H$ of cardinality

$|A| + |B| < \gamma(G - x)\gamma(H) + \gamma(H) = (\gamma(G - x) + 1)\gamma(H) = \gamma(G)\gamma(H)$. This contradicts our assumption that G satisfies Vizing's conjecture, and so we conclude that $G - x$ satisfies Vizing's conjecture. $\square$

Note that, if the converse of this theorem does not hold, we would have a counterexample to Vizing's conjecture. Consider a graph K that satisfies Vizing's conjecture, and let $S \subseteq V(K)$ be a set of vertices such that no vertex of S belongs to any γ-set of K and such that $\gamma(K - S) = \gamma(K)$. We can form a graph G from K by adding a new vertex v and all edges $\{u, v\}$ where u is in S. If the resulting graph G does not satisfy Vizing's conjecture then obviously we have a counterexample. If, on the other hand, we can prove that the graph G satisfies Vizing's conjecture, then this result would contribute to an attempt to prove Vizing's conjecture by using a finite sequence of constructive operations. The idea is to begin with a class C of graphs for which we know Vizing's conjecture is true and find a collection of operations to apply to graphs from C, each of which results in a graph which satisfies Vizing's conjecture. At this point, the goal would be to show that any graph can be obtained from a seed graph in C by applying a finite set of these operations. This type of approach has obviously not yet been successful, but Hartnell and Rall [28] define several operations which could potentially lead to a proof of Vizing's conjecture using a constructive method.

Lemma 2.2 *[20] For any graphs G and H, $\gamma(G\Box H) \geq \min\{|V(G)|, |V(H)|\}$.*

Proof. [20] Let D be a γ-set of the product graph $G\Box H$, and assume to the contrary that $|D| < \min\{|V(G)|, |V(H)|\}$. Then there is a column of vertices $H_u = \{u\} \times V(H)$ and a row of vertices $G_v = V(G) \times \{v\}$ such that $D \cap H_u = D \cap G_v = \emptyset$. But then $(u, v) \notin N[D]$, a contradiction. Therefore, $\gamma(G\Box H) \geq \min\{|V(G)|, |V(H)|\}$. $\square$

The following result providing a lower bound for $\gamma(G\Box H)$ was proved by Jacobson and Kinch [34]. Their proof considers a dominating set for the product graph $G\Box H$ and counts the way the dominating set intersects each set of vertices $V(G) \times \{v\}$, where $v \in V(H)$.

Theorem 2.2 *[34] For any graphs G and H, $\gamma(G\Box H) \geq \frac{|H|}{\Delta(H)+1}\gamma(G)$.*

Observe that this theorem implies Vizing's conjecture holds for cycles of length $3k$. Consider the cycle C_{3k}, for $k \geq 1$ an integer. We have $\Delta(C_{3k}) = 2$ and $\gamma(C_{3k}) = k$, so therefore $\frac{|C_{3k}|}{\Delta(C_{3k})+1} = \frac{3k}{3} = k = \gamma(C_{3k})$.

Theorem 2.3 *[45] For any graphs G and H, $\gamma(G\Box H) \leq \min\{\gamma(G)|V(H)|, |V(G)|\gamma(H)\}$.*

Proof. Let A be a γ-set of G. Now let $D = \{A \times \{v\} \mid v \in V(H)\}$. Then D is a dominating set of $G\Box H$ of cardinality $\gamma(G)|V(H)|$. Similarly, we can let B be a γ-set of H and define $D = \{\{u\} \times B \mid u \in V(G)\}$. Thus, we have $\gamma(G\Box H) \leq \min\{\gamma(G)|V(H)|, |V(G)|\gamma(H)\}$. $\Box$

Theorem 2.4 *[35] For any graphs G and H,*

$$\gamma(G\Box H) \geq \max\{\gamma(G)\rho_2(H), \rho_2(G)\gamma(H)\}.$$

Notice that this result from Jacobson and Kinch can be improved by the following theorem from Chen, Piotrowski and Shreve.

Theorem 2.5 *[13] For any graphs G and H,*

$$\gamma(G\Box H) \geq \gamma(G)\rho_2(H) + \rho_2(G)(\gamma(H) - \rho_2(H)).$$

The earliest significant result related to the domination number of a Cartesian product was produced by Barcalkin and German [4] in 1979. Barcalkin and German studied graphs G which have domination number equal to the chromatic number of $\bar{G}$. Recall that the chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G in such a way that no two adjacent vertices are the same color. Observe that any proper coloring of $\bar{G}$ is a partition of the vertices of G into cliques, or complete subgraphs of G. A single vertex may be chosen from each clique to form a dominating set of G and, therefore, it is always true that $\gamma(G) \leq \chi(\bar{G})$.

Barcalkin and German defined *decomposable graphs* as follows. Let G be a graph with $\gamma(G) = k$, and assume $V(G)$ can be partitioned into k sets $C_1, C_2, ..., C_k$ such that each induced subgraph $G[C_i]$ is a complete subgraph of G. If G satisfies these conditions, then it is a decomposable graph. They also define the *A-class*, which consists of all graphs G' that are spanning subgraphs of a decomposable graph G, where $\gamma(G') = \gamma(G)$. The result of Barcalkin and German's 1979 paper established Vizing's conjecture for any graph which belongs to the A-class. Note that we now commonly refer to this class of graphs as BG-graphs.

Theorem 2.6 *[4] Let G be a decomposable graph and let K be a spanning subgraph of G with $\gamma(G) = \gamma(K)$. Then K satisfies Vizing's conjecture.*

Proof. [28] We assume that G is a decomposable graph with $\gamma(G) = k$. Let $\{C_i \mid G[C_i]$ is a complete subgraph of $G, 1 \leq i \leq k\}$ be a partition of $V(G)$. We now consider the partition $\{C_i \times V(H) \mid i = 1, \ldots, k\}$ of $V(G \square H)$ for H an arbitrary graph. Let D be a γ-set of $G \square H$.

Denote by D_j the set of vertices in D that are also in $C_j \times V(H)$. That is,

$$D_j = D \cap (C_j \times V(H)) \text{ for } j = 1, \ldots, k.$$

Let $u_j \in C_j$ and denote by P_j the projection of vertices in $C_j \times V(H)$ onto $\{u_j\} \times V(H)$.

Let L_j be the set of all vertices v such that (u_j, v) is not dominated by $P_j(D_j)$. That is,

$$L_j = \{v \mid (u_j, v) \notin N[P_j(D_j)]\}.$$

We observe that if $v \in L_i$, then the vertices $C_j \times \{v\}$ are dominated "horizontally". Obviously, if $P_j(D_j)$ dominates $u_j \times V(H)$, $|L_j| = 0$. However, if $|D_j| = \gamma(H) - m$ then we have

$$|D_j| + |L_j| \geq |P_j(D_j)| + |L_j| \geq \gamma(H).$$

This implies that $|L_j| \geq m$.

We now consider $v \in V(H)$ such that $v \in L_i$ for at least one $i = 1, \ldots, k$. Define the sets D_v, S_v, and A_v as follows. We let $S_v = \{C_i \mid v \in L_i$ and $i = 1, \ldots, k\}$. Define A_v to be the set of cliques C_j such that there is at least one edge from a vertex in C_j to a member of S_v and $D \cap (C_j \times \{v\}) \neq \emptyset$. Finally, we let $D_v = \{u \in V(G) \mid (u, v) \in D$ and $u \in C_j \in A_v\}$.

We observe that $|D_v| \geq |S_v| + |A_v|$, for otherwise we would have

$$\hat{D}_v = D_v \cup \{(u_j, v) | C_j \notin S_v \cup A_v\}$$

is a dominating set of $V(G) \times \{v\}$ of cardinality less than k.

Also observe that for each $i = 1, \ldots, k$ either $|D_i| \geq \gamma(H)$, in which case summing over i gives the desired inequality; or $|D_i| = \gamma(H) - m$. In the latter case, we have shown that $|D_v| \geq |S_v| + |A_v|$. From this, we have

$$|S_v| \leq \sum_{u \in D_v} (|D \cap (C_j \times \{u\})| - 1). \tag{2.1}$$

Thus, we have sufficient extra vertices in D in neighboring cliques so that we still have an average of $\gamma(H)$ for each $|D_j|$. We conclude that $\gamma(G \square H) = |D| \geq \gamma(G)\gamma(H)$.

If K is a spanning subgraph of a decomposable graph G satisfying $\gamma(G) = \gamma(K)$, then we apply Lemma 2.1 to prove that K also satisfies Vizing's conjecture. $\square$

Corollary 2.1 *[4] Let G be a graph satisfying $\gamma(G) = 2$ or $\rho_2(G) = \gamma(G)$. Then G satisfies Vizing's conjecture.*

This corollary follows from the previous theorem. Any graph G with $\gamma(G) = 2$ is a subgraph of a decomposable graph. To establish the second part of the corollary, we assume G is a graph satisfying $\gamma(G) = \rho_2(G)$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a 2-packing of G. Then we can add edges to G to make $N[v_1], N[v_2], \ldots, N[v_{k-1}]$ and $V(G) - (N[v_1] \cup N[v_2] \cup \ldots \cup N[v_{k-1}])$ into cliques. The resulting graph is decomposable and still has k pairwise disjoint closed neighborhoods. Hence, it follows from Theorem 2.6 that any graph with $\gamma(G) = \rho_2(G)$ satisfies Vizing's conjecture. An example of this can be seen in Figure 12. The labeled vertices $v_1, v_2,$ and v_3 in G form a 2-packing of the graph. We can add edges as described above to get the decomposable graph H.

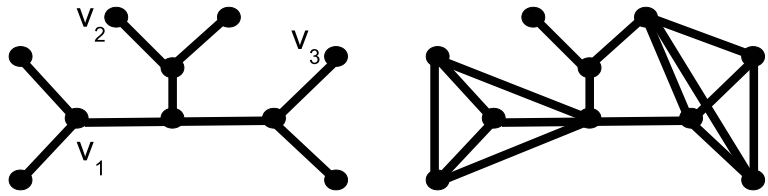

Figure 12.: A graph G with $\gamma(G) = \rho_2(G)$ and a decomposable graph H formed by adding edges to G.

Observe that this corollary implies Vizing's conjecture is true for any tree. We also have the following result from Hartnell and Rall as a corollary of Theorem 2.6 and Corollary 2.1.

Corollary 2.2 *[28] Let G be a graph such that $\bar{G}$ is 3-colorable. Then G satisfies Vizing's conjecture.*

Proof. We consider three cases based on the chromatic number of $\bar{G}$.

- Case 1: $\chi(\bar{G}) = 1$. Then G is a complete graph and the result holds.
- Case 2: $\chi(\bar{G}) = 2$. Then G belongs to the A-class and Vizing's conjecture holds.

- Case 3: $\chi(\bar{G}) = 3$. If $\gamma(G) = 3$ then G is decomposable and result holds by Theorem 2.6. Otherwise $\gamma(G) \leq 2$ and result holds by Corollary 2.1.

$\square$

We now define Type $\mathcal{X}$ graphs, as introduced by Hartnell and Rall [27] in 1995. This class of graphs contains the BG-graphs as a proper subset and, hence, is an improvement of Barcalkin and German's [4] 1979 result. Hartnell and Rall, in defining Type $\mathcal{X}$ graphs, took an approach similar to that of Barcalkin and German in that they considered a particular way of partitioning a graph G. The difference is that not every set in the partition of a Type $\mathcal{X}$ graph induces a complete subgraph.

Type $\mathcal{X}$ graphs are defined as follows. Let k, t, r be nonnegative integers, not all zero. Let G be a graph with $\gamma(G) = k + t + r + 1$ whose vertices can be partitioned as $S \cup SC \cup BC \cup C$, where S, SC, BC, and C satisfy the following.

- Let $BC = B_1 \cup B_2 \cup \ldots \cup B_t$. Each B_i for $i = 1, \ldots, t$ is referred to as a *buffer clique*.
- Let $C = C_1 \cup C_2 \cup \ldots \cup C_r$.
- Each of $SC, B_1, \ldots, B_k, C_1, \ldots, C_r$ induces a clique.
- Every $v \in SC$ has at least one neighbor outside of SC. The set SC is called a *special clique*.
- Each B_i, for $i = 1 \ldots, k$ has at least one vertex which has no neighbors outside of B_i.
- Let $S = S_1 \cup S_2 \cup \ldots \cup S_k$ where each S_i is star-like. That is, each S_i has a vertex v_i which is adjacent to all $v \in S_i - v_i$. The vertex v_i has no neighbors other than those in S_i. Note that S_i does not induce a clique, and no edges may be added to S_i without decreasing the domination number of G.
- There are no edges between vertices in S and vertices in C.

Observe that not every graph that is Type $\mathcal{X}$ has a special clique. We can also have t, r, or k equal to zero. The example in Figure 13, is a Type $\mathcal{X}$ graph with a special clique. In this graph, the blue vertices represent the set S, the red vertices represent the buffer clique B, and the green vertices represent the special clique SC. One can easily verify that this graph satisfies the definition of Type $\mathcal{X}$ graphs above.

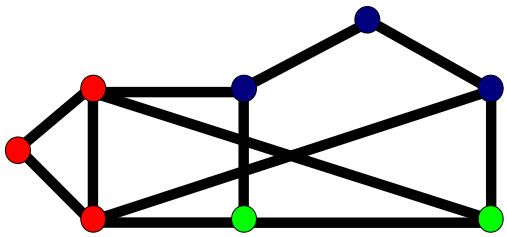

Figure 13.: Example of a Type $\mathcal{X}$ graph with a special clique

Theorem 2.7 *[27] Let G be a Type $\mathcal{X}$ graph. Then for any graph H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.*

The proof of Hartnell and Rall's theorem is similar to the proof that Vizing's conjecture is true for BG-graphs. We partition the vertices of G as indicated by the definition of a Type $\mathcal{X}$ graph and consider any dominating set D of $G\Box H$. Hartnell and Rall used the idea that some vertices in the product graph must be dominated "horizontally" and found $\gamma(G)$ disjoint sets in D, each of which have cardinality at least $\gamma(H)$, thus implying that Vizing's conjecture holds for any Type $\mathcal{X}$ graph.

Theorem 2.8 *[27] Let G be a Type $\mathcal{X}$ graph and let K be a spanning subgraph of G such that $\gamma(G) = \gamma(K)$. Then Vizing's conjecture is true for K.*

This theorem can be proved in the same way we showed that any spanning subgraph K of a decomposable graph G with $\gamma(G) = \gamma(K)$ satisfies Vizing's conjecture.

Hartnell and Rall were also able to show that any graph with domination number one more than its 2-packing number is a Type $\mathcal{X}$ graph and, hence, we have the following result.

Corollary 2.3 *[27] Let G be a graph satisfying $\gamma(G) = \rho_2(G) + 1$. Then Vizing's conjecture is true for G.*

Brešar and Rall [11] recently discovered a new class of graphs which satisfy Vizing's conjecture. They defined fair domination and proved that any graph with fair domination number equal to its domination number satisfies the conjecture. Furthermore, they proved that this class of graphs is an extension of the BG-graphs distinct from Type $\mathcal{X}$ graphs. Their results are presented in Chapter 3.

2.2 Vizing-Like Conjectures for Other Domination Types

As Vizing's conjecture has not yet been proven in general, researchers such as Fisher, Ryan, Domke and Majumdar [25]; Nowakowski and Rall [38]; Brešar [7]; and Dorbec, Henning and Rall [19]

have studied variations of the original problem. These similar problems deal with other types of graph products and different graph parameters. As we will see, several of these variations remain open conjectures, while others have been proven.

Fractional Domination

One of the first Vizing-like results was proved for the fractional domination number. Recall that the fractional domination number of a graph G is the minimum weight of a fractional-dominating function, where the weight of the function is the sum over all vertices of its values. We note that for any graph G, $\gamma_f(G) \leq \gamma(G)$. Fisher, Ryan, Domke, and Majumdar proved the following result in their 1994 paper.

Theorem 2.9 *[25] For any pair of graphs G and H, $\gamma_f(G \square H) \geq \gamma_f(G)\gamma_f(H)$.*

This theorem can be proved by first showing that $\gamma_f(G \boxtimes H) = \gamma_f(G)\gamma_f(H)$. Recall that $G \boxtimes H$ denotes the strong direct product of G and H, which has vertex set $V(G) \times V(H)$ and edge set $E(G \square H) \cup \{\{(u_1, v_1), (u_2, v_2)\} \mid \{u_1, u_2\} \in E(G)$ and $\{v_1, v_2\} \in E(H)\}$. Since $G \square H$ is a subgraph of $G \boxtimes H$, we have $\gamma_f(G \square H) \geq \gamma_f(G \boxtimes H)$.

Fisher [24] also proved the following similar theorem in 1994; an improved proof was given by Brešar [6] in 2001.

Theorem 2.10 *[24] For any pair of graphs G and H, $\gamma(G \square H) \geq \gamma_f(G)\gamma_(H)$.*

An obvious corollary of this theorem is that Vizing's conjecture is true for any graph with fractional domination number equal to domination number.

Integer Domination

A related concept to fractional domination is integer domination, which was studied first by Domke, Hedetniemi, Laskar, and Fricke [18]. We recall that the weight of a $\{k\}$-dominating function is the sum of its function values over all vertices, and the $\{k\}$-domination number of G, $\gamma_{\{k\}}(G)$ is the minimum weight of a $\{k\}$-dominating function of G. Domke, et. al. proved the following theorem relating fractional domination to integer domination.

Theorem 2.11 *[18] For any graph G, $\gamma_f(G) = \min_{k \in \mathbb{N}} \frac{\gamma_{\{k\}}(G)}{k}$.*

The following Vizing-like conjecture for integer domination is from Hou and Lu [33].

Conjecture 2.2 *[33] For any pair of graphs G and H and any integer $k \geq 1$, $\gamma_{\{k\}}(G \Box H) \geq \frac{1}{k}\gamma_{\{k\}}(G)\gamma_{\{k\}}(H)$.*

This conjecture remains open, but Brešar, Henning and Klavžar [9] prove several related results in their 2006 paper. Note that if this conjecture is true for all k, in particular $k = 1$, then Vizing's conjecture is true.

Upper Domination

Nowakowski and Rall's [38] 1996 paper gives results and conjectures on several associative graph products, two of which are the Cartesian product and the categorical product, as previously defined in Section 1.2.

Recall that the upper domination number $\Gamma(G)$ of a graph G is the maximum cardinality of a minimal dominating set of G. Also recall that the minimum cardinality of a maximal independent set is the independent domination number $i(G)$.

Nowakowski and Rall [38] made the following conjectures in their 1996 paper.

- $i(G \times H) \geq i(G)i(H)$
- $\Gamma(G \times H) \geq \Gamma(G)\Gamma(H)$
- $\Gamma(G \Box H) \geq \Gamma(G)\Gamma(H)$

The last of these conjectures was proved by Brešar [7] in 2005. In fact, he provided a slight improvement of the conjectured lower bound.

Theorem 2.12 *[7] For any nontrivial graphs G and H,*

$$\Gamma(G \Box H) \geq \Gamma(G)\Gamma(H) + 1.$$

The proof Brešar provided for this theorem is constructive in nature. He begins with arbitrary graphs G and H and creates a minimal dominating set D of the product graph $G \Box H$ which contains at least $\Gamma(G)\Gamma(H) + 1$ vertices.

Total Domination

Henning and Rall's [30] 2005 paper was the first to introduce results on total domination in Cartesian products of graphs. Recall that a set $D \subset V(G)$ is a total dominating set if $N(D) = V(G)$. The total domination number is the minimum cardinality of a total dominating set of G and is denoted by $\gamma_t(G)$. Henning and Rall conjectured that $2\gamma_t(G\Box H) \geq \gamma_t(G)\gamma_t(H)$ and they proved this inequality holds for certain classes of graphs G with no isolated vertices and any graph H without isolated vertices. This conjecture was proved for graphs without isolated vertices by Ho.

Theorem 2.13 *[32] Let G and H be graphs without isolated vertices. Then*

$$2\gamma_t(G\Box H) \geq \gamma_t(G)\gamma_t(H).$$

Recall that the total $\{k\}$-domination number $\gamma_t^{\{k\}}(G)$ is defined as the minimum cardinality of a total k-dominating set D of a graph. In 2008, Li and Hou [37] proved that for any graphs G and H without isolated vertices, $\gamma_t^{\{k\}}(G)\gamma_t^{\{k\}}(H) \leq k(k+1)\gamma_t^{\{k\}}(G\Box H)$. Note that Theorem 2.13 is easily proved using this inequality.

Upper Total Domination

Recall that we define the upper total domination number of G, denoted by $\Gamma_t(G)$, to be the maximum cardinality of a minimal total dominating set of a graph G. Dorbec, Henning and Rall [19] published results in 2008 on a Vizing-like inequality for the upper total domination number. They achieved the following two results.

Theorem 2.14 *[19] If G and H are connected graphs of order at least 3 and $\Gamma_t(G) \geq \Gamma_t(H)$, then*

$$2\Gamma_t(G\Box H) \geq \Gamma_t(G)(\Gamma_t(H)+1)$$

and this bound is sharp.

In order to prove this theorem we must first define the sets $epn(S,v)$, $ipn(v,S)$, and $pn(v,S)$. Let $S \subset V(G)$ and let $v \in S$. The set $epn(v,S)$ of external private neighbors of v is $epn(v,S) = \{u \in V(G) - S \mid N(u) \cap S = \{v\}\}$. The set of internal private neighbors of $v \in S$ is $ipn(v,S) = \{u \in S \mid N(u) \cap S = \{v\}\}$. We denote the set of all private neighbors of $v \in S$ by $pn(v,S)$. This is the union of all external and internal private neighbors of v. That is, $pn(v,S) = epn(v,S) \cup ipn(v,S)$. Cockayne, et. al. make the following observation.

Observation 2.1 *[14] Let S be a total dominating set in a graph G with no isolated vertices. Then S is a minimal total dominating set if and only if for all $v \in S$,*

1. $epn(v, S) \neq \emptyset$*, or*
2. $pn(v, S) = ipn(v, S) \neq \emptyset$.

We will also need the following lemma.

Lemma 2.3 *[19] Let G be a graph. Every $\Gamma_t(G)$-set contains as a subset a γ-set D such that $|D| \geq \frac{1}{2}\Gamma_t(G)$ and for all $v \in D$, $|epn(v, D)| \geq 1$.*

We will now prove Theorem 2.14.

Proof. [19] We assume G and H are connected graphs with order at least 3, where $\Gamma_t(G) \geq \Gamma_t(H)$. By the above lemma, there is a γ-set S of G with $|S| \geq \frac{1}{2}\Gamma_t(G)$ and for each $v \in S$, $|epn(v, S)| \geq 1$. For each $u \in V(G)$, denote $H_u = \{u\} \times V(H)$. Similarly, for $w \in V(H)$, let $G_w = V(G) \times \{w\}$.

Now, let $D = S \times V(H)$, and observe that D dominates $G\Box H$ since S dominates V(G). Also, for each $u \in S$, the vertices $V(H_u)$ are totally dominated "vertically"; thus, D is a total dominating set of $G\Box H$. We claim that D is a minimal total dominating set of $G\Box H$.

Let $(u, w) \in D$ and consider (u', w), where $u' \in epn(u, S)$ in G. Then $(u', w) \in epn((u, w), D)$ in $G\Box H$. Thus, for all $(u, w) \in D$, $|epn((u, w), D)| \geq 1$. Then, by Observation 2.1, D is a minimal total dominating set of $G\Box H$ and so $\Gamma_t(G\Box H) \geq |D|$. Note that since H is a connected graph with order at least 3, $|V(H)| \geq \Gamma_t(H) + 1$. Therefore,

$$\Gamma_t(G\Box H) \geq |D| = |S| \times |V(H)| \geq \frac{1}{2}\Gamma_t(G)(\Gamma_t(H) + 1).$$

Equality holds when both G and H are *daisies* with $k \geq 2$ *petals*. That is, we begin with k copies of K_3 and identify one vertex from each copy to form a single vertex. The resulting graph is a daisy. Figure 14 shows the daisy with 3 petals. □

The following theorem is easily proved using Theorem 2.14 and the fact that for a graph G with no isolated vertices, $\Gamma_t(G)\Gamma_t(K_2) \leq 2\Gamma_t(G\Box K_2)$. Equality holds if and only if G is a disjoint union of copies of K_2. Let $u \in V(K_2)$. Then $V(G) \times \{u\}$ is a minimal total dominating set of $G\Box K_2$, giving that

$$\Gamma_t(G\Box K_2) \geq |V(G)| \geq \Gamma_t(G) = \frac{1}{2}\Gamma_t(G)\Gamma_t(K_2).$$

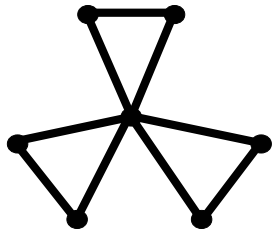

Figure 14.: The daisy with 3 petals

In order for equality to hold, we must have $\Gamma_t(G) = |V(G)|$, and so G must be a disjoint union of copies of K_2.

Theorem 2.15 *[19] If G and H have no isolated vertices, then*

$$2\Gamma_t(G\Box H) \geq \Gamma_t(G)\Gamma_t(H)$$

with equality if and only if both G and H are disjoint unions of copies of K_2.

Paired Domination

Brešar, Henning and Rall [10] published results in 2007 about Vizing-like inequalities for paired domination. Recall that a set $D \subseteq V(G)$ is a paired dominating set of G if D dominates G and the induced subgraph $G[D]$ has a perfect matching. Note that in every graph without isolated vertices, a maximal matching forms a paired dominating set. The paired domination number $\gamma_{pr}(G)$ is the minimum cardinality of a paired dominating set.

The inequalities established by Brešar, Henning and Rall relate the paired domination number of the Cartesian product of G and H to the 3-packing number of G. Recall that a 2-packing of a graph G is a set of vertices $S \subset V(G)$ such that for any vertices u and v in S, $d(u, v) > 2$. We define a 3-packing similarly. That is, a 3-packing of the graph G is a set S of vertices such that the distance between any pair of vertices in S is greater than 3. The 3-packing number of G, denoted $\rho_3(G)$, is the maximum cardinality of a 3-packing in G.

Theorem 2.16 *[10] If G and H are graphs without isolated vertices, then*

$$\gamma_{pr}(G\Box H) \geq \max\{\gamma_{pr}(G)\rho_3(H), \gamma_{pr}(H)\rho_3(G)\}.$$

Brešar, Henning and Rall were also able to show that $\gamma_{pr}(T) = 2\rho_3(T)$ in any nontrivial tree T. Thus, the following result follows from Theorem 2.16.

Theorem 2.17 *[10] Let T be a nontrivial tree. Then for any graph H without isolated vertices, $\gamma_{pr}(T\Box H) \geq \frac{1}{2}\gamma_{pr}(T)\gamma_{pr}(H)$, and this bound is sharp.*

The final major result from Brešar, Henning and Rall in 2007 is the following theorem relating paired domination in the Cartesian product of G and H to the 3-packing numbers of G and H.

Theorem 2.18 *[10] If G and H have no isolated vertices, then $\gamma_{pr}(G\Box H) \geq 2\rho_3(G)\rho_3(H)$.*

Independence Domination

Aharoni and Szabó [2] in 2009 provided a Vizing-like result for the independence domination number. Recall that this is different from the independent domination number; we let the independence domination number $\gamma^i(G)$ denote the maximum, over all independent sets I in G, of the minimum number of vertices required to dominate I. It was proven by Aharoni, Berger and Ziv [1] that $\gamma(G) = \gamma^i(G)$ for any chordal graph G, where a graph is chordal if any cycle of more than four vertices contains at least one chord, or edge connecting vertices that are not adjacent in the cycle. Aharoni and Szabó proved the following theorem.

Theorem 2.19 *[2] For arbitrary graphs G and H, $\gamma(G\Box H) \geq \gamma^i(G)\gamma(H)$.*

Proof. [2] Let G and H be graphs. We may assume that G has no isolated vertices, for if it did have an isolated vertex v then the validity of the theorem for $G - v$ implies the validity for G.

Assume $I \subset V(G)$ is an independent set which requires at least $\gamma^i(G)$ vertices to dominate it. We will show that $\gamma(I\Box H) \geq \gamma^i(G)\gamma(H)$ by showing that $|D| \geq \gamma^i(G)\gamma(H)$, where D is a set that dominates $I \times V(H)$.

Let $\{v_1, v_2, \ldots, v_{\gamma(H)}\}$ be a γ-set of H. Use these vertices to partition $V(H)$ into sets $\{\Pi_i \mid v_i \in \Pi_i$ and $v \in \Pi_i$ if and only if $v = v_i$ or $\{v, v_i\} \in E(H)\}$. Note that, for every $J \subseteq \{1, 2, \ldots, \gamma(H)\}$, we have

$$\gamma(\bigcup_{j\in J} \Pi_j) \geq |J| \tag{2.2}$$

Let $S_u = \{i \mid \{u\} \times \Pi_i$ is dominated vertically by some vertices $(u, v) \in D\}$, and let $S_i = \{u \in I \mid \{u\} \times \Pi_i$ is dominated vertically by some vertices $(u, v) \in D\}$. Summing S_u and S_i, we have

$$\mathcal{S} = \sum_{u\in I} S_u = \sum_{i=1}^{\gamma(H)} S_i$$

By (2.2), for each $u \in I$ we have

$$|D \cap (\{u\} \times V(H))| \geq |S_u|.$$

Sum over $v \in I$ to get

$$|D \cap (I \times V(H))| \geq |\mathcal{S}|. \tag{2.3}$$

Now consider $k \leq \gamma(H)$; each set of vertices $\{u\} \times \Pi_k$ which is not in $\mathcal{S}$ contains at least one vertex (u, v) which is not dominated by any vertex in $\{u\} \times V(H)$. Thus, (u, v) is dominated "horizontally" by some vertex (w, v) where $w = w(v)$. Note that $w \notin I$ since I is independent and so the set $\{w(v) \mid \{v\} \times \Pi_k \notin \mathcal{S}\}$ dominates $|I| - |S_j|$ vertices in I and has cardinality at least $\gamma^i(G) - |S_j|$. Sum over k to get

$$|D \cap ((V(G) - I) \times V(H))| \geq \gamma^i(G)\gamma(H) - |\mathcal{S}|. \tag{2.4}$$

Combine equations (2.3) and (2.4) to get

$$\gamma(G \Box H) \geq \gamma^i(G)\gamma(H).$$

$\Box$

Combining this result with that of Aharoni, Berger and Ziv [1], an obvious corollary is that Vizing's conjecture holds for chordal graphs.

Independent Domination

Brešar, et. al. [8] provide a few open conjectures in their survey paper, including the following.

Conjecture 2.3 *[8] For any graphs G and H, $\gamma(G \Box H) \geq \min\{i(G)\gamma(H), \gamma(G)i(H)\}$.*

The truth of this conjecture would immediately imply Vizing's conjecture holds for any pair of graphs G and H, as $\gamma(G) \leq i(G)$ by Lemma 1.1. We also have the following conjecture, which is implied by Vizing's conjecture. Brešar, et. al. suggest that perhaps this could be established without first proving Vizing's conjecture.

Conjecture 2.4 *[8] For any graphs G and H, $i(G \Box H) \geq \gamma(G)\gamma(H)$.*

In addition, the survey paper makes the following partition conjecture, which would also imply the truth of Vizing's conjecture.

Conjecture 2.5 *[8] Let G and H be arbitrary graphs. There is a partition of $V(G)$ into $\gamma(G)$ sets $\Pi_1, \ldots, \Pi_{\gamma(G)}$ such that there is a minimal dominating set D of $G\Box H$ such that the projection of $D \cap (\Pi_i \times V(H))$ onto H dominates H for all $i = 1, \ldots, \gamma(G)$.*

2.3 Clark-Suen Inequality and Improvement

We have given several results establishing the truth of Vizing's conjecture for classes of graphs satisfying certain properties. Another approach to proving Vizing's conjecture is to find a constant c such that for any graphs G and H, $\gamma(G\Box H) \geq c\gamma(G)\gamma(H)$. Clark and Suen [17] in 2000 proved that this inequality is true for $c = \frac{1}{2}$. Here, we present an improvement of this result.

Theorem 2.20 *For any graphs G and H, $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}$.*

Proof. Let G and H be arbitrary graphs, and let D be a γ-set of the Cartesian product $G\Box H$. Let $\{u_1, u_2, ..., u_{\gamma(G)}\}$ be a γ-set of G. Partition $V(G)$ into $\gamma(G)$ sets $\Pi_{1\bullet}, \Pi_{2\bullet}, ..., \Pi_{\gamma(G)\bullet}$, where $u_i \in \Pi_{i\bullet}$ for all $i = 1, 2, ..., \gamma(G)$ and if $u \in \Pi_{i\bullet}$ then $u = u_i$ or $\{u, u_i\} \in E(G)$.

Let $P_{i\bullet}$ denote the projection of $(\Pi_{i\bullet} \times V(H)) \cap D$ onto H. That is,

$$P_{i\bullet} = \{v \in V(H) \mid (u, v) \in D \text{ for some } u \in \Pi_{i\bullet}\}.$$

Define $C_{i\bullet} = V(H) - N_H[P_{i\bullet}]$ as the complement of $N_H[P_{i\bullet}]$, where $N_H[X]$ is the set of closed neighbors of X in graph H. As $P_i \cup C_i$ is a dominating set of H, we have

$$|P_{i\bullet}| + |C_{i\bullet}| \geq \gamma(H), \qquad i = 1, 2, \ldots, \gamma(G). \tag{2.5}$$

For $v \in V(H)$, let

$$D_{\bullet v} = \{u \mid (u, v) \in D\} \quad \text{and} \quad S_{\bullet v} = \{i \mid v \in C_{i\bullet}\}.$$

Observe that if $i \in S_{\bullet v}$ then the vertices in $\Pi_{i\bullet} \times \{v\}$ are dominated "horizontally" by vertices in $D_{\bullet v} \times \{v\}$. Let S_H be the number of pairs (i, v) where $i = 1, 2, \ldots, \gamma(G)$ and $v \in C_{i\bullet}$. Then obviously

$$S_H = \sum_{v \in V(H)} |S_{\bullet v}| = \sum_{i=1}^{\gamma(G)} |C_{i\bullet}|.$$

Since $D_{\bullet v} \cup \{u_i \mid i \notin S_{\bullet v}\}$ is a dominating set of G, we have

$$|D_{\bullet v}| + (\gamma(G) - |S_{\bullet v}|) \geq \gamma(G),$$

giving that

$$|S_{\bullet v}| \leq |D_{\bullet v}|. \tag{2.6}$$

Summing over $v \in V(H)$, we have

$$S_H \leq |D|. \tag{2.7}$$

We now consider two cases based on (2.5).

Case 1 *Assume* $|P_{i\bullet}| + |C_{i\bullet}| > \gamma(H)$ *for all* $i = 1, ..., \gamma(G)$. *Then as* $|(\Pi_{i\bullet} \times V(H)) \cap D| \geq |P_{i\bullet}|$, *we have*

$$\sum_{i=1}^{\gamma(G)} (|C_{i\bullet}| + |(\Pi_{i\bullet} \times V(H)) \cap D|) \geq \sum_{i=1}^{\gamma(G)} (\gamma(H) + 1),$$

which implies that

$$S_H + |D| \geq \gamma(G)\gamma(H) + \gamma(G). \tag{2.8}$$

Combining (2.7) *and* (2.8) *gives that*

$$\gamma(G \square H) = |D| \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\gamma(G). \tag{2.9}$$

Case 2 *Assume* $|P_{i\bullet}| + |C_{i\bullet}| = \gamma(H)$ *for some* $i = 1, ..., \gamma(G)$. *Note that* $P_{i\bullet} \cup C_{i\bullet}$ *is a* γ*-set of* H. *We now use this* γ*-set of* H *to partition* $V(H)$ *in the same way as* $V(G)$ *is partitioned above. That is, label the vertices in* $P_{i\bullet} \cup C_{i\bullet}$ *as* $v_1, v_2, ..., v_{\gamma(H)}$, *and let* $\{\Pi_{\bullet j} \mid 1 \leq j \leq \gamma(H)\}$ *be a partition of* H *such that for all* $j = 1, ..., \gamma(H)$, $v_j \in \Pi_{\bullet j}$ *and if* $v \in \Pi_{\bullet j}$, *either* $v = v_j$ *or* $\{v, v_j\} \in E(H)$. *We next define the sets* $P_{\bullet j}, C_{\bullet j}, S_{u\bullet}$ *and* $D_{u\bullet}$ *in the same way* $P_{i\bullet}, C_{i\bullet}, S_{\bullet v}$ *and* $D_{\bullet v}$ *are defined above. To be specific, for* $1 \leq j \leq \gamma(H)$, *let*

$$P_{\bullet j} = \{u \in V(G) \mid (u, v) \in D \text{ for some } v \in \Pi_{\bullet j}\}, \quad \text{and} \quad C_{\bullet j} = V(G) - N_G[P_{\bullet j}],$$

and for $u \in V(G)$, *let*

$$D_{u\bullet} = \{v \mid (u, v) \in D\} \quad \text{and} \quad S_{u\bullet} = \{j \mid u \in C_{\bullet j}\}.$$

Similarly, we have

$$S_G = \sum_{u \in V(G)} |S_{u\bullet}| = \sum_{j=1}^{\gamma(H)} C_{\bullet j}.$$

For $u \in V(G)$, *let* $\hat{D}_{u\bullet} = \{v_j \mid (u, v_j) \in D_{u\bullet}, 1 \leq j \leq \gamma(H)\}$. *We claim that*

$$|S_{u\bullet}| \leq |D_{u\bullet}| - |\hat{D}_{u\bullet}|. \tag{2.10}$$

This is because $D_{u\bullet} \cup \{v_j \mid j \notin S_{u\bullet}\}$ *is a dominating set of* H*, with*

$$D_{u\bullet} \cap \{v_j \mid j \notin S_{u\bullet}\} = \hat{D}_{u\bullet},$$

and the argument for proving (2.10) *follows in the same way as* (2.6) *is proved. To make use of the claim, we note that when we partition the vertices of* H*, we have at least* $\gamma(H)$ *vertices in* D *that are of the form* (u, v_k)*. Indeed, for each* $k = 1, 2, \ldots, \gamma(H)$*, either* $v_k \in P_{i\bullet}$*, which implies* $(u, v_k) \in D$ *for some* $u \in \Pi_{i\bullet}$*, or* $v_k \in C_{i\bullet}$*, which implies that the vertices in* $\Pi_{i\bullet} \times \{v_k\}$ *are dominated "horizontally" by some vertices* $(u', v_k) \in D$*. It therefore follows that*

$$\sum_{u \in V(G)} |\hat{D}_{u\bullet}| \geq \gamma(H),$$

and hence summming both sides of (2.10)

$$\sum_{u \in V(G)} |S_{u\bullet}| \leq \sum_{u \in V(G)} (|D_{u\bullet}| - |\hat{D}_{u\bullet}|)$$

gives that

$$S_G \leq |D| - \gamma(H). \tag{2.11}$$

To complete the proof, we note that similar to (2.5)*, we have*

$$|P_{\bullet j}| + |C_{\bullet j}| \geq \gamma(G), \qquad j = 1, 2, \ldots, \gamma(H),$$

and summing over j *gives that*

$$|D| + S_G \geq \gamma(G)\gamma(H). \tag{2.12}$$

Combining (2.11) *and* (2.12)*, we obtain*

$$\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\gamma(H). \tag{2.13}$$

As either (2.9) or (2.13) holds, it follows that

$$\gamma(G \Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.$$

□

This approach may also be used to prove a similar inequality involving the independence number of a graph, where G is a claw-free graph. Recall that the independence number of a graph G is the maximum cardinality of a maximal independent set in G, and is denoted by $\alpha(G)$. Also recall that a graph is claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. Brešar, et. al. [8] proved the following.

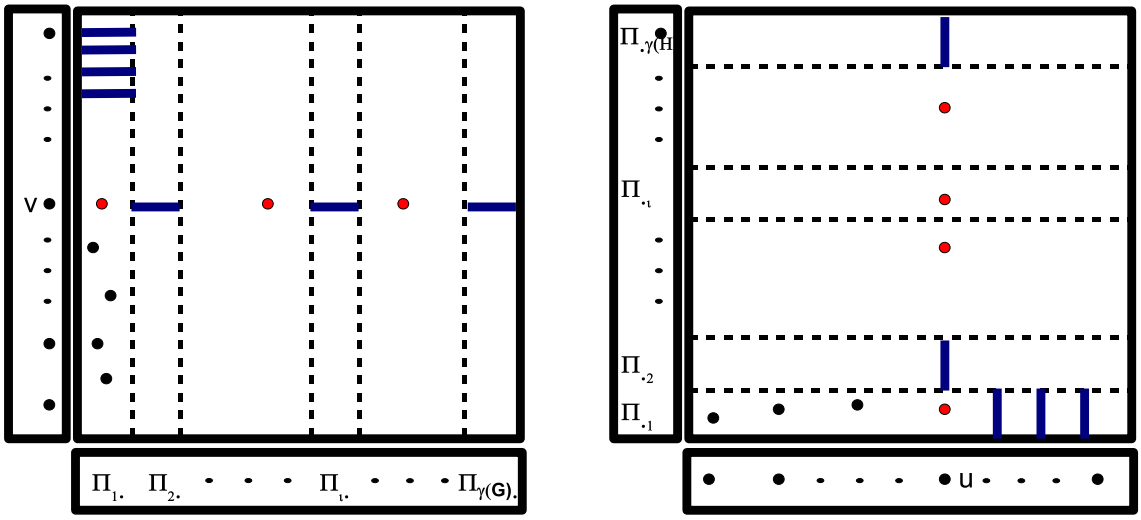

Figure 15.: Partitions $\Pi_{i\bullet}$ and the sets $D_{\bullet v}, S_{\bullet v}$, and $C_{i\bullet}$; and partitions $\Pi_{\bullet j}$ and the sets $D_{u\bullet}, S_{u\bullet}$, and $C_{\bullet j}$

Theorem 2.21 *[8] Let G be a claw-free graph and let H be a graph without isolated vertices. Then*

$$\gamma(G\Box H) \geq \frac{1}{2}\alpha(G)(\gamma(H)+1).$$

Observe that $\gamma(G) \leq \alpha(G)$ for every graph G, so we have the following corollary.

Corollary 2.4 *[8] Let G be a claw-free graph and let H be a graph without isolated vertices. Then*

$$\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)(\gamma(H)+1).$$

From this corollary we can conclude that any claw-free graph satisfying $\alpha(G) = 2\gamma(G)$ satisfies Vizing's conjecture.

Chapter 3

Fair Domination

A recent development in attempts to prove Vizing's conjecture is Brešar and Rall's [11] idea of fair domination. Their 2009 paper defines this concept and establishes the truth of Vizing's conjecture for graphs with fair domination number equal to domination number. Furthermore, they verify that the class of such graphs contains the BG-graphs and is distinct from the Type $\mathcal{X}$ graphs defined by Hartnell and Rall. We will define fair reception and fair domination, provide a proof that Vizing's conjecture holds for the class of graphs with fair domination number equal to domination number, examine fair domination in edge-critical graphs, and summarize some open questions related to fair domination.

3.1 Definition and General Results

A recent paper by Brešar and Rall [11] published in 2009 introduces the concept of fair domination of a graph. Brešar and Rall were able to verify that Vizing's conjecture holds for any graph G with a fair reception of size $\gamma(G)$.

In order to define fair domination, we must first define external domination. We say that a set $X \subset V(G)$ *externally dominates* set $U \subset V(G)$ if $U \cap X = \emptyset$ and for each $u \in U$ there is $x \in X$ such that $\{u, x\} \in E(G)$.

Let G be a graph and let $S_1, ..., S_k$ be pair-wise disjoint sets of vertices of G. Let $\mathcal{S} = S_1 \cup S_2 \cup ... \cup S_k$ and let $Z = V(G) - \mathcal{S}$. The sets $S_1, ..., S_k$ form a *fair reception of size k* if for each $l \in \mathbb{Z}$, $1 \leq l \leq k$, and any choice of l sets $S_{i_1}, ..., S_{i_l}$ the following holds: if D externally dominates $S_{i_1} \cup ... \cup S_{i_l}$ then

$$|D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) \geq l.$$

Notice that on the left-hand side of the above inequality, we count all the vertices of D that are not in $\mathcal{S}$. For vertices of D that are in some S_j, we count all but one from $D \cap S_j$. The largest k such

that there exists a fair reception of size k in graph G is called the *fair domination number* of G and is denoted by $\gamma_F(G)$.

Proposition 3.1 *[11] For any graph G, $\rho_2(G) \leq \gamma_F(G) \leq \gamma(G)$.*

Proof. Let T be a 2-packing of G. Let each S_i consist of exactly one vertex $v \in T$. This gives us a fair reception of size $|T|$. Thus, $\rho_2(G) \leq \gamma_F(G)$. Now assume there exists a graph G with $r = \gamma(G) < \gamma_F(G) = k$. Let D be a γ-set of G and let $S_1, ..., S_k$ form a fair reception of size k in G. Since $r < k$, D must be disjoint from at least one S_i. We assume $D \cap S_i = \emptyset$ for $1 \leq i \leq t$ and $D \cap S_j \neq \emptyset$ for $t+1 \leq j \leq k$. Then D externally dominates $S_1 \cup S_2 \cup ... \cup S_t$, and so by the definition of fair reception, we have

$$t \leq |D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) = |D \cap Z| + \sum_{j=t+1}^{k} |S_j \cap D| - (k-t) = |D| - k + t.$$

Then $k \leq |D|$ and we have a contradiction. Therefore, $\rho_2(G) \leq \gamma_F(G) \leq \gamma(G)$. □

Theorem 3.1 *[11] For any graphs G and H,*

$$\gamma(G \Box H) \geq \max\{\gamma_F(G)\gamma(H), \gamma(G)\gamma_F(H)\} \tag{3.1}$$

Proof. Let G and H be arbitrary graphs. Let D be a γ-set of $G \Box H$ and let the sets $S_1, S_2, ..., S_k$ form a fair reception of H, where $\gamma_F(H) = k$. As in the definition of fair reception, we let $S = \bigcup_{i=1}^{k} S_i$ and $Z = V(H) - S$.

Let $D_{u\bullet}$ be the set of vertices in $\{u\} \times V(H)$ that are also in D and let $P_{u\bullet}$ denote the projection of $D_{u\bullet}$ onto H. That is, $D_{u\bullet} = (\{u\} \times V(H)) \cap D$ and $P_{u\bullet} = \{v \in V(H) \mid \{u, v\} \in \{u\} \times V(H)\}$.

Let $D_{\bullet i}$ be the set of vertices in $V(G) \times S_i$ that are also in D and let $P_{\bullet i}$ denote the projection of $D_{\bullet i}$ onto G. That is, $D_{\bullet i} = (V(G) \times S_i) \cap D$ and $P_{\bullet i} = \{u \in V(G) \mid \{u, v\} \in V(G) \times S_i\}$.

Let $D_{ui} = (\{u\} \times S_i) \cap D$.

Let $D_{\bullet Z} = (V(G) \times Z) \cap D$ and let $D_{uZ} = (\{u\} \times Z) \cap D$.

Now define d_{ui} as follows.

$$d_{ui} = \begin{cases} |D_{ui}| - 1 & \text{if } D_{ui} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Observe that d_{ui} counts the vertices in D_{ui} that are not uniquely projected onto G.

Now we define $T_{\bullet i} = \{u \in V(G) \mid u \notin N[P_{\bullet i}]\}$. Observe that $T_{\bullet i} \cup P_{\bullet i}$ is a dominating set of G and, thus,

$$|T_{\bullet i}| + |P_{\bullet i}| \geq \gamma(G) \tag{3.2}$$

Let $T_{u\bullet} = \{i \mid u \in T_{\bullet i}, i = 1, 2, \ldots, k\}$. By definition of $T_{\bullet i}$ and $T_{u\bullet}$, the following holds

$$\sum_{i=1}^{k} |T_{\bullet i}| = \sum_{u \in V(G)} |T_{u\bullet}| \tag{3.3}$$

Observe that if $i \in T_{u\bullet}$ the vertices $\{u\} \times S_i$ are not dominated by $D_{\bullet i}$, and so $P_{u\bullet}$ externally dominates S_i for all $i \in T_{u\bullet}$. Therefore, by definition of fair reception, we have

$$|D_{uZ}| + \sum_{i=1}^{k} d_{ui} \geq |T_{u\bullet}| \tag{3.4}$$

Now, we have:

$$\begin{aligned}
|D| &= \sum_{i=1}^{k} |D_{\bullet i}| + |D_{\bullet Z}| = \sum_{i=1}^{k} (|P_{\bullet i}| + (|D_{\bullet i}| - |P_{\bullet i}|)) + |D_{\bullet Z}| \\
&= \sum_{i=1}^{k} |P_{\bullet i}| + \sum_{i=1}^{k} \sum_{u \in V(G)} d_{ui} + |D_{\bullet Z}| = \sum_{i=1}^{k} |P_{\bullet i}| + \sum_{u \in V(G)} (\sum_{i=1}^{k} d_{ui} + |D_{uZ}|) \\
&\geq \sum_{i=1}^{k} |P_{\bullet i}| + \sum_{u \in V(G)} |T_{u\bullet}| && (3.5) \\
&= \sum_{i=1}^{k} (|P_{\bullet i}| + |T_{\bullet i}|) && (3.6) \\
&\geq k\gamma(G) = \gamma(G)\gamma_F(H). && (3.7)
\end{aligned}$$

Note, (3.5) holds by (3.4), (3.6) holds by (3.3), and (3.7) holds by (3.2).

Similarly, we define a fair reception of G and repeat the proof with the roles of G and H reversed to conclude that $\gamma(G \Box H) \geq \gamma_F(G)\gamma(H)$. Therefore, we conclude that $\gamma(G \Box H) \geq \max\{\gamma_F(G)\gamma(H), \gamma(G)\gamma_F(H)\}$. □

Corollary 3.1 *Let G be a graph with $\gamma(G) = \gamma_F(G)$. Then G satisfies Vizing's conjecture.*

There are some known examples of graphs G for which $\gamma_F(G) \neq \gamma(G)$. One such example can be seen in Figure 16. It can be easily verified that this graph G has $\gamma(G) = 3$. Brešar, et. al. [8] verified by computer that $\gamma_F(G) = 2$.

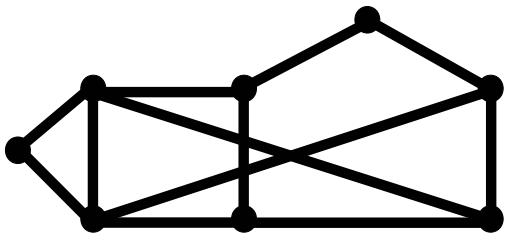

Figure 16.: Example of a graph with $\gamma(G) = \gamma_F(G) + 1$

Brešar and Rall observe that the class of graphs satisfying $\gamma(G) = \gamma_F(G)$ is an extension of the class of BG-graphs which is distinct from Type $\mathcal{X}$ graphs. An open question regarding fair domination is whether a lower bound may be found for $\gamma_F(G)$ in terms of $\gamma(G)$. If, for example, one could find a constant $c > \frac{1}{2}$ such that $\gamma_F(G) \geq c\gamma(G)$, that would improve the Clark-Suen inequality.

3.2 Edge Critical Graphs

There are two classes of graphs that are critical with respect to the domination number: *edge-critical graphs* and *vertex-critical graphs*. In an edge-critical graph, the domination number decreases if an edge is added; in vertex-critical graphs, the domination number decreases if a vertex is deleted. Here, we concentrate on the class of edge-critical graphs.

A graph G is *k-edge-domination-critical*, or simply k-edge-critical if $\gamma(G) = k$ and for every pair of nonadjacent vertices $u, v \in V(G)$, $\gamma(G + \{u, v\}) = k - 1$. In other words, the domination number decreases if any missing edge is added to the graph G.

Note that a graph G is 1-edge-critical if and only if G is a complete graph. It is also straightforward to characterize 2-edge-critical graphs, using the following theorem.

Theorem 3.2 *[43] A graph G is 2-edge-critical if and only if $\bar{G} = \bigcup_{i=1}^{t} K_{1,p_i}$ for some $t \geq 1$.*

In other words, the only 2-edge-critical graphs are complements of unions of stars. Although Vizing's Conjecture has already been established for graphs G with $\gamma(G) = 2$, we can provide a different method of proof for 2-edge-critical graphs. We will show that the domination number equals the fair domination number in a 2-edge-critical graph and, therefore, we can apply Brešar and Rall's [11] result to show that Vizing's conjecture holds.

Theorem 3.3 *For any 2-edge-critical graph G, $\gamma(G) = \gamma_F(G)$.*

Proof. Let G be a 2-edge-critical graph. By Theorem 3.2, every 2-edge-critical graph is the complement of a union of stars. Consider $H = K_{1,n_1-1} \cup K_{1,n_2-1} \cup \ldots \cup K_{1,n_t-1}$, where $|V(K_{1,n_i-1})| = n_i$ and $t \geq 1$. Let v_i be the vertex of maximum degree in K_{1,n_i-1} for $i = 1, 2, \ldots, t$. Now let $G = \bar{H}$. Let $S_1 = \{v_i \mid i = 1, 2, \ldots, t\}$ and let $S_2 = V(G) - S_1$.

We need to show that S_1 and S_2 form a fair reception of G. Consider the set S_1. In order to externally dominate this set, we need at least 2 vertices from S_2. Take $v \in K_{1,n_i-1}$ where $v \neq v_i$. Then v externally dominates v_j for all $j \neq i$. Thus we must choose at least one more vertex from S_2 to externally dominate v_i. This implies $|D \cap S_2| - 1 \geq 1$ for all sets of vertices D that externally dominate S_1.

Now consider S_2. Choose $v_i \in S_1$. Then v_i externally dominates all vertices of S_2 except those that were in the star K_{1,n_i-1} in H. Thus, we must choose an additional vertex $v_j \neq v_i$ to externally dominate those vertices. We have $|D \cap S_1| - 1 \geq 1$.

Therefore, for any 2-edge-critical graph G, $\gamma_F(G) \geq 2$. But $\gamma_F(G) \leq \gamma(G)$ by Proposition 3.1 and since $\gamma(G) = 2$, we have $\gamma_F(G) = \gamma(G) = 2$. □

Since we know that Vizing's conjecture holds for any graph G that has $\gamma(G) = \gamma_F(G)$, this result implies Vizing's conjecture holds for all 2-edge-critical graphs.

Unfortunately, 3-edge-critical graphs are not easily characterized as 1- and 2-edge-critical graphs are. We provide a few examples of 3-edge-critical graphs.

Figure 17 provides seven examples of 3-edge-critical graphs. Observe that we can find a fair reception of size 3 in five of these graphs, as shown in Figure 18; however, it is difficult to tell if there is a fair reception of size 3 in the remaining two graphs in Figure 17. We do have the following result which may help in finding fair reception of size $\gamma(G)$ in an edge-critical graph G.

Theorem 3.4 *Let G be k-edge-critical with $\gamma(G) = \gamma_F(G) = k$. Then if $S_1, S_2, \ldots, S_k$ form a fair reception of G, each S_i for $i = 1, 2, \ldots, k$ is a complete subgraph of G.*

Proof. Assume G is k-edge-critical and that $\gamma(G) = \gamma_F(G) = k$. Let $S_1, S_2, ..., S_k$ form a fair reception of G. Without loss of generality, assume S_1 does not form a complete subgraph of G. For $u, v \in S_1$ such that $\{u, v\} \notin E(G)$, draw the edge $\{u, v\}$. Then we still have a fair reception of size

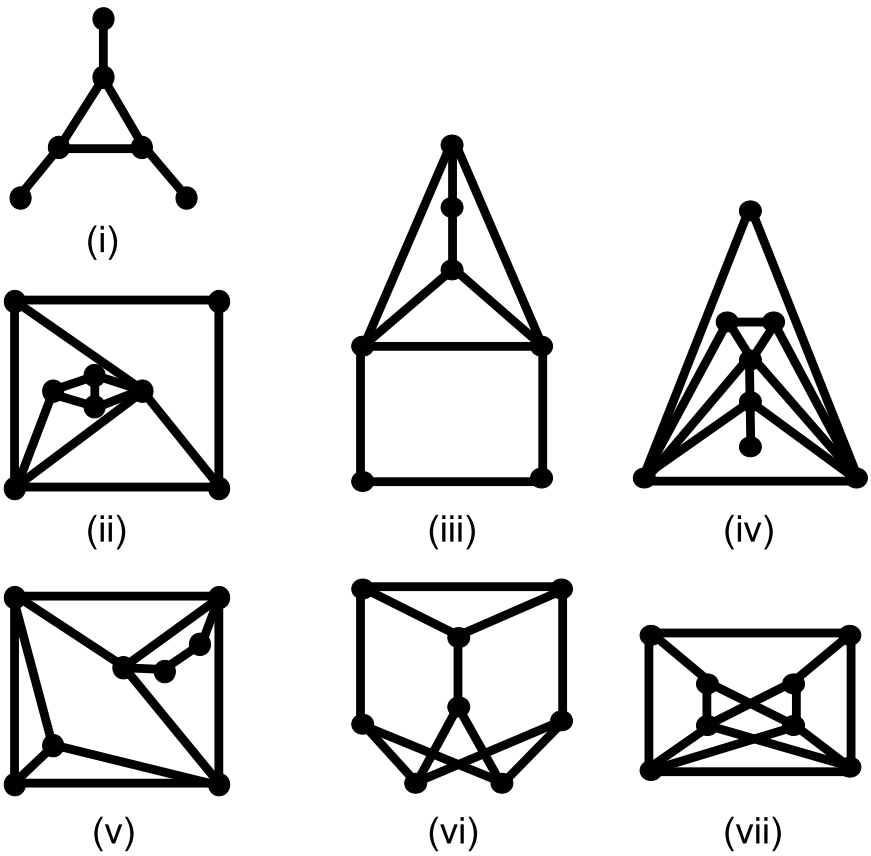

Figure 17.: Examples of 3-edge-critical graphs

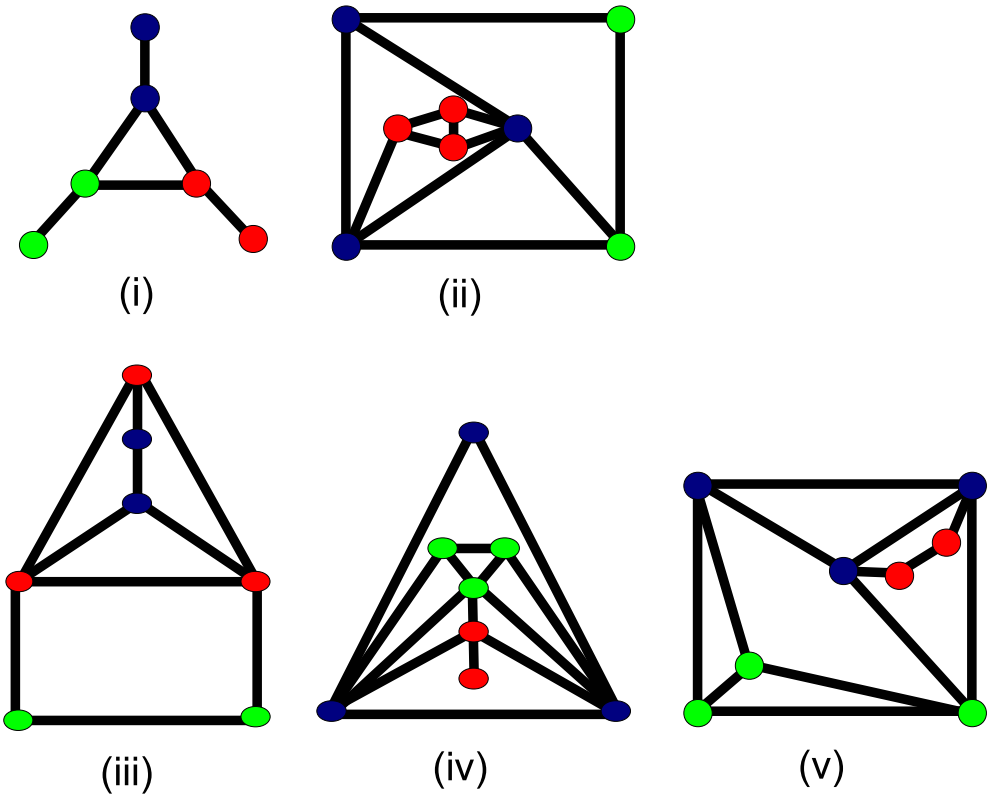

Figure 18.: Fair domination in 3-edge-critical graphs: In each graph, let S_1 = the vertices that are blue, S_2 = set of green vertices, and S_3 = red vertices. These sets form a fair reception of each graph of size 3.

k. But adding $\{u, v\}$ decreases $\gamma(G)$, so now we have $\gamma_F(G) > \gamma(G)$, a contradiction. Therefore, for each $i = 1, 2, ..., k$, S_i forms a complete subgraph of G. $\square$

We define the *line graph* of the complete graph on $[k]$ as follows: let $[k]$ denote the k-set $\{1, 2, \ldots, k\}$ and consider the set of 2-subsets of $[k]$. Let these $\binom{n}{2}$ 2-subsets be the vertices $v_1, v_2, \ldots, v_{\binom{k}{2}}$ of the line graph G_k. There is an edge $\{v_1, v_2\}$ between vertices $v_1, v_2 \in V(G)$ if and only if $v_1 \cap v_2 \neq \emptyset$. For the line graph G_k, $\gamma(G_k) = \lceil \frac{k-1}{2} \rceil$. If k is even then a γ-set of G_k is $\{1, 2\}, \{3, 4\}, \ldots, \{k-1, k\}$, and $\gamma(G) = \lceil \frac{k-1}{2} \rceil$. If k is odd then a γ-set of G_k is $\{1, 2\}, \{3, 4\}, \ldots, \{k-2, k-1\}$, and $\gamma(G) = \frac{k-1}{2}$.

Lemma 3.1 *If k is even, then the line graph G_k is edge-critical.*

Proof. Let D be a dominating set for G_k, where k is even. Without loss of generality, let $D = \{\{1, 2\}, \{3, 4\}, \ldots, \{k-1, k\}\}$. Now add an edge between two vertices in D, say $\{\{1, 2\}, \{3, 4\}\}$ to form the graph G'_k. Then $D' = \{\{1, 2\}, \{5, 6\}, \ldots, \{k-1, k\}\}$ is a dominating set of G'_k and $|D'| = |D| - 1$. Hence, G_k is edge-critical when k is even. $\square$

Consequently, if there is a fair reception of G_k of size $\lceil \frac{k-1}{2} \rceil$, then each set S_i, $i = 1, 2, \ldots, \frac{k-1}{2}$ is a complete subgraph of G_k.

Note that for any k, we can find a fair reception of G_k of size $\lfloor \frac{k}{3} \rfloor$. Consider partitioning the set $[k]$ into 3-subsets; without loss of generality, say we have $\{1, 2, 3\}, \{4, 5, 6\}$, and so on. Then the vertices generated by each set form the sets $S_1, S_2, \ldots, S_{\lfloor \frac{k}{3} \rfloor}$. So we have, for example, $S_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. By forming our sets S_i in this way, we ensure that no vertex in S_j dominates a vertex in S_i for $i \neq j$. We also require at least two vertices from $V(G_k) - S$ to dominate each S_i and so these sets satisfy the criteria to be a fair reception. As $\gamma(G_k) = \lceil \frac{k-1}{2} \rceil$ we have $\gamma_F(G_k) \geq \lfloor \frac{k}{3} \rfloor \geq \frac{2}{3}\gamma(G_k)$. Now, observe that we have a lower bound on $\gamma_F(G_{k+6})$ in terms of $\gamma_F(G_k)$.

Lemma 3.2 *For any k, $\gamma_F(G_{k+6}) \geq \gamma_F(G_k) + 2$.*

Proof. Let $S_1, S_2, \ldots, S_{\gamma_F(G_k)}$ form a fair reception of $\gamma(G_k)$. Now add the 6 points $\{1, 2, 3, 4, 5, 6\}$ to $[k]$ and consider G_{k+6}. We can form a fair reception of this graph by adding $S_{\gamma_F(G_k)+1} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $S_{\gamma_F(G_k)+2} = \{\{4, 5\}, \{4, 6\}, \{5, 6\}\}$ to $S_1, S_2, \ldots, S_{\gamma_F(G_k)}$. Thus, $\gamma_F(G_{k+6}) \geq \gamma_F(G_k) + 2$. $\square$

Note that finding a good upper bound for $\gamma_F(G_k)$ is much more difficult. We know that $\gamma_F(G_k) \leq \gamma(G_k)$. It remains an open problem whether we can improve this upper bound.

We observe that the line graph G_k is claw-free, and so we can apply Corollary 2.4, which states that for a claw-free graph and any graph H without isolated vertices,

$$\gamma(G_k \Box H) \geq \frac{1}{2}\gamma(G_k)(\gamma(H)+1).$$

Note, also, that we can apply Theorem 3.1 to get $\gamma(G_k \Box H) \geq \gamma_F(G_k)\gamma(H) \geq \frac{2}{3}\gamma(G_k)\gamma(H)$.

Chapter 4

Conclusion

Vizing's conjecture, as stated in 1963, is that the domination number of the Cartesian product of two graphs is at least the product of their domination numbers. The first major result related to the conjecture was from Barcalkin and German [4] in 1979 when they defined decomposable graphs and proved Vizing's conjecture holds for the so-called A-class, now commonly called BG-graphs. Hartnell and Rall's [27] 1995 breakthrough established the truth of Vizing's conjecture for what they called Type $\mathcal{X}$ graphs; this class of graphs is an extension of the BG-graphs. Brešar and Rall [11] in 2009 defined fair reception and fair domination. They proved that Vizing's conjecture holds for graphs with domination number equal to fair domination number. The class of such graphs is an extension of the BG-graphs which is distinct from Type $\mathcal{X}$ graphs. We also know that Vizing's conjecture is true for any graph with domination number less than 4; this was proved in 2004 by Sun [44].

Another approach to proving Vizing's conjecture is to find a constant $c > 0$ so that $\gamma(G\Box H) \geq c\gamma(G)\gamma(H)$, with the hope that eventually this constant will improve to 1. Clark and Suen [17] were able to do this in 2000 for $c = \frac{1}{2}$, and we were able to tighten their arguments to prove a slightly improved inequality.

As Vizing's conjecture is not yet proved for all graphs, several researchers have studied Vizing-like conjectures for other graph products and other types of domination. We provided a summary of some Vizing-like results for fractional domination, integer domination, upper domination, upper total domination, paired domination, and independence domination. In addition, we stated a few conjectures which remain open problems and would contribute to efforts to prove Vizing's conjecture. Two of these conjectures involve independent domination, and one is known as the projection conjecture (Conjecture 2.5). A proof of any of these three conjectures would imply the truth of Vizing's conjecture.

We also defined fair reception and fair domination, as introduced by Brešar and Rall, and included a proof of their Vizing-like inequality relating the domination number of the Cartesian product of graphs G and H to the fair domination numbers of G and H. It remains an open question whether we can find a constant $c > \frac{1}{2}$ so that $\gamma_F(G) \geq c\gamma(G)$ for any graph G. We do know that there are graphs for which $\gamma_F(G) = \gamma(G) - 1$, and we believe the line graph G_k could have fair domination number much smaller than domination number; however it remains difficult to find a lower bound on the fair domination number of a graph in terms of the domination number.

Finally, we considered fair domination in edge critical graphs. We found that a fair reception of an edge-critical graph G of size $\gamma(G)$ must have each set S_i induce a complete subgraph of G. We also provided a proof that Vizing's conjecture is true for 2-edge-critical graphs. This result, of course, was already known since we know Vizing's conjecture holds for any graph with domination number less than 4; however, it is an example of how we might use the idea of fair domination to prove that Vizing's conjecture is true for certain graphs.

Note that a common method of proof in most of the Vizing-like results is to partition a dominating set D of $G\Box H$ and project the vertices of D onto G or H. It is unclear whether this particular method will be useful to prove Vizing's conjecture. As long as Vizing's conjecture remains unresolved, possible next steps in attempt to prove it are to continue studying Vizing-like conjectures, particularly those relating domination and independent domination. One might also study fair domination further, with hopes of finding a lower bound on the fair domination number of a graph. We also note that the BG-graphs, Type $\mathcal{X}$ graphs, and graphs with fair domination number equal to domination number are all defined by a partition of the vertex set of a graph. It could be useful to find a new way of partitioning the vertices of a graph in such a way that we can establish the truth of Vizing's conjecture for an even larger class of graphs.

Table 1: Symbols

Symbol	Description
$\gamma(G)$	Domination number
$\gamma_F(G)$	Fair domination number
$\delta(G)$	Minimum vertex degree
$\Delta(G)$	Maximum vertex degree
$\chi(G)$	Chromatic number
$i(G)$	Independent domination number
$\gamma_t(G)$	Total domination number
$\gamma_c(G)$	Connected domination number
$\gamma_{cl}(G)$	Clique domination number
$\rho_2(G)$	2-packing number
$\gamma_f(G)$	Fractional domination number
$\gamma_{\{k\}}(G)$	$\{k\}$-domination number
$\alpha(G)$	Independence number
$\Gamma(G)$	Upper domination number
$\gamma_t^{\{k\}}(G)$	Total $\{k\}$-domination number
$\Gamma_t(G)$	Upper total domination number
$\gamma_{pr}(G)$	Paired domination number
$\gamma^i(G)$	Independence domination number

References

[1] R. Aharoni, E. Berger, and R. Ziv. A tree version of Kőnigs theorem. *Combinatorica*, 22(3):335-343, 2002.

[2] R. Aharoni and T. Szabó. Vizing's conjecture for chordal graphs. *Discrete Math.*, 309(6):1766-1768, 2009.

[3] R. B. Allan and R. C. Laskar. On domination and independent domination numbers of a graph. *Discrete Math.*, 23:73-76, 1978.

[4] A. M. Barcalkin and L. F. German, The external stability number of the Cartesian product of graphs. *Bul. Akad. Štiince RSS Moldoven*. no.1 5-8, 1979.

[5] C. Berge, *Theory of Graphs and its Applications*. Methuen, London, 1962.

[6] B. Brešar. On Vizing's conjecture. *Discuss. Math. Graph Theory*, 21(1):5-11, 2001.

[7] Boštjan Brešar, Vizing-like conjecture for the upper domination of Cartesian products of graphs - the proof. *Electron. J. Combin.*, 12:Note 12, 6 pp. (electronic), 2005.

[8] Boštjan Brešar, Paul Dorbec, Wayne Goddard, Bert L. Hartnell, Michael A. Henning, Sandi Klavžar, and Douglas F. Rall, Vizing's Conjecture: A Survey and Recent Results, 2009.

[9] B. Brešar, M. A. Henning, and S. Klavžar. On integer domination in graphs and Vizing-like problems. *Taiwanese J. Math.*, 10(5):1317-1328, 2006.

[10] B. Brešar, M. A. Henning, and D. F. Rall. Paired-domination of Cartesian products of graphs. *Util. Math.*, 73:255-265, 2007.

[11] Boštjan Brešar and Douglas F. Rall, Fair Reception and Vizing's Conjecture. *Journal of Graph Theory*. 45-54, 2009.

[12] Robert C. Brigham and Julie R. Carrington, Global Domination. *Domination in Graphs*. Edited by Teresa W. Haynes, Stephen T. Hedetniemi and Peter J. Slater. 301-320, 1998.

[13] G. Chen, W. Piotrowski and W. Shreve, A partition approach to Vizing's conjecture. *J. Graph Theory* 21 103-111, 1996.

[14] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, Total domination in graphs. *Networks*, 10:211-219, 1980.

[15] E. J. Cockayne, T. W. Haynes and S. T. Hedetniemi, Extremal graphs for inequalities involving domination parameters. *Discrete Math.*, 2000.

[16] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs. *Networks*, 7:247-261, 1977.

[17] W. Edwin Clark and Stephen Suen, An Inequality Related to Vizing's Conjecture. *The Electronic Journal of Combinatorics*, 2000.

[18] G. S. Domke, S. T. Hedetniemi, R. C. Laskar, and G. Fricke. Relationships between integer and fractional parameters of graphs. *Graph theory, combinatorics, and applications, Vol. 1* (Kalamazoo, MI, 1988), Wiley-Intersci. Publ., 371-387. Wiley, New York, 1991.

[19] Paul Dorbec, Michael A. Henning, Douglas F. Rall, On the upper total domination number of Cartesian products of graphs. *J. Comb. Optim.* 16: 68-80, 2008.

[20] M. El-Zahar and C. M. Pareek, Domination number of products of graphs. *Ars Combin.* 31 223-227, 1991.

[21] C. F. De Jaenisch, *Applications de l'Analyse mathematique an Jen des Echecs*, 1862.

[22] R. J. Faudree, R. H. Schelp and W. e. Shreve, The domination number for the product of graphs. *Congr. Numer.* 79 29-33, 1990.

[23] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, On graphs having domination number half their order. *Period. Math. Hungar.*, 16: 287-293, 1985.

[24] D. C. Fisher, Domination, fractional domination, 2-packing, and graph products. *SIAM J. Discrete Math.*, 7(3):493-498, 1994.

[25] D. C. Fisher, J. Ryan, G. Domke, and A. Majumdar, Fractional domination of strong direct products. *Discrete Appl. Math.*, 50(1):89-91, 1994.

[26] B. L. Hartnell and D. F. Rall, On Vizing's conjecture. *Congr. Numer.* 82 87-96, 1991.

[27] B. L. Hartnell and D. F. Rall, Vizings conjecture and the one-half argument. *Discuss. Math. Graph Theory*, 15(2):205-216, 1995.

[28] Bert Hartnell and Douglas F. Rall, Domination in Cartesian Products: Vizing's Conjecture. *Domination in Graphs*. Edited by Teresa W. Haynes, Stephen T. Hedetniemi and Peter J. Slater. 163-189, 1998.

[29] Teresa W. Haynes, Stephen T. Hedetniemi and Peter J. Slater, *Fundamentals of Domination in Graphs*. 41-65, 155-181, 1998.

[30] Henning MA, Rall DF On the total domination number of Cartesian products of graph. *Graphs Comb.* 21:63-69, 2005.

[31] Michael A. Henning, Distance Domination in Graphs. *Domination in Graphs*. 321-349, 1998.

[32] Pak Tung Ho, A note on the total domination number. *Util. Math.* 77:97-100, 2008.

[33] Xinmin Hou and You Lu, On the $\{k\}$-domination number of Cartesian products of graphs. *Discrete Math.*, 309: 3413-3419, 2009.

[34] M. S. Jacobson and L. F. Kinch, On the domination number of products of graphs: I. *Ars Combin.* 18: 33-44, 1983.

[35] M. S. Jacobson and L. F. Kinch, On the domination of the products of graphs II: trees. *J. Graph Theory* 10 97-106, 1986.

[36] F. Jaeger and C. Payan, Relations du type Nordhaus-Gaddum pour le nombre d'absorption d'un graphe simple. *C. R. Acad. Sci. Paris*, 274: 728-730, 1972.

[37] Ning Li and Xinmin Hou, On the total $\{k\}$-domination number of Cartesian products of graphs. *Journal of Combinatorial Optimization*, 2009.

[38] Richard J. Nowakowski and Douglas F. Rall, Associative graph products and their independence, domination and coloring numbers. *Discuss. Math. Graph Theory* 16:53-79, 1996.

[39] O. Ore, Theory of Graphs. *Amer. Math. Soc. Colloq. Publ.*, 38 (Amer. Math. Soc., Providence, RI), 1962.

[40] C. Payan and N. H. Xuong, Domination-balanced graphs. *J. Graph Theory*, 6: 23-32, 1982.

[41] B. Randerath and L. Volkmann, Characterization of graphs with equal domination and covering number. *Discrete Math.* 191:159-169, 1998.

[42] W. W. Rouse Ball, *Mathematical Recreation and Problems of Past and Present Times*, 1892.

[43] D. Sumner and P. Blitch, Domination critical graphs. *J. Combin. Theory Ser. B* 34:65-76, 1983.

[44] L. Sun. A result on Vizing's conjecture. *Discrete Math.*, 275(1-3):363-366, 2004.

[45] V. G. Vizing, The Cartesian product of graphs. *Vyčisl. Sistemy* 9: 30-43, 1963.

[46] V. G. Vizing, Some unsolved problems in graph theory. *Uspehi Mat. Nauk* 23:117-134, 1968.

[47] H. B. Walikar, B. D. Acharya, and E. Sampathkumar. Recent developments in the theory of domination in graphs. *MRI Lecture Notes in Math., Mahta Research Instit., Allahabad*, volume 1, 1979.

[48] A. M. Yaglom and I. M. Yaglom. Challenging mathematical problems with elementary solutions. *Volume 1: Combinatorial Analysis and Probability Theory*, 1964.